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LIGHT SOLITONS IN NONLINEAR MEDIA. LIGHT PROPAGATION IN FIBER-LIKE MEDIA

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Abstract. The light beam behaviour in nonlinear media is considered in the case when the nonlinear polarization contains susceptibilities of third and fifth order. A number of soliton solutions of the nonlinear differential equations describing self-action effects is obtained in analytical form. The existence of the solitary pulses obtained is discussed from energetically point of view.

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1. Introduction

In the paper [1] some nonlinear light phenomena with regard to the high power light density have been considered. For dielectric media which possess a centre of symmetry the polarization $\mathbf{P}$ has been presented in the form

$$\mathbf{P} = \kappa\mathbf{E} + \chi^{(3)}\mathbf{E}^3 + \chi^{(5)}\mathbf{E}^5 \tag{1}$$

where $\mathbf{E}$ denotes the electric field, κ is the linear susceptibility and $\chi^{(3)}$ and $\chi^{(5)}$ are the nonlinear ones of third and fifth order, respectively. Using (1) together with Maxwell's equations, the following nonlinear equation of motion was obtained in [1]

$$\left(\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)\mathbf{E} = \frac{4\pi}{c^2}\left[\kappa\frac{\partial^2\mathbf{E}}{\partial t^2} + \chi^{(3)}\frac{\partial^2\mathbf{E}^3}{\partial t^2} + \chi^{(5)}\frac{\partial^2\mathbf{E}^5}{\partial t^2}\right]. \tag{2}$$

If the radiation field is taken to be of the form

$$\mathbf{E} = \mathcal{E}(\mathbf{r}, t)\, \mathrm{e}^{\mathrm{i}(kz-\omega t)} + \text{c.c.}, \tag{3}$$

and the complex amplitude $\mathcal{E}(\mathbf{r},t)$ is supposed to vary slowly, the following nonlinear differential equation was obtained in [1]

$$\begin{gathered}\frac{1}{c^{*2}}\frac{\partial^2\mathcal{E}}{\partial t^2}-\frac{\partial^2\mathcal{E}}{\partial x^2}-\frac{\partial^2\mathcal{E}}{\partial y^2}-\frac{\partial^2\mathcal{E}}{\partial z^2}-\mathrm{i}\left(\frac{2\omega}{c^{*2}}\frac{\partial\mathcal{E}}{\partial t}+2k\frac{\partial\mathcal{E}}{\partial z}\right)\\ -\left(\frac{\omega^2}{c^{*2}}-k^2\right)\mathcal{E}-\frac{4\pi\omega^2}{c^2}\left[3\chi^{(3)}|\mathcal{E}|^2+10\chi^{(5)}|\mathcal{E}|^4\right]\mathcal{E}=0\end{gathered} \tag{4}$$

where $\mathcal{E}=\mathcal{E}(x,t)$ and

$$c^*=c/\sqrt{1+4\pi\kappa}\,. \tag{5}$$

Equation (4) was analyzed in [1] for some cases of self-focusing of a light beam, self-modulation of optical pulses and transmission of the light. In [1] the dispersion relation

$$ck=\omega\sqrt{1+4\pi\kappa} \tag{6}$$

was supposed to be valid and $\chi^{(3)}$ was taken to be positive.

In the considerations made in [1] more realistic cases of the dispersion relations $\omega(k)$ were used in which ω depended on $|E|$. The problem was situated on the base of the Lagrangian formalism of the field theory. At that, in [1] Eq. (4) is analyzed and solved for the cases when the electric field amplitude $\mathcal{E}$ is a function of the coordinates (x,z or y,z).

In the present paper $\mathcal{E}$ is supposed to be weakly dependent on x and y as well as dependent on z and t. Then it can be presented in the form

$$\mathcal{E}(x,y,z,t)=\Phi(x,y)\tilde{\mathcal{E}}(z,t)\,. \tag{7}$$

After the substitution of (7) into (4), neglecting the second derivatives of $\Phi(x,y)$ and integrating over x and y, we obtain

$$\begin{gathered}\frac{1}{c^{*2}}\frac{\partial^2\tilde{\mathcal{E}}}{\partial t^2}-\frac{\partial^2\tilde{\mathcal{E}}}{\partial z^2}-\mathrm{i}\left(\frac{2\omega}{c^{*2}}\frac{\partial\tilde{\mathcal{E}}}{\partial t}+2k\frac{\partial\tilde{\mathcal{E}}}{\partial z}\right)\\ -\left(\frac{\omega^2}{c^{*2}}-k^2\right)\tilde{\mathcal{E}}-\frac{4\pi\sigma\omega^2}{c^2}\left[3\chi^{(3)}|\tilde{\mathcal{E}}|^2+10\rho\chi^{(5)}|\tilde{\mathcal{E}}|^4\right]\tilde{\mathcal{E}}=0\end{gathered} \tag{8}$$

where σ and $\sigma\rho$ are the mean values of Φ^2 and Φ^4 over the beam cross section.

Using the method based on the Lagrangian formalism in the field theory, the following expression for the energy density of the system considered can be obtained (see [1])

$$w(z,t)=\frac{c^2}{4\pi\omega^2}\left[\frac{1}{c^{*2}}|\frac{\partial\tilde{\mathcal{E}}}{\partial t}|^2+|\frac{\partial\tilde{\mathcal{E}}}{\partial z}|^2+\mathrm{i}k(\tilde{\mathcal{E}}^*\frac{\partial\tilde{\mathcal{E}}}{\partial z}-\tilde{\mathcal{E}}\frac{\partial\tilde{\mathcal{E}}^*}{\partial z})+\tilde{U}\right] \tag{9}$$

where

$$\tilde{U} = -\frac{c^2}{4\pi\omega^2}\left[(\frac{\omega^2}{c^{*2}} - k^2)|\tilde{\mathcal{E}}|^2 + \frac{2\pi\sigma\omega^2}{c^2}(3\chi^{(3)}|\tilde{\mathcal{E}}|^4 + \frac{20}{3}\rho\chi^{(5)}|\tilde{\mathcal{E}}|^6)\right] \quad (10)$$

plays the role of a potential energy. The light propagation will be considered as a motion determined by the extremum of $\tilde{U}$ with respect to $|\mathcal{E}|$. Such a manner the condition

$$\frac{\mathrm{d}\tilde{U}}{\mathrm{d}|\tilde{\mathcal{E}}|^2} = 0 \quad (11)$$

together with the series expansion (1) (in which the E^5-term is supposed to be smaller than $\tilde{E}^3$-one), shall lead to some restrictions upon the parameters of the solutions obtained. In particular, (11) leads to

$$|\tilde{\mathcal{E}}|^2 = \frac{3}{20\rho}\left|\frac{\chi^{(3)}}{\chi^{(5)}}\right|(1 \pm \sqrt{1-Q}) \quad (12)$$

where

$$Q \equiv \frac{10\rho\chi^{(5)}(\omega^2 - k^2c^{*2})c^2}{9\pi(\chi^{(3)})^2\omega^2c^{*2}} \,. \quad (13)$$

(At that sign plus in (12) has to be used for which $\tilde{U} = \tilde{U}_{\min}$). On the other hand the nonlinear terms in (8) lead to

$$|\tilde{\mathcal{E}}_{\max}| < \sqrt{\frac{3}{10\rho}\left|\frac{\chi^{(3)}}{\chi^{(5)}}\right|} \quad (14)$$

which must be satisfied if E^5-term is smaller compared to E^3-one [1].

