Bulg. J. Phys. **29** (2002) 91–96

HOMOGENEOUS KANTOWSKI–SACHS MODEL IN LYRA GEOMETRY

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Abstract. Cosmological models for homogeneous Kantowski–Sachs space–tume within the framework of Lyra geometry in the presence of a massless scalar field with flat potential are presented. Also some physical features of the model are discussed.

PACS number: 98.80.cq, 04.20.jb, 04.50

1. Introduction

The origin of the Universe is one of the greatest cosmological mysteries even today. So the aim of cosmology is to determine the large-scale structure of the physical Universe ? for that different cosmological models have been proposed. Astronomical observations have indicated that Universe on the large scale can be treated as isotropic and homogeneous, but it is not so at small scales. In the early stage, the Universe did not have the same property of isotropy as we have found today. Homogeneous space–times belong to the Bianchi type I-IX or to the Kantowski–Sachs class and are often interpreted as cosmological models. Besides the Bianchi type metrics, the Kantowski–Sachs model also describes a spatially homogeneous one. Several authors found an anisotropic asymptotic behaviour of the model [1].

Lyra [2] has proposed a modification of Riemannian geometry by introducing a gauge function into the structureless manifold that bears a close resemblance to Weyl's geometry. Several authors have studied cosmological models based on Lyra's manifold with constant and time-dependent displacement vectors [3].

Soleng [3] has pointed out that the constant displacement field in Lyra's geometry will either include a creation and be equal to Hoyle's creation field cosmology or contain a special vacuum field which together with the gauge vector term may be considered as a cosmological term. Halford [3] has pointed out that the constant displacement field ϕ_i in Lyra's geometry plays the role of cosmological constant. Λ in the normal general relativistic treatment. According to Halford the present theory predict the same effects within observational limits, as far as the classical solar system tests are concerned, as well as tests based on the linearised form of field equations.

Several authors [4] have studied cosmological models for Kantowski–Sachs space–time. In 1922, Singh and Singh [3] have investigated Kantowski–Sachs cosmological model in Lyra geometry. by taking the energy momentum tensor in an ad hoc manner. They have obtained a solution in the case of a Kantowski–Sachs Universe filed with Zeldovich fluid. In this paper we would like to consider Kantowski–Sachs model in the presence of massless scalar field with a flat potential with time dependent displacement vector on Lyra's geometry in normal gauge.

2. Field Equations

The Einstein field equation based on Lyra's geometry which in normal gauge may be written as [2]

$$R_{ik} = \frac{1}{2}g_{ik}R + \frac{3}{2}\phi_i\phi_k - \frac{3}{4}g_{ik}\phi_m\phi^m = -8\pi G T_{ik} \tag{1}$$

where $\phi_i = (0,0,0,\beta(t))$ is the displacement vector and other symbols have their usual meaning as in Riemannian geometry.

The Lagrangian will be that of gravity minimally coupled to scalar field $\phi(t)$ with $V(\phi)$ is [6]

$$S = \int \sqrt{-g}\left[R - \frac{1}{2}g^{ab}\partial_a\phi\partial_b\phi - V(\phi)\right]\,\mathrm{d}^4x\,. \tag{2}$$

The energy momentum tensor is [6]

$$T_{ab} = \frac{1}{2}\partial_a\phi\partial_b\phi - \left[\frac{1}{4}(\partial_k\phi\partial^k\phi) + \frac{1}{2}V(\phi)\right]g_{ab}\,. \tag{3}$$

The field equation is [6]

$$\left(\frac{1}{\sqrt{-g}}\right)\partial_a\left(\sqrt{-g}\partial^a\phi\right) = -\frac{\mathrm{d}V(\phi)}{\mathrm{d}\phi}\,. \tag{4}$$

The Kantowski–Sachs metric is taken as

$$\mathrm{d}s^2 = \mathrm{d}\tau^2 - A^2(t)\,\mathrm{d}r^2 - B^2(t)(\,\mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\varphi^2)\,. \tag{5}$$

The field equation (1) reduces to (ϕ will be resealed by 2ϕ)

$$2\frac{A'B'}{AB} + \frac{B'^2}{B^2} + \frac{1}{B^2} = \varphi'^2 + \frac{1}{2}V(\phi) + \frac{3}{4}\beta^2 \tag{6}$$

$$2\frac{B''}{B} + \frac{B'^2}{P}B^2 + \frac{1}{B^2} = -\phi'^2 + \frac{1}{2}V(\phi) - \frac{3}{4}\beta^2 \tag{7}$$

$$\frac{B''}{B} + \frac{A''}{A} + \frac{A'B'}{AB} = -\phi'^2 + \frac{1}{2}V(\phi) - \frac{3}{4}\beta^2 \tag{8}$$

$$\phi'' + \phi'\left[\frac{A'}{A} + 2\frac{B'}{B}\right] = \frac{\mathrm{d}V(\phi)}{\mathrm{d}\phi} \tag{9}$$

(the prime denotes the differentiation in respect to t).

