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Quantum Mechanics on a Noncommutative Geometry

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Abstract. Quantum mechanics in its presently known formulation requires an external classical time for its description. A classical spacetime manifold and a classical spacetime metric are produced by classical matter fields. In the absence of such classical matter fields, quantum mechanics should be formulated without reference to a classical time. If such a new formulation exists, it follows as a consequence that standard linear quantum mechanics is a limiting case of an underlying non-linear quantum theory. A possible approach to the new formulation is through the use of noncommuting spacetime coordinates in noncommutative differential geometry. Here, the non-linear theory is described by a non-linear Schrodinger equation which belongs to the Doebner-Goldin class of equations, discovered some years ago.

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1 Why Quantum Mechanics without Classical Spacetime?

Quantum mechanics in its presently known formulation requires an external classical time for its description. A classical spacetime manifold and a classical spacetime metric are produced by classical matter fields. In the absence of such classical matter fields, quantum mechanics should be formulated without reference to a classical time.

The new formulation should have the following two properties. Firstly, in the limit in which the quantum system under consideration becomes macroscopic, the ‘quantum spacetime’ should become the standard classical spacetime described by the laws of special and general relativity; and the quantum dynamics should reduce to standard classical dynamics on this spacetime. Secondly, consider the situation in which a dominant part of the quantum system becomes macroscopic and classical, and a sub-dominant part remains microscopic and quantum. By virtue of the first property, the dominant part should look like our classical Universe – classical matter existing in a classical spacetime. Seen from the viewpoint of this dominant part, the quantum dynamics of the sub-dominant part should be the same as the standard quantum mechanics on an external classical spacetime.

The paper is organized as follows. In Section 2 we give an argument as to why quantum gravity should be a non-linear theory – this argument is independent of any specific mathematical structure one might use to describe a quantum theory of gravity, and is hence generic. In the third section we propose that noncommutative differential geometry is the appropriate language for the new formulation of quantum mechanics. Noncommutative geometry provides a natural generalization of general covariance to include noncommuting coordinate systems. We propose a description of the quantum Minkowski spacetime using noncommuting coordinates and explain how standard quantum dynamics can be recovered as an approximation to the dynamics on the noncommutative Minkowski spacetime. In Section 4 we develop a description for a nonlinear quantum mechanics, to which the linear theory is an approximation. This nonlinear Schrodinger equation we arrive at belongs to the Doebner-Goldin class of non-linear Schrodinger equations – this aspect is discussed in Section 5.

2 Why a Quantum Theory of Gravity Should Be Nonlinear?

Consider the aforesaid thought experiment, wherein we consider a box of point particles, having masses m_i. The only known fundamental mass with which these masses can be compared is the Planck mass $m_{Pl} \equiv (\hbar c/G)^{1/2} \sim 10^{-5}$ grams. Since we know from observations that microscopic masses obey quantum mechanics and macroscopic masses obey classical mechanics, we will assume a particle behaves quantum mechanically if its mass is much smaller than Planck mass, and classically if its mass is much greater than Planck mass. In order to argue that quantum gravity should be a non-linear theory we construct a series of thought experiments where the values of the masses are made to vary, from one experiment to the next.

We shall first consider the case where each of the masses, as well as the total mass of the system, is much, much smaller than Planck mass; say typically in the atomic mass range, or smaller. If we observe this box from an external classical spacetime, the dynamics of the particles will obey the rules of quantum mechanics. We also assume that the total mean energy associated with the system is much smaller than the Planck energy scale $E_p \sim 10^{19}$ GeV. Since both Planck-mass and Planck-energy scale inversely with the gravitational constant, the approximation we are considering is equivalent to considering the limit $m_{Pl} \to \infty$, or letting $G \to 0$. It is thus reasonable, in this approximation, to neglect the gravitational field produced by the particles inside the box. The reason for doing so is the same, for example, as to why we ignore the gravitational field of the hydrogen atom while studying its spectrum. What we thus have in the box is a collection of particles obeying quantum dynamics in an external spacetime, and the gravitation of these particles can be neglected.

