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BOSONIC AND FERMIONIC ENTROPY FOR ROTATING $U(1) \otimes U(2)$ DILATON BLACK HOLE

ZHAO REN, WU YUEQIN and ZHANG LICHUN
Department of Physics, Yanbei Normal Institute
037000 Datong, P.R. China

Abstract. By using the method of quantum statistics, we directly derive the partition function of bosonic and fermionic field under the background of rotating $U(1) \otimes U(2)$ Dilaton black hole. The difficulty in solving the wave equation is avoided. Then via the improved brick-wall method, membrane model, we calculate the entropy of bosonic and fermionic field. We obtain that the entropy of black hole is proportional to the area of horizon. The stripped term and the divergent logarithmic term in the original brick-wall method no longer exist. Why the entropy of the scalar or Dirac field outside the event horizon is the entropy of black hole is solved. We offer a new simple and direct way of calculating the entropy of various complicated black hole.

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1. Introduction

The entropy of a black hole is one of the important subjects in theoretical physics. Because entropy has statistical meaning, understanding the entropy of the black hole involves the microscopic essence of the black hole. To understand the entropy, we need a good theory of quantum gravitation. At present, the statistical origin of the black hole is not solved [1]. On the other hand, the result that the entropy of a black hole is proportion to the area off horizon was put forward in a lot of papers [2–9]. The most frequently used method is the brick-wall method advanced by G't Hooft [7]. This method is used to study the statistical properties of a free scalar and Dirac field in various black holes [10–14] and it is found that the general expression of the black hole's entropy consists of a term which is proportional to the area of its event horizon and a divergent logarithmic term which is not proportional to the area of its event horizon. However it is doubted that: first, the entropy of the scalar or Dirac field outside the event horizon is the entropy of the black hole; second, to obtain the result that the entropy of a black hole is proportional to the area of its event horizon, we have to detract the logarithmic term and take L^3 term as the contribution of vacuum surrounding the system at long distance; third, solving the wave function of the scalar or Dirac field by WKB approximation is complicated. The above mentioned problems with the original brick-wall method are unnatural.

It is well-known that the entropy of a black hole is proportional to the area of horizon. The existence of horizon is the basic property of the black hole. It has already been proved that the general existence of horizon leads to the Hawking effect [15]. And whether there is the black hole's entropy or not relates to the existence of horizon [16]. Then it reveals that it is a natural disposition that the entropy of the black hole is proportional to the area of horizon. Its value has nothing to do with the radiation field outside the horizon. And the horizon only has the property of the two-dimensional membrane in a three-dimensional space. Does the number of quantum states of the two-dimensional membrane correspond to the entropy of the black hole? If it is done, calculating the entropy of the membrane will be the key issue.

We derive the bosonic and fermionic partition functions in axisymmetric rotating $U(1) \otimes U(2)$ Dilaton black hole directly by using the quantum statistical method and then obtain the integral expression of the system's entropy [17]. The integral expression of bosonic field is in correspondence with the expression of Ref. [10]. Then we use the membrane model to calculate entropy [17–20]. As a result, the left out term in original brick-wall method no longer exists. The problem that the state density near the event horizon is divergent does not exist either. We also consider the spinning degeneracy of radiational particles. In the whole process, we avoid the difficulty in solving wave equation. The physics idea is clear, calculation is simple and the result is reasonable. It offers a neat way of studying the black hole's entropy. In this article, we take the simplest functional form of the temperature ($C = \hbar = K_B = 1$).

2. Bosonic Entropy

The linear element of the rotating $U(1) \otimes U(2)$ dilaton space-time is given by [21]

$$\begin{aligned} \mathrm{d}S^2 &= g_{tt}\,\mathrm{d}t^2 + 2g_{t\varphi}\,\mathrm{d}t\,\mathrm{d}\varphi + g_{rr}\,\mathrm{d}r^2 + g_{\theta\theta}\,\mathrm{d}\theta^2 + g_{\varphi\varphi}\,\mathrm{d}\varphi^2 \\ &= \frac{\Delta - a^2\sin^2\theta}{\Sigma}\,\mathrm{d}t^2 - \frac{\Sigma}{\Delta}\,\mathrm{d}r^2 - \Sigma\,\mathrm{d}\theta^2 \\ &\quad - \frac{(r^2+a^2-D^2)^2 - \Delta a^2\sin^2\theta}{\Sigma}\sin^2\theta\,\mathrm{d}\varphi^2 \\ &\quad - \frac{2a\sin^2\theta[(r^2+a^2-D^2)-\Delta]}{\Sigma}\,\mathrm{d}t\,\mathrm{d}\varphi \end{aligned} \tag{1}$$

where Δ and Σ are defined as $\Delta = r^2 + a^2 + Q^2 + P^2 - D^2 - 2Mr = (r - r_+)(r - r_-)$, $\Sigma = r^2 - D^2 + a^2\cos^2\theta$, M, a, D, P and Q are mass, the specific angular momentum, the dilaton charge, the magnetic charge and the electric charge of the black hole, respectively.