As in the previous considerations [1] the most interesting cases $\chi^{(3)} > 0$, $\chi^{(5)} < 0$ and $\chi^{(3)} < 0$, $\chi^{(5)} > 0$ will be considered separately.

1.1. Case $\chi^{(3)} > 0$, $\chi^{(5)} < 0$

In this case Eq. (8) has the following soliton solutions.

1.1.1. *Kink (anti-kink) solution*

$$\begin{aligned}\tilde{\mathcal{E}}(z,t) = \tilde{\mathcal{E}}_0\, \mathrm{e}^{\mathrm{i}(qz-\Omega t+\varphi)} \sinh\frac{z-z_0-vt}{\mathcal{L}} \\ \times\left(1 + \sec^2\eta \sinh^2\frac{z-z_0-vt}{\mathcal{L}}\right)^{-\frac{1}{2}}\end{aligned} \quad (15)$$

where

$$q = -k - \Delta q\,,$$

$$\Delta q = \pm \frac{k\beta^*}{\sqrt{b^{*2}-1}}\sqrt{1-\frac{\omega^2}{k^2c^{*2}}} \qquad (\beta^* \equiv \frac{v}{c^*} > 1)\,, \tag{16}$$

$$\Omega = -\omega - \Delta\Omega, \qquad \Delta\Omega = \pm\sqrt{\frac{k^2c^{*2}-\omega^2}{\beta^{*2}-1}}\,, \tag{17}$$

$$\tilde{\mathcal{E}}_0 = \pm\frac{3}{2}\tan\sqrt{\frac{\chi^{(3)}}{5\rho|\chi^{(5)}|(2-\cos 2\eta)}}\,, \tag{18}$$

$$\mathcal{L} = \frac{2c(2-\cos 2\eta)}{3\omega\chi^{(3)}\sin 2\eta}\sqrt{\frac{10\rho|\chi^{(5)}|(2-cos2\eta)}{3\pi\sigma}}\,, \tag{19}$$

$$\eta = \arccos\left(\frac{3\sqrt{1-Q}}{2\sqrt{1-Q}+1}\right)^{\frac{1}{2}}\,. \tag{20}$$

The soliton (15) represents a kink (sign plus in (14)) or anti-kink (sign minus in (14)) which moves along the z-axis with a velocity v greater than the light phase velocity c^* in the medium without dispersion or radiation ($\beta^* \equiv v/c^* > 1$). Naturally, it must be $v < c$ (c being the light velocity in vacuum), i. e.

$$c^* < v < c^*\sqrt{1+4\pi\kappa}\,. \tag{21}$$

The wave vector component q as well as the frequency Ω also depend on v. At that the kink frequency $\Delta\Omega$ can be very small. The parameter $\mathcal{L}$ characterizes the length of the formation (15). In fact $2\mathcal{L}$ plays the role of the effective width of the kink. The constant z_0 appears as a result of the translational invariance of the system and represents the center of the solitary formation (15); φ is a phase constant.

The parameter η is determined with the help of the extremum condition (11). Taking into account that $\tilde{U}$ has a minimum when $|\tilde{\mathcal{E}}|^2$ takes its maximal value

$$|\tilde{\mathcal{E}}|^2 = \tilde{\mathcal{E}_0}^2\cos^2\eta$$

and using (12) (with sign plus, which corresponds to $\tilde{U}_{\min}$) the expression (20) can be obtained. The condition (14) leads to the inequality

$$0 < k^2c^{*2} - \omega^2 < \frac{9\pi\sigma(\chi^{(3)})^2\omega^2c^{*2}}{10\rho|\chi^{(5)}|c^2} \tag{22}$$

which must be satisfied. It must be noted that Eq. (8) has also a solution of the type

$$\tilde{\mathcal{E}}_1(z,t) = \tilde{\mathcal{E}}_0\, \mathrm{e}^{\mathrm{i}(qz-\omega t+\varphi)} \tag{23}$$

where q, Ω, $\tilde{\mathcal{E}}_0$ and η have the same values as the kink (15). To compare (15) and (23) we shall calculate the energies W (for the kink) and W_1 (for the plane-wave (23)). Using the energy density (14), we find that

$$W = \int\limits_{-\infty}^{+\infty} W(z,t)\,\mathrm{d}z \tag{24}$$

contains divergent terms. The calculation of W_1 leads to an expression which contains the same divergent terms as W. Thus the energy difference

$$\begin{aligned} \Delta W \equiv W - W_1 = &-\frac{3c^3\eta}{2\pi\omega c^{*2}\sqrt{30\pi|\chi^{(5)}|(\beta^{*2}-1)}} \\ &\times\Bigg[\frac{k^2c^{*2}}{\omega^2} - 1 \mp \sqrt{\left(\frac{k^2c^{*2}}{\omega^2}-1\right)(\beta^{*2}-1)} \\ &+ \frac{27\pi c^{*2}\sigma(\chi^{(3)})^2}{8c^2\rho|\chi^{(3)}|(\beta^{*2}-1)}\,\frac{1-2\cos 2\eta}{(2-\cos 2\eta)^2}\left(1+\frac{\sin 2\eta}{2\eta}\,\frac{2-\cos 2\eta}{1-2\cos 2\eta}\right)\Bigg]. \end{aligned} \tag{25}$$

The analysis of (25) shows that ΔW can be negative. Thus the solitary formation (15) can be energetically more favourable than the plane-wave (23).

