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The Influence of the Quadrupole Pairing on the Non-rotational States of Deformed Odd-Nuclei

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(Summary)

The influence of the quadrupole pairing on the characteristics of the one-quasiparticle states of the deformed odd-nuclei is investigated in the free-quasiparticle approximation. Results concerning ^{163}Dy and ^{163}Ho nuclei show a better agreement between the theoretical and experimental spectra of the non-rotational states when quadrupole components of the pairing are taken into account. The $|\Delta N|=2$-mixing between positive parity states ($K^{\pi}\downarrow=1/2^{+}$) can be treated properly in this case, i.e. without artificial change of the deformation parameters.

Analytical Approximate Eigenvalues of Bounded Quantum Systems

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A general systematic way of obtaining analytical approximate expressions for the eigenvalues of bounded quantum systems is proposed. The method is based on a nonlinear transformation of the perturbation parameter that leads to improved perturbation series. The bounded harmonic oscillator is discussed as an illustrative example.
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1. Introduction

Bounded quantum systems prove to be very useful in simulating physical phenomena [1, 2] (and references therein). To this end, it is helpful to have analytical expressions for the eigenvalues as functions of the boundary that take into account the main physical features of the problem as accurately as possible. Attempts have certainly been made in the case of the bounded harmonic oscillator and hydrogen atom which are not altogether satisfactory [2—4].

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To see this more clearly, let us focus our attention on the bounded harmonic oscillator (BHO). In appropriate units the Schrödinger equation can be written $H\Psi_n = E_n\Psi_n$, where

$$H = -(1/2)d^2/dx^2 + x^2/2, \tag{1}$$

and $\Psi_n(\pm x_0) = 0$, $(x_0 > 0)$.

For small enough x_0 values, the eigenenergies can be expanded as [1, 2]

$$x_0^2 E_n(x_0) = \sum_{j=0}^{\infty} E_n^{(j)} x_0^{4j}, \tag{2}$$

where the perturbation coefficients $E_n^{(j)}$ are easily calculated by means of the hypervirial perturbative method [1, 2]. This expansion is the basis for most analytical approaches [2—5].

An acceptable analytical expression for the eigenvalues of the BHO should satisfy the expansion (2) as accurately as possible and lead to the eigenvalues of the harmonic oscillator $E_n(\infty) = n + 1/2$, $n = 0, 1, \ldots$, as x_0 tends to infinity [2—5]. Besides, from a physical point of view it is of utmost importance to take into account that

$$\partial E_n(x_0)/\partial x_0 < 0, \tag{3}$$

for all x_0 values [2] (and references therein). It is known that the analytical formulas obtained from the Padé approximants do not always obey Eq. (3) for all x_0 values [2]. On the other hand, the more accurate Vawter's method [5] is not general enough since it applies to a restricted class of problems [2].

The purpose of the present communication is to suggest a simple systematic way of obtaining analytical expressions for the eigenvalues of bounded systems that fulfil the abovementioned requirements. For the sake of simplicity the BHO is considered as an illustrative example. A quite general approach based on a nonlinear mapping is developed in Section 2 and the coth method [5] is improved in Section 3. Conclusions are found in Section 4.

2. The nonlinear mapping

Since $E_n(x_0)$ behaves as in Eq. (2) for small enough x_0 values and tends to $E_n(\infty) = n + 1/2$ as $x_0 \to \infty$, we can write

$$U(x_0) = K^{-1/2}(1-a)^{-1/2} \sum_{j=0}^{\infty} U_j(K)\, a^j, \tag{4}$$

where $U(x_0) = x_0^2 E_n(x_0)$,

$$a = Kx_0^4/(1 + Kx_0^4), \tag{5}$$

and K is an adjustable parameter. Clearly, Eq. (5) maps $0 \leq x_0 < \infty$ onto $0 \leq a < 1$. The expansion of the r. h. s. of Eq. (4) in powers of x_0^4 shows that the coefficients U_j are given by

$$U_j = \sum_{i=0}^{j} (-1)^{j-i} \binom{1/2-i}{j-i} K^{1/2-i} E_n^{(i)}. \tag{6}$$