Let us imagine now that the external classical spacetime is not there, and that the 'box' of particles is the whole universe. The arguments of the previous section imply that there is no longer any classical spacetime manifold available, and one should describe the quantum dynamics without reference to it. We assume that there is a concept of 'quantum spacetime' associated with the system, and since the associated gravitational field can be neglected we call this quantum spacetime the 'quantum Minkowski spacetime'. The classical analog of this situation is a set of particles of small mass, whose gravitation can be neglected, existing in Minkowski spacetime.

The new quantum description of the dynamics of the particles in the box should become equivalent to standard quantum mechanics as and when a classical spacetime manifold becomes externally available. A classical spacetime would become externally available if outside the box there are classical matter fields which dominate the Universe.

Consider next the box universe, in the case in which, to a first order approximation, the values of the masses m_i (as well as the total mass and the mean energy of the system) in the box are no longer negligible, compared to Planck mass. We need to now take into account the 'quantum gravitational field' produced by these masses, and denoted by a set of variables, say η. There is still no background classical spacetime manifold, but only a background quantum Minkowski spacetime. Let us associate with the system a physical state $\Psi(\eta, m_i)$. It is plausible that to this order of approximation the physical state is determined by a linear equation

$$\hat{O}\Psi(\eta, m_i) = 0 \tag{1}$$

where the operator $\hat{O}$, defined on the background quantum Minkowski spacetime, depends on the gravitational field variables η only via the linearized departure of η from its 'quantum Minkowski limit' and furthermore, does not have any dependence on the physical state Ψ. The classical analog of this situation is a linearized description of gravity, obtained from general relativity, when the spacetime metric is a small departure from Minkowski spacetime, and the gravitation of the matter sources cannot be entirely neglected.

We now consider the case of central interest to us, where we see departure from linear quantum theory, and argue that quantum gravity should be non-linear. Let the masses m_i in the box (as well as the total mass and the mean energy) be comparable to Planck mass. The behavior of the particles will still be quantum mechanical and there is still no background classical spacetime manifold available. Furthermore, the 'quantum gravitational field' of the system can no longer be neglected, nor is it a first order departure from the 'quantum Minkowski spacetime'. As a result, the quantum gravitational field described by the physical state $\Psi(\eta, m_i)$ will have to be taken into account, and will act as a source for itself. This will happen recursively, so that as a consequence the operator $\hat{O}$ in (1) will itself depend non-linearly on the state Ψ. In this sense quantum gravity should be a non-linear theory, because the quantum gravitational field will act as a source for itself.

It can now be shown that if the quantum theory of gravity is non-linear, the equation of motion which describes the quantum dynamics of the particles in the box is also non-linear, and provides a non-linear generalization of the Schrodinger equation [1]

It should be mentioned that one has to distinguish clearly between gravity as a nonlinear theory, and its quantisation, which could have another type of nonlinearity, through the quantisation method. As discussed above, our proposal in this work is concerned with the latter kind of non-linearity.

3 Noncommutative Geometry and Quantum Minkowski Spacetime

Noncommutative differential geometry is an abstract, but in retrospect a rather natural, extension of Riemannian geometry, that includes the latter as a special case. A short but highly readable account is by Martinetti [2]. Our proposal is that noncommutative differential geometry is an appropriate framework for a formulation of quantum mechanics which does not refer to a classical (i.e. Riemannian) spacetime manifold. We will motivate this approach, and describe its present status, in the following sections. Our present understanding, though well-motivated on physical grounds, is still partly heuristic, and does not yet make contact with the rigorous formulation of noncommutative geometry, for instance with regard to construction of the spectral triple, and relating the noncommutative metric introduced below to the one formally defined in noncommutative geometry. Nonetheless the ideas developed below hold out the promise that the formal connections of these ideas with noncommutative geometry will eventually get developed.