For the case of Rb-vapor with atom density $N = 10^8\ \mathrm{cm}^{-3}$, assuming the atomic susceptibilities to be $\chi^{(3)} = 4\times10^{-31}$ esu and $\chi^{(5)}_{at} = -2.1\times10^{-41}$ esu [2] and taking the light wavelength $\lambda = 1.06\times10^{-4}$ esu and $\eta \approx 0.52$, we obtain

$$\tilde{\mathcal{E}}_0 = 4.3\times10^{-4}\ \text{esu}\,, \qquad \mathcal{L} = 8.5\sqrt{\frac{\rho}{\sigma}(\beta^{*2}-1)}\ \text{cm}.$$

1.1.2. *"Bell" soliton solution*

$$\begin{aligned} \tilde{\mathcal{E}}(z,t) = \tilde{\mathcal{E}}_0\, \mathrm{e}^{\mathrm{i}(qz-\Omega t+\varphi)} \cosh\frac{z-z_0-vt}{\mathcal{L}} \\ \times\left(\operatorname{cosec}^2\eta\cosh^2\frac{z-z_0-vt}{\mathcal{L}} - 1\right)^{-\frac{1}{2}} \end{aligned} \tag{26}$$

where

$$q = -k - \Delta q\,,$$

$$\Delta q = \pm \frac{k\beta^*}{\sqrt{b^{*2}-1}} \sqrt{1 - \frac{\omega^2}{k^2 c^{*2}} - \frac{4(\beta^{*2}-1)}{k^2 \mathcal{L}^2} + \frac{R}{k^2}}\,, \tag{27}$$

$$\Omega = -\omega - \Delta\Omega\,, \qquad \Delta\Omega = v\Delta q\,, \tag{28}$$

$$R \equiv \frac{\omega^2}{c^{*2}} - k^2 + \frac{12\pi\chi^{(3)}\omega^2 \sin^2\eta}{c^2}\tilde{\mathcal{E}}_0^2 - \frac{40\pi\rho|\chi^{(5)}|\omega^2 \sin^4\eta}{c^2}\tilde{\mathcal{E}}_0^4\,, \tag{29}$$

$$\tilde{\mathcal{E}}_0 = \pm \frac{\cot\eta}{2}\sqrt{\frac{3\chi^{(3)}}{5\rho|\chi^{(5)}|}}\,, \tag{30}$$

$$\mathcal{L} = \frac{c}{\omega\chi^{(3)}\sin 2\eta}\sqrt{\frac{10\rho|\chi^{(5)}|(\beta^{*2}-1)}{3\pi\sigma}}\,, \tag{31}$$

$$\beta^{*2} \equiv (\frac{v}{c^*})^2 > 1\,. \tag{32}$$

As one can see from (26) the bell soliton will exist if

$$\operatorname{cosec}^2\eta > 1\,. \tag{33}$$

The analysis of the extremal condition (11) shows that taking into account (33) the electric amplitude must be of the form $\tilde{\mathcal{E}}(r,t)\,\mathrm{e}^{\mathrm{i}\varphi}$, where the phase constant φ can be taken the same as in (26). Such a manner the value

$$\tilde{\mathcal{E}}(z,t) = \left[\frac{3\chi^{(3)}}{20\rho|\chi^{(5)}|}\left(1 - \sqrt{1 - \frac{10\rho|\chi^{(5)}|(k^2c^{*2}-\omega^2)c^2}{9\pi\sigma(\chi^{(3)})^2\omega^2c^{*2}}}\right)\right]^{\frac{1}{2}} \tag{34}$$

corresponds to the extremum (minimum) of $\tilde{U}$. On the other hand, as it can be seen from (26), $|\tilde{\mathcal{E}}(z,t)|$ takes the value $\tilde{\mathcal{E}}_0 \sin\eta$ at infinity (where (26) turns into a plane wave). Hence, from (30) and (34) the following expression for η can be obtained:

$$\eta = \arcsin[1 - \frac{10\rho|\chi^{(5)}|(k^2c^{*2}-\omega^2)c^2}{9\pi\sigma(\chi^{(3)})^2\omega^2c^{*2}}]^{\frac{1}{4}}\,. \tag{35}$$

Also, the condition (14) leads to the inequality

$$0 < k^2c^{*2} - w^2 < \frac{9\pi\sigma(\chi^{(3)})^2\omega^2c^{*2}}{10\rho|\chi^{(5)}|c^2}\,. \tag{36}$$

One can see that for η given by (35) $R = 0$. According to the consideration made in Sec. 3 of [1] the condition $R = 0$ plays the role of the dispersion relation between ω, k and $\tilde{\mathcal{E}}_0$. The substitution of $R = 0$ into (27) and (28) gives the final forms of Δq and $\Delta\Omega$:

$$\Delta q = \pm \frac{k\beta^*}{\sqrt{\beta^{*2}-1}} \sqrt{1 - \frac{\omega^2}{k^2 c^{c*2}} - \frac{4(\beta^{*2}-1)}{k^2 \mathcal{L}^2}}\,, \tag{37}$$

$$\Delta\Omega = v\Delta q\,. \tag{38}$$

The condition $R = 0$ coupled with (36) gives the following final form of the inequality which must be satisfied:

$$\frac{3\pi\sigma(\chi^{(3)})^2\omega^2 c^{*2}}{10\rho|\chi^{(5)}|c^2} \sin^2 2\eta < k^2 c^{*2} - \omega^2 < \frac{9\pi\sigma(\chi^{(3)})^2\omega^2 c^{*2}}{10\rho|\chi^{(5)}|c^2}\,. \tag{39}$$

As it is easy to see, the condition (14) is always satisfied.

Equation (8) has also a homogeneous plane-wave solution of the form

$$\tilde{\mathcal{E}}(z,t) = \mathcal{E}_0\, \mathrm{e}^{\mathrm{i}(qz-\Omega t+\varphi)} \tag{40}$$

with the same parameters as the bell soliton (26). The subtraction of the energy W_1, corresponding to (40), from the energy W (corresponding to (26)) leads to the result:

$$\begin{aligned} \Delta W \equiv W - W_1 = -&\frac{3k^2c^3\eta}{\pi\omega^3\sqrt{30\pi\sigma\rho|\chi^{(5)}|(\beta^{*2}-1)}} \\ \times \Bigg[1 - &\frac{\omega^2}{k^2c^{*2}}(1 + \frac{3\pi\sigma(\chi^{(3)})^2\sin^2 2\eta}{10\rho|\chi^{(5)}|}) \\ \pm &\frac{\omega}{kc^*}\sqrt{\left[1 - \frac{\omega^2}{k^2c^{*2}}\left(1 + \frac{3\pi\sigma(\chi^{(3)})^2\sin^2 2\eta}{10\rho|\chi^{(5)}|}\right)\right](\beta^{*2}-1)} \\ + &\frac{3\pi\sigma(\chi^{(3)})^2\omega^2(2-\beta^{*2})}{80\rho|\chi^{(5)}|k^2c^2}\left(1 + \frac{\sin 2\eta}{2\eta}\,\frac{2-5\cos 2\eta}{1+2\cos 2\eta}\right)\Bigg]\,. \end{aligned} \tag{41}$$

The analysis of (41) shows that ΔW can be negative. Hence, the soliton formation (26) can be more favourable than (40). Besides, the soliton can propagates with a velocity v higher than the phase velocity c^* in the medium considered (see (32)) and without "smearing", i. e. in such nonlinear media the propagation of wave packets with velocity v in the interval

$$c^* < v < c$$

and without radiation takes place. Looking at (38) one can see that self-modulation effect takes place: the phase depends on the amplitude through the soliton length $\mathcal{L}$ and vice versa. For Rb-vapor, considered in the previous

Subcase 1, when $\eta \approx \frac{\pi}{2}$ the solitary formation (26) has a very small amplitude and $\mathcal{L} \approx 73.76\sqrt{\frac{\rho}{\sigma}(\beta^{*2}-1)}$ cm.