The radiational temperature of black hole is

$$T_+ = \frac{r_+ - r_-}{4\pi(r_+^2 - D^2 + a^2)} \tag{2}$$

where $r_\pm = M \pm \sqrt{M^2 + D^2 - a^2 - P^2 - Q^2}$ are locations of outer and inner horizon of the black hole, respectively. The area of horizon of the black hole is

$$A_+ = 4\pi(r_+^2 + a^2 - D^2)\,. \tag{3}$$

Based on the theory of general relativity, an observer at rest at an infinite distance gets the frequency move of the particles from the surface of a star as follows:

$$\nu = \nu_0\sqrt{-g'_{tt}}$$

where ν_0 is the natural frequency of the atoms on the surface of star and ν is the natural frequency of the particles at r point observed by the observer at rest at an infinite distance.

In the view of the Refs [10, 22], the natural radiational temperature got by observer at rest at an infinite distance is as follows:

$$T = \frac{T_+}{\sqrt{-g'_{tt}}} \tag{4}$$

where T_+ is the equilibrium temperature

$$\begin{aligned} g'_{tt} &= \frac{g_{tt}g_{\varphi\varphi} - g_{t\varphi}^2}{g_{\varphi\varphi}} \\ &= -\frac{(r-r_+)(r-r_-)(r^2-D^2+a^2\cos^2\theta)}{(r^2+a^2-D^2)^2-(r-r_+)(r-r_-)a^2\sin^2\theta}\,. \end{aligned} \tag{5}$$

For bosonic gas, we calculate the partition function as follows:

$$\ln Z = -\sum_i g_i \ln(1-\mathrm{e}^{-\beta\varepsilon_i}) \tag{6}$$

In unit volume, the number of quantum states with the energy between ε and $\varepsilon + \mathrm{d}\varepsilon$ or the frequency between ν and $\nu + \mathrm{d}\nu$ is as follows:

$$g(\nu)\,\mathrm{d}\nu = j4\pi\nu^2\,\mathrm{d}\nu \tag{7}$$

where j is the spinning degeneracy of the particles. Since in space-time (1) the area of two-dimensional curved surface at random point r is

$$A(r) = \int \mathrm{d}A = \int \sqrt{g}\,\mathrm{d}\theta\,\mathrm{d}\varphi \tag{8}$$

where $g = \begin{vmatrix} g_{\theta\theta} & g_{\theta\varphi} \\ g_{\varphi\theta} & g_{\varphi\varphi} \end{vmatrix} = g_{\theta\theta}g_{\varphi\varphi}$. The volume element of the lamella at random point r outside the horizon is as follows:

$$\mathrm{d}V = A(r)\sqrt{g}_{rr}\,\mathrm{d}r\,. \tag{9}$$

So, the partition function of the system at the lamella with random thickness at point r outside the horizon is as follows:

$$\begin{aligned} \ln Z &= \int A(r)\sqrt{g}_{rr}\,\mathrm{d}r \sum_i g_i \sum_{n=1} \frac{1}{n}\,\mathrm{e}^{-n\beta\varepsilon_i} \\ &= j4\pi \int A(r)\sqrt{g}_{rr}\,\mathrm{d}r \sum_{n=1} \frac{1}{n} \int_0^\infty \mathrm{e}^{-\frac{nh\nu}{T}}\,\nu^2\,\mathrm{d}\nu \\ &= j\,\frac{1}{90}\pi^2 \int \frac{A(r)\sqrt{g}_{rr}\,\mathrm{d}r}{\beta^3} \\ &= j\,\frac{\pi^2}{90} \int \frac{\sqrt{g_{\theta\theta}g_{\varphi\varphi}g_{rr}}\,\mathrm{d}r d\theta d\varphi}{\beta^3} \end{aligned} \tag{10}$$

where $\frac{1}{\beta} = T$. Using the relation between entropy and partition function

$$S = \ln Z - \beta_0 \frac{\partial \ln Z}{\partial \beta_0} \tag{11}$$

we have:

$$S_b = j\,\frac{2}{45}\pi^2\frac{1}{\beta_0^3}\int\frac{\sqrt{g_{\theta\theta}g_{\varphi\varphi}g_{rr}}}{(-g'_{tt})^{3/2}}\,\mathrm{d}r = j\,\frac{2\pi^2}{45\beta_0^3}\int\mathrm{d}\theta\,\mathrm{d}\varphi \times\int\frac{[(r^2+a^2-D^2)^2-(r-r_+)(r-r_-)a^2\sin^2\theta]^2}{(r-r_+)^2(r-r_-)^2(r^2-D^2+a^2\cos^2\theta)}\sin\theta\,\mathrm{d}r\,. \tag{12}$$