We begin by suggesting the notion of a noncommuting coordinate system, which is to be thought of as 'covering' a noncommutative manifold, and wherein the commutation relations between the coordinates are to be introduced on physical grounds. The new formulation of quantum dynamics is to be given in such a coordinate system, and is to be invariant under transformation (an automorphism) from the given coordinate system to different noncommuting coordinate systems. This is a generalization of general covariance to the noncommutative case, and one is proposing that a physical theory should be invariant under transformations of noncommuting coordinates.

As mentioned above, a quantum Minkowski spacetime will arise if no external classical spacetime is available, and if the masses of all the particles in the box Universe (as well as the total mass and energy scale of the box) are much smaller than Planck mass. In order to motivate our construction of the quantum Minkowski spacetime using noncommuting coordinates, we first briefly recall the relativistic Schrodinger equation for a free particle in quantum mechanics, and for simplicity of presentation we assume only one space dimension. Subsequently we will generalize to the case of many particles, and also to 3+1 spacetime. We chose to consider the relativistic case, as opposed to the non-relativistic one, only because the available space-time symmetry makes the analysis more transparent. Later, we will consider also the non-relativistic limit.

The relativistic Schrodinger equation for a particle of mass m in 2-d spacetime

$$-\hbar^2 \left(\frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial x^2} \right) \psi = m^2 \psi \tag{2}$$

can be rewritten, after the substitution $\psi = e^{iS/\hbar}$, as

$$\left(\frac{\partial S}{\partial t} \right)^2 - \left(\frac{\partial S}{\partial x} \right)^2 - i\hbar \left(\frac{\partial^2 S}{\partial t^2} - \frac{\partial^2 S}{\partial x^2} \right) = m^2. \tag{3}$$

The object $S(t, x)$ introduced above, which we will call the complex action, will play a central role for us. In the classical limit, it will become real and coincide with the classical action. We would like to suggest that quantum dynamics could alternately be described in terms of this complex action, via Equation (3), which should be thought of as the quantum Hamilton-Jacobi equation. The quantum corrections of course come from the $\hbar$-dependent terms, which correct the classical Hamilton-Jacobi equation. In terms of the complex action $S(t, x)$ we define the generalized momenta

$$p^t = -\frac{\partial S}{\partial t}, \qquad p^x = \frac{\partial S}{\partial x} \tag{4}$$

in terms of which Eqn. (3) reads

$$p^\mu p_\mu + i\hbar \frac{\partial p^\mu}{\partial x^\mu} = m^2. \tag{5}$$

where the index μ takes the values 1 and 2. The momenta defined above as gradients of the complex action have an obvious parallel with classical dynamics. In terms of the complex action $S(t, x)$ we have a uniform description of classical and quantum dynamics. Below, we will make the case that Eqn. (3) can be derived from the fundamental assumptions of an underlying noncommutative theory.

A remark on the interpretation of the generalised momentum. We propose, exactly as in quantum mechanics, that experiments measure expectation values of the generalized momentum, where expectation values are defined as in quantum mechanics, but by rewriting the wave function in terms of the complex action. Reality of the expectation value is ensured by demanding that the expectation value of the gradient of the imaginary part of the complex action be zero. This is a constraint on the allowed solutions.

In standard quantum mechanics the origin of the $\hbar$-dependent term in (5) of course lies in the position-momentum commutation relation, and this term provides a correction to the classical energy-momentum relation $E^2 - p^2 = m^2$.

Following this lead, we now suggest a model for the dynamics of the 'quantum Minkowski spacetime' by assuming that there exist two noncommuting coordinates $\hat{x}$ and $\hat{t}$, and that the quantum mechanical particle lives in this noncommutative spacetime. We ascribe to the particle a 'generalized momentum' $\hat{p}$, having two components $\hat{p}^t$ and $\hat{p}^x$, which do not commute with each other. The noncommutativity of these momentum components is assumed to be a consequence of the noncommutativity of the coordinates, as the momenta are defined to be the partial derivatives of a complex action $S(\hat{x}, \hat{t})$, with respect to the corresponding noncommuting coordinates. We will propose a dynamics for the particle, analogous to classical special relativity, in these noncommuting coordinates. We will then argue that seen from an external classical spacetime (as and when the latter exists) this noncommutative dynamics looks the same as standard quantum mechanics as described by Equations (2) and (3).