1.1.3. *"Bell" soliton solution*

$$\tilde{\mathcal{E}}(z,t) = \tilde{\mathcal{E}}_0\, \mathrm{e}^{\mathrm{i}(qz-\Omega t+\varphi)} \left(1 + \operatorname{sech} \frac{z - z_0 - vt}{\mathcal{L}}\right)^{\frac{1}{2}} \tag{42}$$

(compare with Eq. (58) in [1], where

$$q = -k \pm \Delta q\,,$$

$$\Delta q = \frac{\beta^*}{c^*}\sqrt{\frac{k^2c^{*2}-\omega^2}{\beta^{*2}-1}}\,, \qquad (\beta^* > 1)\,, \tag{43}$$

$$\Omega = -\omega \pm \Delta\Omega\,, \qquad \Delta\Omega = \frac{c^*}{\beta^*}\Delta q\,, \tag{44}$$

$$\tilde{\mathcal{E}}_0 = \pm\frac{3}{4}\sqrt{\frac{\chi^{(3)}}{5\rho|\chi^{(5)}|}}\,, \tag{45}$$

$$\mathcal{L} = \frac{c^*}{2}\sqrt{\frac{\beta^{*2}-1}{k^2c^{*2}-\omega^2}}\,, \tag{46}$$

$$k^2c^{*2} - \omega^2 = \frac{27\pi\sigma(\chi^{(3)})^2\omega^2c^{*2}}{32\rho|\chi^{(5)}|c^2}\,. \tag{47}$$

Equation (8) has also the plane-wave solution

$$\tilde{\mathcal{E}}_1 = \tilde{\mathcal{E}}_0\, \mathrm{e}^{\mathrm{i}(qz-\Omega t+\varphi)} \tag{48}$$

with the same parameters as for (42).

In this case the energy difference $\Delta W \equiv W - W_1$ is equal to

$$\Delta W = \frac{3c^3}{8\omega^3\sqrt{30\pi\sigma\rho|\chi^{(5)}|c^2}} \times \left[\frac{9\pi+4}{10\pi}\left(k^2 - \frac{\omega^2}{c^{*2}}\right) \mp \sqrt{\left(k^2 - \frac{\omega^2}{c^{*2}}\right)(\beta^{*2}-1)}\right]. \tag{49}$$

It is easy to see that $\Delta W > 0$, i. e. in this case the plane-wave (48) is energetically more favourable than the soliton.

1.1.4. *"Inverse bell" soliton solution*

$$\tilde{\mathcal{E}}(z,t) = \tilde{\mathcal{E}}_0\, \mathrm{e}^{\mathrm{i}(qz-\Omega t+\varphi)} \left(1 - \operatorname{sech} \frac{z - z_0 - vt}{\mathcal{L}}\right)^{\frac{1}{2}} \tag{50}$$

where q, σ, $\mathcal{E}_0$ and $\mathcal{L}$ are given by (43), (44), (45) and (46), respectively. Taking into account that (48) is also solution of (8), for the energy difference $\Delta W = W - W_1$ one obtains:

$$\begin{aligned} \Delta W = &- \frac{3c^3}{8\omega^3 \sqrt{30\pi\sigma\rho|\chi^{(5)}|(\beta^{*2}-1)}} \\ &\times \left[\frac{9\pi - 4}{10\pi}(k^2 - \frac{\omega^2}{c^{*2}}) \pm \frac{\omega}{c^*}\sqrt{(k^2 - \frac{\omega^2}{c^{*2}})(\beta^{*2}-1)}\right]. \end{aligned} \tag{51}$$

Obviously $\Delta W < 0$, i. e. the solitary formation (50) is energetically more favourable than the plane-wave. The soliton (50) moves with velocity $v > c^*$ ($c^* < v < c$).

When the field amplitude varies slowly in time so as

$$\left|\frac{\partial^2 \tilde{\mathcal{E}}}{\partial t^2}\right| \ll \omega \left|\frac{\partial \tilde{\mathcal{E}}}{\partial t}\right| \tag{52}$$

is satisfied, (8) turns into the following nonlinear Schrödinger equation with nonlinearities of third- and fifth-power with regard to the function:

$$\begin{aligned} &\mathrm{i}\left(\frac{2\omega}{c^{*2}}\frac{\partial \tilde{\mathcal{E}}}{\partial t} + 2k\frac{\partial \tilde{\mathcal{E}}}{\partial z}\right) + \frac{\partial^2 \tilde{\mathcal{E}}}{\partial z^2} + \left(\frac{\omega^2}{c^{*2}} - k^2\right)\tilde{\mathcal{E}} \\ &+ \frac{4\pi\sigma\omega^2}{c^2}(3\chi^{(3)}|\tilde{\mathcal{E}}|^2 + 10\rho\chi^{(5)}|\tilde{\mathcal{E}}|^4)\tilde{\mathcal{E}} = 0 \end{aligned} \tag{53}$$

which has the following soliton solutions when $\chi^{(3)} > 0$ and $\chi^{(5)} < 0$.