Here the potential can be approximated by constant value (c.f. Stain-Schabes [6])

$$V(\phi) = 2\lambda\,. \tag{10}$$

It may be noted that the coefficient of φ' in Eq. (9) acts as a friction term and it is larger for an isotropic model. So, the φ-field moves slowly in an anisotropic space–time.

3. Solutions to the Field Equations

Using (10) we get from Eq. (9)

$$\varphi'' + \varphi'\left[\frac{A'}{A} + 2\frac{B'}{B}\right] = 0\,. \tag{11}$$

Solving this we get

$$\varphi' = \frac{\varphi_0}{AB^2} \tag{12}$$

(φ_0 is an integration constant).

This implies

$$\varphi = \int \frac{\varphi_0}{AB^2}\,\mathrm{d}t + \varphi_{00} \tag{13}$$

(φ_{00} is an integration constant).

From equations (7) and (8) we get

$$\frac{B''}{B}\,\frac{B'^2}{B^2} + \frac{1}{B^2} - \frac{A''}{A} - \frac{A'B'}{AB} = 0\,. \tag{14}$$

We solve this equation by assuming the following relation between the metric coefficients as

$$A = A_0 B^n \tag{15}$$

(A_0 and n are arbitrary constants).

By using (15) we get from (14)

$$\frac{B''}{B}\,\frac{(n+1)B'^2}{B^2} = \frac{1}{n-1}B^2\,. \tag{16}$$

This equation has a first integral form of B as

$$\int \left[\frac{1}{n^2-1} + DB^{-(2n-2)}\right]^{\frac{1}{2}} \mathrm{d}B = \pm(t-t_0) \tag{17}$$

(t_0 is another integration constant).

This equation can not be solved for arbitrary values of n. Here we can find the exact solutions only for $n = 0, -2, -\dfrac{3}{2}$.

Case I: $n = 0$

in this case the expression for B from the integral (18) will be

$$B^2 = D - (t-t_0)^2\,. \tag{18}$$

This model is a contracting model of the Universe and is not of physical interest.

Case I: $n = -2$

We obtain from (18) the expression for B as

$$B = \frac{1}{\sqrt{3D}}\sinh(\sqrt{D})(t-t_0)\,. \tag{19}$$

Here we see that $\sqrt{-g}$ is constant. So proper volume of the Universe is constant and also nave not much more physical interest.

Case I: $n = -\frac{3}{2}$

Here we get the following solution of B:

$$B = \frac{1}{4}D(t-t_0)^2 - \frac{4}{5}D\,. \tag{20}$$

The other parameters are:

$$A = A_0\left[\frac{1}{4}D(t-t_0)^2 - \frac{4}{5}D\right]^{\frac{3}{2}} \tag{21}$$

$$\begin{aligned}\frac{3}{4}\beta^2 = &\left[\frac{1}{4}D(t-t_0)^2 - \frac{4}{5}D\right]^{-2}\\ &-\lambda - \frac{A_0^2}{\phi_0^2}\left[\frac{1}{4}D(t-t_0)^2 - \frac{4}{5}D\right]^{-1}\\ &-\frac{D^2}{2}(t-t_0)^2\left[\frac{1}{4}D(t-t_0)^2 - \frac{4}{5}D\right]^{-2}\,.\end{aligned} \tag{22}$$

Also solving (13) we get

$$\varphi = \frac{2\varphi_0}{A_0\sqrt{D}}\cosh^{-1}\left[\frac{\sqrt{5D}}{4}(t-t_0)\right] + \varphi_{00}\,. \tag{23}$$

The physical quantities that are of importance in cosmology are proper volume R^3, expansion scalar θ and shear scalar σ^2 and they have the following expressions for the above solutions:

$$R^3 = A_0\left[\frac{1}{4}D(t-t_0)^2 - \frac{4}{5}D\right]^{\frac{1}{2}} \tag{24}$$

$$\theta = \left[\frac{1}{4}D(t-t_0)^2 - \frac{4}{5}D\right]^{-1} \tag{25}$$

$$\sigma^2 = \frac{25D}{24}(t-t_0)^2\left[\frac{1}{4}D(t-t_0)^2 - \frac{4}{5}D\right]^{-2}\,. \tag{26}$$

4. Discussion

From the above solutions, we note that at the epoch $(t-t_0)^2 = \frac{16}{5}D^2$ the Universe starts with an initial singularity $\sqrt{-g} \to 0$ while σ^2 and θ diverge. This is a line singularity. Thus Universe starts with an infinite rate of expansion and anisotropy and then expands indefinitely. We also see that the ration of σ to θ is constant. This implies that there is no possibility that the models may get isotropized at some later time, i.e. it remains anisotropic for all times. At

$t \to \infty$, β becomes imaginary, i.e. the concept of Lyra geometry will not linger for infinite times.

Acknowledgements

F.R. is thankful to UGC for financial support and IUCAA for providing research facilities. We are grateful to the Referee for his constructive suggestions.

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