From a practical viewpoint we have to truncate the series (4) at (say) $j = N$. If our approach is to give the exact energy value as x_0 tends to infinity, it is necessary that

Table 1

Lowest energy level of the bounded harmonic oscillator for several x_0 values

x_0	$K=K_1^a$	$K=K_0^b$	coth[c]	exact[d]
0.50	9.90226100	9.90225865	9.90225864	9.90225865
1.00	2.60310334	2.59691591	2.59691991	2.59691967
1.50	1.43072247	1.37736116	1.37787471	1.37786351
2.00	1.14771820	1.06870123	1.07502796	1.07497192
2.50	1.06201611	0.99059242	1.01018344	1.00990321
3.00	1.03018333	0.97104505	1.00101812	1.00078216
3.50	1.01635736	0.96578377	1.00014993	1.00003610
4.00	1.00960710	0.96419757	1.00009051	1.00000010
4.50	1.00600389	0.96366061	1.00000435	1.00000000
5.00	1.00394154	0.96345888	1.00000000	1.00000000

[a] $K_1=0.95081306$ ($N=3$ in Eq. (7)); [b] $K_0=0.078370491$ ($N=3$); [c] $C_3=1.0347\times10^{-9}$ (from Eq. (10)); [d] Power series method [6]

$$\sum_{j=0}^{N} U_j(K)=n+1/2, \tag{7}$$

which determines the appropriate K value. From all the roots of Eq. (7) only those satisfying Eq. (3) are acceptable. It is found that there exists a positive real number K_0 so that Eq. (3) is obeyed only for $K\geqq K_0$. The choice $K=K_0$ seems to be appropriate in those cases where the asymptotic limit $E_n(\infty)$ is unknown.

Table 1 shows the ground-state energy level of the BHO calculated by means of K_0 and the smallest root of Eq. (7) (say K_1) larger than K_0. In the former case the approximate eigenvalue decreases too quickly leading to a rather small estimate of $E_n(\infty)$ whereas in the latter one the approximate eigenvalue approaches too slowly to the exact limit. However, these results are acceptable and the method seems to be promising since only the first 3 perturbation corrections have been taken into account. Owing to the fact that they are analytically calculated in terms of the quantum numbers [2] the excited states can be treated in the same way. Similar conclusions are drawn for them.

3. The coth method

Some time ago Vawter [5] proposed the following approach:

$$E_n(x_0)\simeq E_n(\infty)\coth f_M(x_0), \tag{8}$$

where

$$f_M(x_0)=\sum_{j=0}^{M} C_j x_0^{4j+2}, \tag{9}$$

and the coefficients C_j can be written in terms of the $E_n^{(i)}$'s [2, 5]. Vawter's equation yields accurate results for all the x_0 values when $M=1$ [2, 5]. Unfortunately, C_2 is negative and Eq. (8) is useless in the large-x_0 regime when $M=2$ [2]. However, the coth approximation can be improved by adding a term with $j=M+1$ to Eq. (9) and determining C_{M+1} so that

$$\partial f_{M+1}(x_0)/\partial x_0>0 \tag{10}$$

for all x_0 values.

The results for $M=2$ shown in Table 1 are in good agreement with the "exact" eigenvalues obtained by means of the power series method [6].

4. Further comments and conclusions

This communication is intended to suggest a way of dealing with bounded systems. Although the coth method proves to be more accurate, it is restricted to the example discussed here. The nonlinear-mapping approach, on the other hand, is more general since it does not require the knowledge of $E_n(\infty)$ and, in fact, it was successfully applied before to a number of problems leading to asymptotic expansions for small and large values of the expansion variable [7] (and references therein).

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Аналитические аппроксимированные собственные значения связанных квантовых систем

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(Резюме)*

Предлагается обобщенный систематический способ нахождения аналитических аппроксимированных выражений для собственных значений связанных квантовых систем. Метод основан на нелинейной трансформации параметра возмущения, что приводит к улучшению серий возмущения. В качестве иллюстрационного примера обсуждается связанный гармонический осциллятор.

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