1.1.5. *First type kink (anti-kink) solution*

$$\begin{aligned} \tilde{\mathcal{E}}(z,t) = &\,\mathcal{E}_0\, \mathrm{e}^{\mathrm{i}(qz-\Omega t+\varphi)} \sinh\frac{z - z_0 - vt}{\mathcal{L}} \\ &\times \left(1 + \operatorname{sech}^2\eta \sinh^2\frac{z - z_0 - vt}{\mathcal{L}}\right)^{-\frac{1}{2}} \end{aligned} \tag{54}$$

where

$$\begin{aligned} q &= -k + \Delta q\,, \\ \Delta q &= \frac{\omega}{c^*}\beta^*\,, \end{aligned} \tag{55}$$

$$\Omega = -\omega + \Delta\Omega\,, \qquad \Delta\Omega = \frac{\omega}{2}(2+\beta^* - \frac{k^2c^{*2}}{\omega^2})\,, \tag{56}$$

$$\mathcal{E}_0 = \pm\frac{\coth\eta}{2}\sqrt{\frac{3\chi^{(3)}}{5\rho|\chi^{(5)}|}}\,, \tag{57}$$

$$\mathcal{L} = \frac{2c(\cos 2\eta - 2)}{3\omega\chi^{(3)}\sinh 2\eta}\sqrt{\frac{10\rho|\chi^{(5)}|}{3\pi\sigma}}\,, \tag{58}$$

$$\eta = \operatorname{arccos}h\Big(\frac{3\sqrt{1-Q}}{2\sqrt{1-Q}-1}\Big)^{\frac{1}{2}}\,, \tag{59}$$

Q being given by (13). The condition (14) leads to the inequalities

$$0 < k^2c^{*2} - \omega^2 < \frac{27\pi(\chi^{(3)})^2\omega^2c^{*2}}{40\rho|\chi^{(5)}|c^2}\,. \tag{60}$$

It must be noted that (54) differs from (15) by the replacement $\sec\eta \to \operatorname{sech}\eta$. We do not repeat the procedure of finding η described also in Sec. 1 of [1]. Taking into account that (53) has also a plane-wave solution of the type

$$\mathcal{E}_1(z,t) = \mathcal{E}_0\cosh\eta\,\mathrm{e}^{\mathrm{i}(qz-\Omega t+\varphi)} \tag{61}$$

with the same parameters q, ω, $\mathcal{E}_0$, η as (55), (56), (57), (59), respectively, for the energy difference $\Delta W \equiv W - W_1$ we obtain

$$\begin{aligned}\Delta W = &-\frac{3c^3\eta}{4\pi c^{*2}\omega\sqrt{30\pi|\chi^{(5)}|}}\Big[(\beta^* + \frac{2kc^*}{\omega})\beta^* - \frac{2k^2c^{*2}}{\omega^2}\\ &-\frac{27\pi c^{*2}\sigma(\chi^{(3)})^2(2\cosh 2\eta - 1)}{40c^2\rho|\chi^{(5)}|(\cosh 2\eta - 2)^2}\\ &\times\Big(1+\frac{\sinh 2\eta}{2\eta}\,\frac{\cosh 2\eta - 2}{2\cosh 2\eta - 1}\Big)\Big]\,.\end{aligned} \tag{62}$$

It must be noted that neglecting the second time derivatives of $\tilde{\mathcal{E}}$, the parameters of the soliton are changed drastically: trigonometric functions are changed by hyperbolic ones! Also $\Delta W > 0$ when $\beta^* \leq 1$ and a soliton formation is not energetically favourable in comparison with the plane-wave (61).

For Rb-vapor (see above) $\Delta q \approx 2.2\times10^5\beta^*$ cm, $\Delta\Omega \approx 8.9(1+\beta^*2)\ \mathrm{s}^{-1}$, $\mathcal{E}_0 \approx 6.6\times10^4$ esu, $\mathcal{L} \approx 8.13$ cm (when $\omega \approx kc^*$, $\eta \approx 1.146$).

1.1.6. *Second type kink (anti-kink) solution*

$$\tilde{\mathcal{E}}(z,t) = \tilde{\mathcal{E}}_0\,\mathrm{e}^{\mathrm{i}(qz-\Omega t+\varphi)}\Big(1+\,\mathrm{e}^{\pm\frac{z-z_0-vt}{\mathcal{L}}}\Big)^{-\frac{1}{2}} \tag{63}$$

where q is given by (55),

$$\Omega = -\omega + \Delta\Omega, \qquad \Delta\Omega = \frac{\omega}{2}\left(1 + \beta^{*2} - \frac{27\pi c^{*2}(\chi^{(3)})^2}{40\rho c^2|\chi^{(5)}|}\right), \tag{64}$$

$$\mathcal{E}_0 = \pm\frac{3}{2}\sqrt{\frac{\chi^{(3)}}{10\rho|\chi^{(5)}|}}, \tag{65}$$

$$\mathcal{L} = \frac{c}{3\omega\chi^{(3)}}\sqrt{\frac{10\rho|\chi^{(5)}|}{3\pi\sigma}} \tag{66}$$

and the inequality (14) is always satisfied.

The signs plus and minus in (63) correspond to anti-kink and kink, respectively. When $v < c^*$ the frequency of the kink $\Delta\Omega \approx \omega/2$ i. e. the generation of subharmonics is possible. Equation (53) has also a plane-wave solution of the form

$$\tilde{\mathcal{E}}(z,t) = \tilde{\mathcal{E}}_0\,\mathrm{e}^{\mathrm{i}(qz-\Omega t+\varphi)} \tag{67}$$

with the same parameters q, Ω and $\mathcal{E}_0$ as (55), (64) and (65), respectively. The light can have the form of a step, e. g. "anti-kink-like step", i. e. when $-\infty < z - z_0 - vt < 0$ $\tilde{\mathcal{E}}_1$ is given by (67) and when $0 < z - z_0 - vt < \infty$, $\tilde{\mathcal{E}}_1 = 0$. The comparison of the energy W_1 of such a formation with the energy W of the anti-kink (63) gives for the difference $\Delta W \equiv W - W_1$ the result

$$\Delta W = \frac{81\pi\sigma(\chi^{(3)})^2}{640\sqrt{30\pi\sigma}\,\omega(\rho|\chi^{(5)}|)^{\frac{3}{2}}} > 0\,.$$

Thus the step function is energetically more favourable than (63).

It must be noted that a solution of the type (63) does not exist for the "full equation" (8). Neglecting the second time derivative of the electric field we can obtain solutions (in particular solitons) which cannot exist in reality. Nevertheless, we shall give such solutions which exist according to (52).