A noncommutative coordinate system is a set of noncommuting variables satisfying the fundamental Planck scale commutation relations [1], and having the important property that these commutation relations are invariant under automorphisms. A noncommutative line-element introduces the notion of a distance in a noncommutative space, by incorporating the non-commutativity of the coordinate differentials dx and dt. A noncommutative flat metric is deduced therefrom.

Thus, on the quantum Minkowski spacetime we introduce the following noncommutative flat metric

$$\hat{\eta}_{\mu\nu} = \begin{pmatrix} 1 & 1 \\ -1 & -1 \end{pmatrix} \tag{6}$$

and assume that there exists a corresponding noncommutative line-element

$$ds^2 = \hat{\eta}_{\mu\nu} d\hat{x}^\mu d\hat{x}^\nu = d\hat{t}^2 - d\hat{x}^2 + d\hat{t}d\hat{x} - d\hat{x}d\hat{t}. \tag{7}$$

In analogy with special relativity, the metric and the line-element are assumed to be invariant under a class of coordinate transformations which generalize Lorentz transformations. Thus this line-element is left invariant by the transformation

$$\hat{x}' = \gamma(\hat{x} - \beta\hat{t}), \qquad \hat{t}' = \gamma(\hat{t} - \beta\hat{x}) \tag{8}$$

where $\gamma = (1 - \beta^2)^{-1/2}$. In the commutative case, β has the interpretation of velocity: $\beta = v/c$. On a noncommutative space, β could be thought of as defining a 'rotation' in the noncommutative space, by an angle θ such that $\beta = \tanh\theta$.

The above metric, which is obtained by adding an antisymmetric component to the standard Minkowski metric, is assumed to generalize the Minkowski metric to the noncommutative case. It is non-Hermitian, and has zero determinant – below we consider also a Hermitian modification of this metric. Our discussion here, though, is not obstructed by the fact that the metric is not Hermitian.

In order to motivate a noncommutative dynamics, we note that, from (7), one could heuristically define a velocity $\hat{u}^i = d\hat{x}^i/ds$, which satisfies the relation

$$1 = \hat{\eta}_{\mu\nu}\frac{d\hat{x}^\mu}{ds}\frac{d\hat{x}^\nu}{ds} = (\hat{u}^t)^2 - (\hat{u}^x)^2 + \hat{u}^t\hat{u}^x - \hat{u}^x\hat{u}^t. \tag{9}$$

This suggests a definition of the generalized momentum as $\hat{p}^i = m\hat{u}^i$, which hence satisfies

$$\hat{p}^\mu\hat{p}_\mu = m^2 \tag{10}$$

Here, $\hat{p}_\mu = \hat{\eta}_{\mu\nu}\hat{p}^\mu$ is well-defined. Written explicitly, this equation becomes

$$(\hat{p}^t)^2 - (\hat{p}^x)^2 + \hat{p}^t\hat{p}^x - \hat{p}^x\hat{p}^t = m^2 \tag{11}$$

Eqn. (10) appears an interesting and plausible proposal for the dynamics, because it generalizes the corresponding special relativistic equation to the noncommutative case. The noncommutative Hamilton-Jacobi equation is constructed from (11) by defining the momentum as gradient of the complex action function. We are proposing here that in the absence of a classical spacetime manifold, quantum dynamics should be described using this Hamilton-Jacobi equation – this is the noncommutative analog of the quantum Hamilton-Jacobi equation (3).