where $\beta_0 = \dfrac{1}{T_+} = \dfrac{4\pi(r_+^2-D^2+a^2)}{r_+-r_-}$, $\beta=\beta_0\sqrt{-g'_{tt}}$. Eq. (12) is in correspondence with (53) of Ref. [10]. In the above integral with respect to r, we take the integral region $[r_++\varsigma,\, r_++N\varsigma]$. is a small non-negative quantity and N is a constant larger than one. We have

$$\begin{aligned} S_b &= j\,\frac{4\pi^3}{45\beta_0^3} \times\int_0^\pi\mathrm{d}\theta\int_{r_++\varsigma}^{r_++N\varsigma}\frac{[(r^2+a^2-D^2)^2-(r-r_+)(r-r_-)a^2\sin^2\theta]^2}{(r-r_+)^2(r-r_-)^2(r^2-D^2+a^2\cos^2\theta)}\sin\theta\,\mathrm{d}r\\ &= j\,\frac{4\pi^3}{45\beta_0^3}\int_0^\pi\frac{(r_+^2+a^2-D^2)^4}{(r_+-r_-)^2(r_+^2-D^2+a^2\cos^2\theta)}\left[\frac{N-1}{N\varsigma}\right]\sin\theta\,\mathrm{d}\theta \\ &\quad + G(r_+,N,\varsigma)\\ &= j\,\frac{8\pi^3}{45\beta_0^3}\frac{(r_+^2+a^2-D^2)^3}{(r_+-r_-)^2}\frac{\alpha}{\sin\alpha\cos\alpha}\left[\frac{N-1}{N\varsigma}\right]+G(r_+,N,\varsigma)\end{aligned} \tag{13}$$

where

$$\begin{aligned} &G(r_+,N,\varsigma) = \\ &j\,\frac{4\pi^3}{45\beta_0^3}\int_0^\pi[frac8r_+(r_+^2+a^2-D^2)^3(r_+-r_-)^2(r_+^2-D^2+a^2\cos^2\theta)\\ &-\frac{2r_+(r_+^2+a^2-D^2)^4}{(r_+-r_-)^2(r_+^2-D^2+a^2\cos^2\theta)^2}\\ &-\frac{2(r_+^2+a^2-D^2)^4}{(r_+-r_-)^3(r_+^2-D^2+a^2\cos^2\theta)}\\ &-\left.\frac{2(r_+^2+a^2-D^2)^2a^2\sin^2\theta}{(r_+-r_-)(r_+^2-D^2+a^2\cos^2\theta)}\right]\sin\theta\,\mathrm{d}\theta\ln N\\ &+j\,\frac{4\pi^3}{45\beta_0^3}\int_0^\pi\sin\theta\,\mathrm{d}\theta\int_{r_++\varsigma}^{r_++N\varsigma}F(r)\,\mathrm{d}r\,,\end{aligned} \tag{14}$$

$$F(r)=\sum_{n=2}^{\infty}\frac{f_1^{(n)}(r_+)}{n!}(r-r_+)^{n-2}-\sum_{n=1}^{\infty}\frac{f_2^{(n)}(r_+)}{n!}(r-r_+)^{n-1}+f_3(r) \quad (15)$$

$$\begin{aligned}
f_1(r)&=\frac{(r^2+a^2-D^2)^4}{(r-r_-)^2(r^2-D^2+a^2\cos^2\theta)}, \qquad f_1'(r)=\frac{\mathrm{d}f_1}{\mathrm{d}r}\\
f_2(r)&=\frac{2(r^2+a^2-D^2)^2a^2\sin^2\theta}{(r-r_-)(r^2-D^2+a^2\cos^2\theta)}, \qquad f_2'(r)=\frac{\mathrm{d}f_2}{\mathrm{d}r}\\
f_3(r)&=\frac{a^4\sin^4\vartheta}{r^2-D^2+a^2\cos^2\theta}, \qquad \alpha=\arctan\frac{a}{\sqrt{r_+^2-D^2}}
\end{aligned}$$

From (3.17) in the Ref. [7] we know that when $N\varsigma = L \gg r_+$ (that is $N \gg 1$), if we take $\varsigma=\frac{T_+}{90}$ as ultraviolet cutoff, the main part of the black hole's entropy is proportional to the area of horizon, which is the result of 't Hooft brick-wall. To ensure the radiational field and black hole are in steady equilibrium state [23], the infrared cutoff should not satisfy $L \gg r_+$.