1.1.7. *"Inverse bell" soliton solution of the form*

$$\begin{aligned}\tilde{\mathcal{E}}(z,t) = \mathcal{E}_0\,\mathrm{e}^{\mathrm{i}(qz-\Omega t+\varphi)}\cosh\frac{z-z_0-vt}{\mathcal{L}} \\ \times\left(1 + \operatorname{cosech}^2\eta\cosh^2\frac{z-z_0-vt}{\mathcal{L}}\right)^{-\frac{1}{2}}\end{aligned} \tag{68}$$

where q and Ω are given by (55) and (56), respectively,

$$\mathcal{E}_0 = \frac{3}{2}\coth\eta\sqrt{\frac{\chi^{(3)}}{5\rho|\chi^{(5)}|(2+\cosh 2\eta)}}, \tag{69}$$

$$\mathcal{L} = \frac{2c(2+\cosh 2\eta)}{3\omega\chi^{(3)}\sinh 2\eta}\sqrt{\frac{10\rho|\chi^{(5)}|}{3\pi\sigma}}\,, \tag{70}$$

$$\eta = \operatorname{arcsin} h\left(\frac{3\sqrt{1-Q}}{1-2\sqrt{1-Q}}\right)^{\frac{1}{2}}, \tag{71}$$

Q being given by (13), and

$$\frac{27\pi(\chi^{(3)})^2\omega^2 c^{*2}}{10\rho|\chi^{(5)}|c^2} < \omega^2 - k^2c^{*2} < \frac{18\pi(\chi^{(3)})^2\omega^2 c^{*2}}{5\rho|\chi^{(5)}|c^2}$$

has to be satisfied. Besides (68), Eq. (53) has a plane-wave soliton

$$\tilde{\mathcal{E}}_1(z,t) = \tilde{\mathcal{E}}_0 \sinh\eta\, \mathrm{e}^{\mathrm{i}(qz-\Omega t+\varphi)} \tag{72}$$

with q, Ω, $\mathcal{E}_0$ and η given by (55), (56), (59) and (71), respectively (the parameters of the soliton formation above). The comparison of the energies W (related to (68)) and W_1(related to (72)) gives

$$\begin{aligned}\Delta W \equiv W - W_1 = &-\frac{3c^3\eta}{4\pi\omega c^{*2}\sqrt{30\pi\sigma\rho|\chi^{(5)}|}} \\ &\left[(\beta^{*2} - \frac{k^2c^{*2}}{\omega^2} + \frac{27\pi\sigma c^{*2}(\chi^{(3)})^2(1+2\cosh 2\eta)}{40c^2\rho|\chi^{(5)}|(2+\cosh 2\eta)^2}\right. \\ &\left.\times\left(1 - \frac{\sinh 2\eta}{2\eta}\,\frac{2+\cosh 2\eta}{1+2\cosh 2\eta}\right)\right].\end{aligned} \tag{73}$$

The energy difference (73) can be negative ($\beta^* \geq 1$), i. e. (68) can be energetically more favourable than (72).

In fact, the inverse bell soliton above is a "dark" soliton. For it

$$|E|^2 \sim \frac{\cosh^2 \frac{z-z_0-vt}{\mathcal{L}}}{1+\operatorname{cosech}^2\eta\cosh^2\frac{z-z_0-vt}{\mathcal{L}}}\,,$$

i. e. the light is expelled from the soliton region.

1.1.8. *"Bell" soliton solution*

$$\tilde{\mathcal{E}}(z,t) = \mathcal{E}_0\, \mathrm{e}^{\mathrm{i}(qz-\Omega t+\varphi)}\left(1+\operatorname{sech}\eta\cosh\frac{z-z_0-vt}{\mathcal{L}}\right)^{-\frac{1}{2}} \tag{74}$$

where q and $\mathcal{E}_0$ are given by (55) and (65), respectively,

$$\Omega = -\omega + \Delta\Omega, \qquad \Delta\Omega = \frac{\omega}{2}\left(1+\beta^{*2} - \frac{c^{*2}}{\omega^2\mathcal{L}^2}\right), \tag{75}$$

and

$$\mathcal{L} = \frac{c \coth \eta}{3\omega\chi^{(3)}} \sqrt{\frac{10\rho|\chi^{(5)}|}{3\pi\sigma}} \, . \tag{76}$$

All of the quantities obtained have the same meaning as above but the parameter η cannot be determined by minimization of the potential energy $\tilde{U}$ (10). The parameter η can be connected with a given amplitude $\mathcal{E}_0$ (or the peak power density of the light beam). Another possibility is to normalize the solution (74) according to the condition

$$N = \int_{-\infty}^{+\infty} |E|^2 \, \mathrm{d}x \, \mathrm{d}y \, \mathrm{d}z$$

where N can be interpreted as a number of photons in the soliton region. The integration leads to

$$\mathcal{E}_0 = \left(\frac{N}{4\sigma\mathcal{L}} \frac{\coth \eta}{\eta} \right)^{\frac{1}{2}} \tag{77}$$

which gives

$$\eta = N \frac{\omega}{c} \sqrt{\frac{10\pi\rho|\chi^{(5)}|}{3\sigma}} \, . \tag{78}$$

On the whole $\mathcal{E}_0$ can be treated as a known quantity. In the case considered, the energy of the system W has the finite value

$$\begin{aligned} \Delta W = & - \frac{3c^3\eta}{8\pi c^{*2}\omega\sqrt{30\pi\rho|\chi^{(5)}|}} \bigg[1 - \beta^{*2} - \frac{27\pi c^{*2}\sigma(\chi^{(3)})^2 \tanh^2\eta}{40c^2\rho|\chi^{(5)}|} \\ & \times [1 - 4\coth^2\eta + \frac{6\coth\eta}{\eta^2}(1 - \eta\coth\eta)] \bigg] \, . \end{aligned} \tag{79}$$

The bell soliton (74) is favourable energetically if $\beta^* \ll 1$.

It must be noted that in this subcase the self-modulation effect also takes place. The frequency Ω (75) depends on the soliton length $\mathcal{L}$ and

$$\mathcal{E}_0^2 \mathcal{L}^2 = \frac{c^2}{12\pi\sigma\omega^2\chi^{(3)}} \, . \tag{80}$$

So the amplitude drives the phase and vice versa.

Equation (75) shows also that if β^* is small (v/c^*) $\Delta\Omega$ can be changed drastically ($\Omega \to \omega/2$) and the effect of self-induced transparency can take place.

It must be noted also that the full equation (8) (with the second time derivative) has not a solution which can be turned into (74) when the condition (52) and (53) are used (see the end of the Subcase 6 above). The condition (52) and Eq. (53) are self-consistent when $\beta^* \ll 1$. The analysis made leads to the well-known fact that neglecting of the higher derivative can drastically change the physical results. As it was mentioned in [1] the presence of the second time derivative imposed also a restriction on the soliton propagation velocity. In the solutions of the Schrödinger-like equations the velocity v is a free parameter. The analysis made here also has for an object to show that the use of such type equations (such considerations occur in the literature) is illegal although very often the form of the solutions is apparently the same as for the case when the second time derivative is not neglected.