Let us consider now that an external classical Universe becomes available (in the next section we will discuss how a classical Universe could arise from the quantum dynamics). Seen from this classical Universe the quantum dynamics should be as described by Eqn. (3). Thus, we need to justify the following correspondence rule, when the noncommutative Hamilton-Jacobi equation is examined from our standard spacetime point of view:

$$(\hat{p}^t)^2 - (\hat{p}^x)^2 + \hat{p}^t\hat{p}^x - \hat{p}^x\hat{p}^t = (p^t)^2 - (p^x)^2 + i\hbar\frac{\partial p^\mu}{\partial x^\mu} \tag{12}$$

Here, p is the 'generalized momentum' of the particle as seen from the commuting coordinate system, and is related to the complex action by Eqn. (4). This equality should be understood as an equality between the two equivalent equations of motion for the complex action function S – one written in the noncommuting coordinate system, and the other written in the standard commuting coordinate system.

The idea here is that by using the Minkowski metric of ordinary spacetime one does not correctly measure the length of the 'momentum' vector, because the noncommuting off-diagonal part is missed out. The last, $\hbar$ dependent term in (12) provides the correction – the origin of this term's relation to the commutator $\hat{p}^t\hat{p}^x - \hat{p}^x\hat{p}^t$ can be understood as follows.

Let us write this commutator by scaling all momenta with respect to Planck momenta: define $\hat{P}^\mu = \hat{p}^\mu/P_{pl}$. Also, all lengths are scaled with respect to Planck length: $\hat{X}^\mu = \hat{x}^\mu/L_{pl}$. Let the components of X^μ be denoted as $(\hat{T}, \hat{X})$. The commutator $\hat{P}^T\hat{P}^X - \hat{P}^X\hat{P}^T$ represents the 'non-closing' of the basic quadrilateral when one compares (i) the operator obtained by moving first along $\hat{X}$ and then along $\hat{T}$, with (ii) the operator obtained by moving first along $\hat{T}$ and then along $\hat{X}$. When seen from a commuting coordinate system, this deficit (i.e. non-closing) can be interpreted as a result of moving to a neighboring point, and in the infinitesimal limit the deficit will be the sum of the momentum gradients in the various directions. The deficit is thus given by $i\partial P^\mu/\partial X^\mu = i(L_{Pl}/P_{Pl})\partial p^\mu/\partial x^\mu$. This gives that $\hat{p}^t\hat{p}^x - \hat{p}^x\hat{p}^t = i\hbar\partial p^\mu/\partial x^\mu$. Hence, since the relation (12) holds, there is equivalence between the noncommutative description (10) and standard quantum dynamics given by (5).

We would like to remark on the existence of a possible Hilbert space in our formulation. The proposal in this paper is that the underlying formulation is in the language of a complex action, whose dynamics is described by the noncommutative Hamilton-Jacobi equation, and the dynamics takes place on a configuration space defined by the noncommutative spacetime. It is suggested that the standard description of dynamics in terms of the wave function and a Hilbert space is available only in the presence of a classical spacetime.

4 A Non-Linear Schrodinger Equation

In Section 2 we argued that when the particle masses become comparable to Planck mass, the quantum equation of motion for the particles should become non-linear. We now present an approximate construction for such a non-linear equation, based on the dynamics for the 'quantum Minkowski spacetime' proposed above. Since the quantum Minkowski spacetime and its metric were obtained as a noncommutative generalization of special relativity one can expect that there will also be a generalization of the quantum Minkowski spacetime to a 'quantum curved spacetime' which will be equivalent to a noncommutative generalization of general relativity. This will be the origin of non-linearity; and since standard quantum dynamics corresponds to the dynamics on the quantum Minkowski spacetime, one can expect that the dynamics on the quantum curved spacetime will be a non-linear generalization of standard quantum dynamics.

We once again start by considering the case of a single particle in 2-d spacetime. The starting point for our discussion will be Eqn. (10) above – we will assume that a natural generalization of this equation describes quantum dynamics when the particle mass m is comparable to the Planck mass m_{Pl}, and the effects of the particle's own gravity cannot be ignored. In this case, we no longer expect the metric η to have the 'flat' form given in (6) above. The symmetric components of the metric are of course expected to start depending on m (that is gravity) and in general the antisymmetric components can also be expected to depend on m – so long as the antisymmetric components are non-zero, we can say that quantum effects are present. As m goes to infinity, the antisymmetric component should go to zero – since in that limit we should recover classical mechanics. In fact the antisymmetric part (we will call it $\theta_{\mu\nu}$) should already start becoming ignorable when m exceeds m_{Pl}. It is interesting to note that the symmetric part should grow with m, while the antisymmetric part should fall with increasing m. There probably is some deep reason why this is so.