Now we use membrane model to do discussion. We take the ultraviolet cutoff as

$$\varsigma=\frac{T_+}{90}\,\frac{\alpha}{\sin\alpha\cos\alpha}\,\frac{N-1}{N}\,, \quad (16)$$

and we take $N\varsigma$ as infrared cutoff, so that our result is irrelevant to the parameters N and ς. We take

$$S_b=j\pi(r_+^2+a^2-D^2)+G(r_+,N,\varsigma)=j\,\frac{1}{4}A_+ +G(r_+,N,\varsigma)\,. \quad (17)$$

When $N\to\infty$, that is $N\varsigma=L\gg r_+$ and $j=1$, the result returns to the result of Ref. [10]. When $N\to 1$, $\varsigma\to 0$, and $N\varsigma\to 0$. That is, both the ultraviolet cutoff and infrared cutoff tend to the outer horizon. The entropy of black hole is

$$S_b=\frac{1}{4A_+}\,. \quad (18)$$

In the calculation we use $\lim_{N\to 1} G(r_+,N,\varsigma)\to 0$. And $A_+=4\pi(r_+^2+a^2-D^2)$ is the area of horizon. When both the ultraviolet cutoff and infrared cutoff tend to the outer horizon, our entropy is irrelevant to the radiation field outside the black hole. So the entropy given by (18) should be the entropy of black hole.

3. Fermionic Entropy

For fermionic gas, the partition function is as follows:

$$\ln Z = \sum_i g_i \ln\left(1 + \mathrm{e}^{-\beta\varepsilon_i}\right) . \tag{19}$$

From (7), we obtain

$$\begin{aligned}\ln Z &= \int A(r)\sqrt{g_{rr}}\,\mathrm{d}r \sum_i g_i \sum_{n=1} \frac{(-1)^{n-1}}{n}\,\mathrm{e}^{-n\beta\varepsilon_i} \\ &= \omega 4\pi \int A(r)\sqrt{g_{rr}}\,\mathrm{d}r \sum_{n=1} \frac{(-1)^{n-1}}{n} \int\limits_0^\infty \mathrm{e}^{-\frac{nh\nu}{T}}\,\nu^2\,\mathrm{d}\nu \\ &= \omega\frac{\pi^2}{90}\frac{7}{8}\int \frac{A(r)\sqrt{g_{rr}}\,\mathrm{d}r}{\beta^3} = \omega\frac{\pi^2}{90}\frac{7}{8}\int \frac{\sqrt{g_{\theta\theta}g_{\varphi\varphi}g_{rr}}\,\mathrm{d}r\,\mathrm{d}\theta\,\mathrm{d}\varphi}{\beta^3}\end{aligned} \tag{20}$$

where ω is spinning degeneracy of fermionic particles.

Using the result of Section 2, we can get the fermionic entropy of axisymmetric rotating $U(1)\otimes U(2)$ Dilaton black hole is as follows:

$$S_f = \omega\frac{7}{8}\frac{1}{4}A_+ \tag{21}$$

4. Conclusion

In the above analysis, we derive partition functions of various fields in axisymmetric rotating $U(1)\otimes U(2)$ Dilaton black hole directly by using the statistical method. We avoid the difficulty in solving wave equation. Since we use the improved brick-wall method, membrane model, to calculate the entropy of various fields, the problem that the state density is divergent around horizon does not exist any more. In order to ensure the entropy of black hole is irrelevant to the random parameters, the parameters N and ς in integral expression have the forms of (16). The entropy is the inherent property of black hole. From (16), we know the cutoff ς is irrelevant to angle θ. The result is different from the result of Ref. [14]. In Ref. [14], the cutoff is related to angle θ. In our calculation, as $N \to 1$, $\varsigma \to 0$ and $N\varsigma \to 0$, that is, the ultraviolet cutoff and infrared cutoff both approach the outer horizon of black hole. From (18) and (21), we know that the divergent logarithmic term and L^3 term in the original brick-wall method no longer exist. The obtained entropy is proportional to the area of its horizon, so it can be taken as black hole's entropy.

In above analysis, we know that by using the statistical and membrane model methods, the doubt that why the entropy of the scalar or Dirac field outside the event horizon is the entropy of black hole in the original brick-wall method does not exist and the complicated approximations in solution are avoided. In the whole process, the physics idea is clear; the calculation is simple; and the result is reasonable. We also consider the influence of the spinning degeneracy of particles on the entropy. For calculating the entropy in various space-times, we only need to change the red-shift factor, but the others are the same. Especially for unspherically symmetric space-times, we can directly derive the entropy of various quantum particles without solving the complicated wave equation. We offer a new neat way of studying the entropy of different kinds of complicated black holes.

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