1.2. The Case $\chi^{(3)} < 0$, $\chi^{(5)} > 0$

In this case Eq. (8) has the following soliton solutions

1.2.1. *Kink (anti-kink) solution*

$$\begin{aligned}\tilde{\mathcal{E}}(z,t) = \mathcal{E}_0\, \mathrm{e}^{\mathrm{i}(qz-\Omega t+\varphi)} \sinh \frac{z-z_0-vt}{\mathcal{L}} \\ \times \Big(1+\sec^2\eta \sinh^2 \frac{z-z_0-vt}{\mathcal{L}}\Big)^{-\frac{1}{2}}\end{aligned} \tag{81}$$

where

$$\begin{aligned} q &= -k - \Delta q\,, \\ \Delta q &= \pm \frac{k\beta^*}{\sqrt{1-\beta^2}} \sqrt{\frac{\omega^2}{k^2 c^{*2}} - 1}\,, \end{aligned} \tag{82}$$

$$\Omega = -\omega - \Delta\Omega, \qquad \Delta\Omega = \Delta q c^*/\beta^*\,, \tag{83}$$

$$\mathcal{E}_0 = \pm\frac{3}{2}\tan\eta \sqrt{\frac{|\chi^{(3)}|}{5\rho\chi^{(5)}(2-\cos 2\eta)}}\,, \tag{84}$$

$$\mathcal{L} = \frac{2c(2-\cos 2\eta)}{3\omega|\chi^{(3)}|\sin 2\eta}\sqrt{\frac{10\rho\chi^{(5)}(1-\beta^{*2})}{3\pi\sigma}}\,, \tag{85}$$

$$\eta = \arccos\Big(\frac{3\sqrt{1-Q}}{1+2\sqrt{1-Q}}\Big)^{\frac{1}{2}}\,, \tag{86}$$

Q being given by (13) and

$$0 < \omega^2 - k^2 c^{*2} < \frac{9\pi\sigma(\chi^{(3)})^2\omega^2 c^{*2}}{10\rho\chi^{(5)} c^2} \tag{87}$$

$$\beta \equiv \frac{v}{c^*} < 1\,. \tag{88}$$

Equation (81) looks as (15) but the parameters of the kink are changed due to the condition (88). The formation (81) propagates with a velocity $v < c^*$. Analogous to the case $\chi^{(3)} > 0$, $\chi^{(5)} < 0$, (8) has also a plane-wave solution of the type (23) with the parameters (82), (83), (84) and (86).

In this case the energy difference $\Delta W \equiv W - W_1$ (the self-kink energy) is

$$\begin{aligned}\Delta W = &- \frac{3c^3\eta}{2\pi\omega c^{*2}\sqrt{30\pi\sigma\rho\chi^{(5)}(1-\beta^{*2})}}\Bigg[1 - \frac{k^2c^{*2}}{\omega^2}\\ &\pm\sqrt{(1-\frac{k^2c^{*2}}{\omega^2})(1-\beta^{*2})} - \frac{27\pi\sigma c^{*2}(\chi^{(3)})^2(1-2\cos 2\eta)}{80\rho\chi^{(5)}c^2(2-\cos 2\eta)^2}\\ &\times\left(1 + \frac{\sin 2\eta}{2\eta}\,\frac{2-\cos 2\eta}{1-2\cos 2\eta}\right)\Bigg]\,.\end{aligned} \tag{89}$$

(As it was many times mentioned above W corresponds to the solitary formation and W_1 to the corresponding plane-wave). The energy difference $\Delta W < 0$, i. e. such a kink (anti-kink) formation is energetically more favourable than the corresponding homogeneous plane-wave (23).

For Cs-vapor with atom density $N_0 = 10^{17}\,\text{cm}^{-3}$ assuming that $\chi^{(3)} = -9.2\times10^{-34}$ esu, $\chi^{(5)}_{at} = 7.4\times10^{-41}$ esu $\lambda = 1.06\times10^{-4}$ cm and $\eta \approx 0$ one obtains

$$\mathcal{E}_0 \approx 2408\eta\,\text{esu}\,,\qquad \mathcal{L} = \frac{0.63\sqrt{1-\beta^{*2}}}{\eta}\,\text{cm}\,,\qquad (\Delta q \approx 0\,,\quad \Delta\Omega \approx 0)\,.$$

1.2.2. *"Bell" soliton solution*

$$\begin{aligned}\tilde{\mathcal{E}}(z,t) = \mathcal{E}_0\,\mathrm{e}^{\mathrm{i}(qz-\Omega t+\varphi)}\cosh\frac{z-z_0-vt}{\mathcal{L}}\\ \times\left(\operatorname{cosec}^2\eta\cosh^2\frac{z-z_0-vt}{\mathcal{L}} - 1\right)^{-\frac{1}{2}}\end{aligned} \tag{90}$$

where

$$q = -k - \Delta q\,,$$

$$\Delta q = \pm\frac{k\beta^*}{\sqrt{1-\beta^{*2}}}\sqrt{\frac{\omega^2}{k^2c^{*2}} - 1 - \frac{4(1-\beta^{*2})}{k^2\mathcal{L}^2}}\,, \tag{91}$$

$$\Omega = -\omega - \Delta\Omega\,,\qquad \Delta\Omega = \Delta q c^*/\beta^*\,, \tag{92}$$

$$\mathcal{E}_0 = \pm\frac{\cot\eta}{2}\sqrt{\frac{3|\chi^{(3)}|}{5\rho\chi^{(5)}}}\,, \tag{93}$$

$$\mathcal{L} = \frac{2c}{\omega|\chi^{(3)}|\sin 2\eta}\sqrt{\frac{10\rho\chi^{(5)}(1-\beta^{(*2)})}{3\pi\sigma}}\,, \tag{94}$$

$$\eta = \arcsin\Big(1-Q\Big)^{\frac{1}{4}}\,, \tag{95}$$

Q is given by (13) and the inequalities (87) and (88) must be satisfied.

Equation (8) has also a homogeneous solution of the form (67) with the parameters of (90). The energy difference is

$$\begin{aligned}\Delta W = &-\frac{3c^3\eta}{2\pi\omega c^{*2}\sqrt{30\pi\sigma\rho\chi^{(5)}(1-\beta^{*2})}}\Bigg[1-\frac{k^2c^{*2}}{\omega^2}\\ &\mp\sqrt{(1-\frac{k^2c^{*2}}{\omega^2})(1-\beta^{*2})-\frac{3\pi c^{*2}\sigma(\chi^{(3)})^2\sin^2 2\eta}{10c^2\rho\chi^{(5)}}}\\ &-\frac{3\pi c^{(*2)}\sigma(\chi^{(3)})^2}{80c^2\rho\chi^{(5)}}\\ &\times\Bigg[5-8\cos^4\eta+2\beta^{*2}(3\sin^2 2\eta-2\sin^2\eta)\\ &+\frac{\sin 2\eta}{2\eta}(3\cos 2\eta-4(1-\beta^{*2})\sin^2\eta)\Bigg]\Bigg]\,.\end{aligned} \tag{96}$$

The analysis of (96) shows that ΔW can be negative and solitary formation can be energetically more favourable than (67). The condition (14) is always satisfied.