If $\hat{h}_{\mu\nu}$ is the noncommutative metric which generalizes $\hat{\eta}_{\mu\nu}$ we can write it as

$$\hat{h}_{\mu\nu} = \begin{pmatrix} \hat{g}_{tt} & \hat{\theta} \\ -\hat{\theta} & -\hat{g}_{xx} \end{pmatrix} \tag{13}$$

There is a corresponding generalization of the line-element,

$$ds^2 = \hat{h}_{\mu\nu} d\hat{x}^\mu d\hat{x}^\nu = \hat{g}_{tt} d\hat{t}^2 - \hat{g}_{xx} d\hat{x}^2 + \hat{\theta}[d\hat{t}d\hat{x} - d\hat{x}d\hat{t}]. \tag{14}$$

while the velocity and momentum are defined as in the previous section. The energy-momentum relation will now be

$$\hat{h}_{\mu\nu}\hat{p}^\mu\hat{p}^\nu = m^2 \tag{15}$$

and instead of Eqn. (12), we will have the dynamical equation

$$\hat{g}_{tt}(\hat{p}^t)^2 - \hat{g}_{xx}(\hat{p}^x)^2 + \hat{\theta}\left(\hat{p}^t\hat{p}^x - \hat{p}^x\hat{p}^t\right) = m^2 \tag{16}$$

and the correspondence rule

$$\hat{g}_{tt}(\hat{p}^t)^2 - \hat{g}_{xx}(\hat{p}^x)^2 + \hat{\theta}\left(\hat{p}^t\hat{p}^x - \hat{p}^x\hat{p}^t\right) = g_{tt}(p^t)^2 - g_{xx}(p^x)^2 + i\hbar\theta\frac{\partial p^\mu}{\partial x^\mu} \tag{17}$$

and as seen from an external classical spacetime the new dynamical equation

$$g_{tt}(p^t)^2 - g_{xx}(p^x)^2 + i\hbar\theta\frac{\partial p^\mu}{\partial x^\mu} = m^2 \tag{18}$$

which replaces Eqn. (3) as the dynamical equation.

Let us return now to the consideration of the dynamical equation (18) and its implications for quantum mechanics. We recall that the complex action $S(x,t)$ is related to the wave-function by the definition $\psi = e^{iS/\hbar}$, and now, because of gravitational corrections which become important at the Planck scale, the complex action satisfies the new dynamical equation (18) and not the equation (3). If we transform back from (18) by substituting for S in terms of ψ we will find that ψ satisfies a non-linear equation, instead of the linear Klein-Gordon equation (2). This non-linearity is being caused by the presence of the non-trivial metric components g and θ, and is in principle observable.

We should now make simplifying assumptions, in order to construct a simple example of a non-linear Schrodinger equation. We are interested in the case $m \sim m_{Pl}$. The symmetric part of the metric – g – should resemble the Schwarzschild metric, and assuming we are not looking at regions close to the Schwarzschild radius (which is certainly true for objects of such masses which we expect to encounter in the laboratory) we can approximately set g to unity. The key quantity then is $\theta = \theta(m/m_{Pl})$ - we have no proof one way or the other whether θ should be retained. We will assume here that θ should be retained, and work out its consequences.

$\theta(m/m_{Pl})$ in principle should also depend on the quantum state via the complex action S, but we can know about the explicit dependence of θ on m and S only if we know the dynamical equations which relate θ to m, which at present we do not. It is like having to know the analog of the Einstein equations for θ. But can we extract some useful conclusions just by retaining θ and knowing its asymptotic behavior? Retaining θ, the above dynamical equation can be written in terms of the complex action S as follows:

$$\left(\frac{\partial S}{\partial t}\right)^2 - \left(\frac{\partial S}{\partial x}\right)^2 - i\hbar\theta(m/m_{Pl})\left(\frac{\partial^2 S}{\partial t^2} - \frac{\partial^2 S}{\partial x^2}\right) = m^2. \tag{19}$$

This is the equation we would like to work with. We know that $\theta = 1$ is quantum mechanics, and $\theta = 0$ is classical mechanics. We expect θ to decrease from one to zero, as m is increased. It is probably more natural that θ continuously decreases from one to zero, as one goes from quantum mechanics to classical mechanics, instead of abruptly going from one to zero. In that case we should expect to find experimental signatures of θ when it departs from one. By substituting the definition $S = -i\hbar \ln \psi$ in (19) we get the following non-linear equation for the Klein-Gordon wave-function ψ:

$$-\hbar^2\left(\frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial x^2}\right)\psi + \frac{\hbar^2}{\psi}\left(1 - \frac{1}{\theta}\right)\left(\dot{\psi}^2 + \psi'^2\right) = m^2\psi \tag{20}$$

We are interested in working out the consequences of the non-linearity induced by θ, even though we do not know the explicit form of θ.

Let us go back to the equation (19). In general θ will also depend on the state S but for all states θ tends to zero for large masses, and if we are looking at large masses we may ignore the dependence on the state, and take $\theta = \theta(m)$. Let us define an effective Planck's constant $\hbar_{eff} = \hbar\theta(m)$, i.e. the constant runs with the mass m. Next we define an effective wave-function $\psi_{eff} = e^{iS/\hbar_{eff}}$. It is then easy to see from (19) that the effective wave function satisfies a linear Klein-Gordon equation

$$-\hbar_{eff}^2\left(\frac{\partial^2\psi_{eff}}{\partial t^2} - \frac{\partial^2\psi_{eff}}{\partial x^2}\right) = m^2\psi_{eff} \tag{21}$$

and is related to the usual wave function ψ through

$$\psi_{eff} = \psi^{1/\theta(m)} \tag{22}$$

For small masses, the effective wave-function approaches the usual wave function, since θ goes to unity.

We would now like to obtain the non-relativistic limit for this equation. Evidently this limit is

$$ih_{eff}\frac{\partial\psi_{eff}}{\partial t} = -\frac{h_{eff}^2}{2m}\frac{\partial^2\psi_{eff}}{\partial x^2}. \tag{23}$$

By rewriting ψ_{eff} in terms of ψ using the above relation we arrive at the following non-linear Schrodinger equation

$$i\hbar\frac{\partial\psi}{\partial t} = -\frac{\hbar^2}{2m}\frac{\partial^2\psi}{\partial x^2} + \frac{\hbar^2}{2m}(1-\theta)\left(\frac{\partial^2\psi}{\partial x^2} - [(\ln\psi)']^2\psi\right). \tag{24}$$

It is reasonable to propose that if the particle is not free, a term proportional to the potential, $V(q)\psi$, can be added to the above non-linear equation.

In terms of the complex action function S defined above (3) as $\psi = e^{iS/\hbar}$ this non-linear Schrodinger equation is written as

$$\frac{\partial S}{\partial t} = -\frac{S'^2}{2m} + \frac{i\hbar}{2m}\theta(m)S''\,. \tag{25}$$

This equation is to be regarded as the non-relativistic limit of Eqn. (19).

5 A Comparison with the Doebner-Goldin Equation

When we found the non-linear equation (24) we did not know that this equation already exists in the literature. Only subsequently we learned, from a review article by Svetlichny [3], that many years ago Doebner and Goldin arrived at a very similar equation from an apparently different (but possibly related) approach. Considering that the same equation has been arrived at independently by two different routes, and considering that the approach of Doebner and Goldin is on firmer ground (as compared to our partly heuristic analysis) we are led to believe that this non-linear equation deserves some serious attention, and should be tested in the laboratory. We will briefly review the Doebner-Goldin equation here, along with its possible implications, and its possible connection with our work. Actually there is an entire class of D-G equations, and we will begin by recalling the first non-linear equation derived by them.