As it is seen from (92) and (91), the frequency of the soliton Ω depends on the soliton "length" $\mathcal{L}$ in z-direction. Hence the effect of self-modulation can take place. As in the other bell soliton solution above the phase can be changed with the amplitude.

For Cs-vapor:

a) at $\eta \approx \frac{\pi}{2}$, $\Delta q \approx \Delta\Omega \approx 0$, $\mathcal{E}_0$ is very small, $\mathcal{L} \approx 3.78\sqrt{1-\beta^{*2}}$ cm;

b) $\omega = kc^*$ gives $\eta \approx 1.31$, $\mathcal{E}_0 \approx 3576$ esu, $\mathcal{L} \approx 2\sqrt{1-\beta^{*2}}$ cm.

1.2.3. *"Bell" soliton solution*

$$\tilde{\mathcal{E}}(z,t) = \mathcal{E}_0\, \mathrm{e}^{\mathrm{i}(qz-\Omega t+\varphi)}\left(1+\operatorname{sech}\frac{z-z_0-vt}{\mathcal{L}}\right)^{-\frac{1}{2}} \tag{97}$$

where q and Ω have the form (82) and (83), respectively,

$$\mathcal{E}_0 = \pm\frac{3}{4}\sqrt{\frac{|\chi^{(3)}|}{5\rho\chi^{(5)}}}\,, \tag{98}$$

$$\mathcal{L} = \frac{2c}{3\omega|\chi^{(3)}|}\sqrt{\frac{10\rho\chi^{(5)}(1-\beta^{*2})}{3\pi\sigma}}\,. \tag{99}$$

Taking into account that there exists a plane-wave solution of the form (67) with the parameters of (97), we find

$$\begin{aligned}\Delta W = {} & \frac{3c^3}{8\omega^3\sqrt{30\pi\sigma\rho\chi^{(5)}(1-\beta^{*2})}} \\ & \times\left[\frac{9\pi+4}{10\pi}\left(\frac{\omega^2}{c^{*2}}-k^2\right)\right. \\ & \left.\mp\frac{\omega}{c^*}\sqrt{\left(\frac{\omega^2}{c^{*2}}-k^2\right)(1-\beta^{*2})}\right] > 0\,.\end{aligned} \tag{100}$$

Hence, the plane-wave solution (67) is energetically more favourable than (97).

1.2.4. *"Inverse" bell soliton solution*

$$\tilde{\mathcal{E}}(z,t) = \mathcal{E}_0\,\mathrm{e}^{\mathrm{i}(qz-\Omega t+\varphi)}\left(1-\operatorname{sech}\frac{z-z_0-vt}{\mathcal{L}}\right)^{\frac{1}{2}} \tag{101}$$

where q, Ω, $\mathcal{E}_0$ and $\mathcal{L}$ are the same is in Subsection 3, the plane-wave solution has the form (67) and

$$\begin{aligned}\Delta W = {} & \frac{3c^3}{8\omega^3\sqrt{30\pi\sigma\rho\chi^{(5)}(1-\beta^{*2})}} \\ & \times\left[\frac{9\pi-4}{10\pi}\left(\frac{\omega^2}{c^{*2}}-k^2\right)\right. \\ & \left.\mp\frac{\omega}{c^*}\sqrt{\left(\frac{\omega^2}{c^{*2}}-k^2\right)(1-\beta^{*2})}\right] > 0\,.\end{aligned} \tag{102}$$

In this subcase the soliton formation (101) is energetically more favourable than the plane-wave (67).

When the electric field amplitude varies so slowly that (52) is supposed to take place, Eq. (8) turns into (53) for which the following soliton solutions are found.

1.2.5. *Kink (anti-kink) solution of the form (81)*

q, Ω, $\mathcal{E}_0$, η and Q are given by (55), (56), (84), (85) (where $\beta^* = 0$), (86) and(13), respectively. Equation (53) has also a homogeneous solution of the form (67) with the parameters of the soliton. In this case

$$\begin{aligned}\Delta W = &-\frac{3c^3\eta}{4\pi\omega c^{*2}\sqrt{30\pi\sigma\rho\chi^{(5)}}}\left[\beta^{*2} - \frac{k^2c^{*2}}{\omega^2}\right.\\ &-\frac{27\pi c^{*2}\sigma(\chi^{(3)})^2}{40c^2\rho\chi^{(5)}}\frac{(1-2\cos 2\eta)}{(2-\cos 2\eta)^2}\\ &\left.\times\left(1+\frac{\sin 2\eta}{2\eta}\frac{2-\cos 2\eta}{1-2\cos 2\eta}\right)\right].\end{aligned} \tag{103}$$

According to (103) ΔW can take negative values which means that the solitary formation can be more favourable energetically.

1.2.6. *"Bell" soliton solution of the form (90)*

q, Ω, $\mathcal{E}_0$, $\mathcal{L}$, η and Q are given by (55), (64), (84), (85) (with $\beta^* = 0$), (86) and (13), respectively.

Finally, taking into account that (53) has a homogenous plane-wave solution of the form (67), for the energy difference $\Delta W \equiv W - W_1$ one obtains

$$\begin{aligned}\Delta W = &-\frac{3c^3\eta}{4\pi\omega c^{*2}\sqrt{30\pi\sigma\rho\chi^{(5)}}}\left[\beta^{*2} - \frac{k^2c^{*2}}{\omega^2}\right.\\ &-\frac{3\pi c^{*2}\sigma(\chi^{(3)})^2(3-8\sin^4\eta)}{40c^2\rho\chi^{(5)}}\\ &\left.\times\left(1-\frac{\sin 2\eta}{2\eta}\frac{3(1+\sin^4\eta)-14\sin^2\eta}{3-8\sin^4\eta}\right)\right].\end{aligned} \tag{104}$$

2. Conclusions

In this paper a wide variety of solitary formations is shown to exist and propagate in nonlinear media for which the polarisation has the form (1). In the most interesting cases when $\chi^{(3)}$ and $\chi^{(5)}$ have opposite signs different kind kinks (anti-kinks), bell (bright) solitons and inverse bell (dark) solitons take place. Some solitary formations can propagate with velocities greater than the phase light velocity in the medium and without radiation. For the solitary pulses the effect of self-modulation can take place (the phase depends on the soliton amplitude). The comparison of the solitary formations and plane-wave ones is made from an energetical point of view.

The analytical solutions obtained for the nonlinear equations deduced as well as the Lagrangian description of the problem can be used also in the classical and quantum field theory.

References

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