Doebner and Goldin inferred their equation from a study [4] of representations of non-relativistic current algebras. This involves examining unitary representations of an infinite-dimensional Lie algebra of vector fields $Vect(R^3)$ and group of diffeomorphisms $Diff(R^3)$. These representations provide a way to classify physically distinct quantum systems. There is a one-parameter family, labeled by a real constant D, of mutually inequivalent one-particle representations of the Lie-algebra of probability and current densities. The usual one-particle Fock representation is the special case $D = 0$. The probability density ρ and the current density $\mathbf{j}$ satisfy, not the continuity equation, but a Fokker-Planck equation

$$\frac{\partial \rho}{\partial t} = -\nabla . \mathbf{j} + D\nabla^2 \rho. \tag{26}$$

A linear Schrodinger equation cannot be consistent with the above Fokker-Planck equation with $D \neq 0$, but Doebner and Goldin found that the following is one of the non-linear Schrodinger equations which leads to the above Fokker-Planck equation

$$i\hbar \frac{\partial \psi}{\partial t} = -\frac{\hbar^2}{2m}\nabla^2 \psi + iD\hbar \left(\nabla^2 \psi + \frac{|\nabla \psi|^2}{|\psi|^2}\psi \right). \tag{27}$$

The Doebner-Goldin equation should be compared with the equation (24) found by us. Although we have considered a 2-d case, and although there are some differences, the similarity between the two equations is striking, considering that the two approaches to this non-linear equation are, at least on the face of it, quite different. It remains to be seen as to what is the connection between the representations of $Diff(R^3)$, quantum mechanics, and the antisymmetric part θ of the asymmetric metric introduced by us.

The comparison between the two non-linear equations suggests the following relation between the new constants D and θ

$$D \sim \frac{\hbar}{2m}(1 - \theta). \tag{28}$$

There is a significant difference of an i factor in the correction terms in the two equations, and further, the relative sign of the two correction terms is different in the two equations, and in the last term we do not have absolute values in the numerator and denominator, unlike in the Doebner-Goldin equation. Despite these differences, the similarity between the two equations is noteworthy and we believe this aspect should be explored further. It is encouraging that there is a strong parallel between the limits $D \to 0$ and $\theta \to 1$ – both limits correspond to standard linear quantum mechanics. In their paper Doebner and Goldin note that the constant D could be different for different particle species. In the present analysis we clearly see that $\theta(m)$ is labeled by the mass of the particle.

For comparison with Doebner and Goldin we write down the corrections to the continuity equation which follow as a consequence of the non-linear terms in (24). These can be obtained by first noting, from (23), that the effective wave-function ψ_{eff} obeys the following continuity equation

$$\frac{\partial}{\partial t}\left(\psi^*_{eff}\psi_{eff}\right) - \frac{i\hbar_{eff}}{2m}\left(\psi^*_{eff}\psi'_{eff} - \psi_{eff}\psi^{*'}_{eff}\right)' = 0. \tag{29}$$

By substituting $\psi_{eff} = \psi^{1/\theta(m)}$ and $\hbar_{eff} = \hbar\theta(m)$ in this equation we get the following corrections to the continuity equation for the probability and current density constructed from $\psi(x)$

$$\frac{\partial}{\partial t}\left(\psi^*\psi\right) - \frac{i\hbar}{2m}\left(\psi^*\psi' - \psi\psi^{*'}\right)' = \frac{\hbar(1-\theta)}{m}|\psi|^2\phi''. \tag{30}$$

where ϕ is the phase of the wave-function ψ, i.e. $\phi = Re(S)/\hbar$. It is interesting that the phase enters in a significant manner in the correction to the continuity equation. This equation should be contrasted with Eqn. (26). The fact that the evolution is not norm-preserving when the mass becomes comparable to Planck mass suggests that the appropriate description should be in terms of the effective wave-function ψ_{eff}.

A more detailed version of this article is available on the archives [1].

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