

Poisson-Lie Symmetry and q - WZW Model

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Abstract. We review the notion of (anomalous) Poisson-Lie symmetry of a dynamical system and we outline the Poisson-Lie symmetric deformation of the standard WZW model from the vantage point of the twisted Heisenberg double.

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1 Introduction

The concept of Poisson-Lie symmetry [1] of a dynamical system is a generalization of ordinary Hamiltonian symmetry and it is characterized by the presence of two Lie groups in the story. One of those groups is called a symmetry group and it acts on a phase space of a dynamical system. The structure of the other group (called cosymmetry group) underlies the way how the action of the symmetry group is expressed in terms of Poisson brackets. If the cosymmetry group is Abelian the Poisson-Lie symmetry is nothing but the ordinary Hamiltonian symmetry known from every textbook on classical mechanics. Sometimes the Poisson-Lie symmetry can be “switched off” and it becomes the ordinary symmetry. This happens for one-parameter families of Poisson-Lie symmetric dynamical systems such that for a particular value of the parameter the cosymmetry group becomes Abelian (see Example in Section 3). In such case we say that the corresponding ordinary symmetry is deformable. This short contribution has two purposes. First of all it aims to be a very concise review of the concept of the (anomalous) Poisson-Lie symmetry and, secondly, it wants to present the main results of the paper [5] from the vantage point of the so-called twisted Heisenberg double [2].

In Section 2, we review the definition of the Poisson-Lie group and in Section 3, we describe and illustrate the concept of the Poisson-Lie symmetry. In Section 4, we review the construction of the twisted Heisenberg double due to Semenov-Tian-Shansky and we show that the structure of the standard WZW model can be understood in its terms. Finally, in Section 5, we show how to choose the twisted Heisenberg double in order to obtain the q -deformation of the WZW model.

2 Poisson-Lie Groups

Let B be a Lie group and $Fun(B)$ the algebra of functions on it. It is well known that the group structure on B can be (dually) described by the so called coproduct $\Delta : Fun(B) \rightarrow Fun(B) \otimes Fun(B)$, the antipode $S : Fun(B) \rightarrow Fun(B)$ and the counit $\varepsilon : Fun(B) \rightarrow \mathbf{R}$ given, respectively, by the formulae

$$\Delta y(b_1, b_2) = y'(b_1)y''(b_2) = y(b_1b_2), \quad S(y)(b) = y(b^{-1}), \quad \varepsilon(y) = y(e_B).$$

Here $y \in Fun(B)$, $b, b_1, b_2 \in B$, e_B is the unit element of B and we use the Sweedler notation for the coproduct:

$$\Delta y = \sum_{\alpha} y'_{\alpha} \otimes y''_{\alpha} \equiv y' \otimes y''.$$

Poisson-Lie group is a Lie group equipped with a Poisson-bracket $\{.,.\}_B$ such that

$$\Delta\{y_1, y_2\}_B = \{y'_1, y'_2\}_B \otimes y''_1 y''_2 + y'_1 y'_2 \otimes \{y''_1, y''_2\}_B, \quad y_1, y_2 \in Fun(B). \quad (1)$$

Consider the linear dual \mathcal{B}^* of the Lie algebra $\mathcal{B} = Lie(B)$. The Poisson-Lie bracket $\{.,.\}_B$ induces a natural Lie algebra structure $[\cdot, \cdot]^*$ on \mathcal{B}^* . Let us explain this fact in more detail: First of all recall that \mathcal{B}^* can be identified with the space of right-invariant 1-forms on the group manifold B . We have a natural (surjective) map $\phi : Fun(B) \rightarrow \mathcal{B}^*$ defined by

$$\phi(y) = dy'S(y''), \quad y \in Fun(B).$$

Note that the 1-form $\phi(y)$ is right-invariant therefore it is indeed in \mathcal{B}^* . Let $u_1, u_2 \in \mathcal{B}^*$ and $y_1, y_2 \in Fun(B)$ such that $u_j = \phi(y_j), j = 1, 2$. Then we have

$$[u_1, u_2]^* = \phi(\{y_1, y_2\}_B).$$

It is the Poisson-Lie property (1) of $\{.,.\}_B$ which ensures the independence of $[u, v]^*$ on the choice of the representatives y_1, y_2 . In what follows, the Lie algebra $(\mathcal{B}^*, [\cdot, \cdot]^*)$ will be denoted by the symbol \mathcal{G} and G will be a (connected simply connected) Lie group such that $\mathcal{G} = Lie(G)$. We note that G is often referred to as the dual group of B .

3 Poisson-Lie Symmetry

A dynamical system is a triple (M, ω, v) where M is a manifold, ω is a non-degenerate closed 2-form on it and v is a vector field on M defining the time evolution. One often considers the case when the vector field v is Hamiltonian. This means that it exists a function $H \in Fun(M)$ such that

$$v = \Pi_M(\cdot, dH).$$

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Here Π_M is the so-called Poisson bivector (i.e. an antisymmetric contravariant tensor field) on M obtained by the inversion of the symplectic form ω . An expression $\Pi_M(df, dg)$ for $f, g \in Fun(M)$ is nothing but the Poisson bracket $\{f, g\}_M$.

Basic definition:

Let B be a Poisson-Lie group and G its dual group. We call the dynamical system (M, ω, v) (G, B) -Poisson-Lie symmetric if G acts on M and if it exists a surjection $\mu : M \rightarrow B$ (called the momentum map) such that

i) $Im(\mu^*)$ is the Poisson subalgebra of $Fun(M)$ stable with respect to the evolution vector field v .

ii) The map $w : Fun(B) \rightarrow Vect(M)$ given by

$$w(y) \equiv \Pi_M(\cdot, \mu^* \phi(y)), \quad y \in Fun(B)$$

is the homomorphism of Lie algebras fulfilling $Im(w) = Lie(G)$.

Remarks and explanations:

a) The group G is called the symmetry group since it is the one which acts on the phase space M . B is called the cosymmetry group and it underlies (via ϕ) the way how the G -action is described via the Poisson bracket on M .

b) Both $Fun(B)$ and $Vect(M)$ are naturally Lie algebras, the former with respect to the Poisson bracket $\{., .\}_B$, the latter with respect to the Lie bracket of vector fields on M . When writing $Im(w) = Lie(G)$ we mean that, using the action of G on M , $Lie(G)$ is embedded in $Vect(M)$.

c) If the dual map $\mu^* : Fun(B) \rightarrow Fun(M)$ is a Poisson morphism (i.e. if $\{\mu^* y_1, \mu^* y_2\}_M = \mu^* \{y_1, y_2\}_B$) the Poisson-Lie symmetry is called non-anomalous. If μ^* is not the Poisson morphism the symmetry is called anomalous.

Example of non-anomalous Poisson-Lie symmetry:

For G we take the group $SU(2)$ whose elements are matrices of the form $\begin{pmatrix} \alpha & -\bar{\beta} \\ \beta & \bar{\alpha} \end{pmatrix}$, $\alpha, \beta \in \mathbf{C}$, $\alpha\bar{\alpha} + \beta\bar{\beta} = 1$. For B we take a three-dimensional manifold \mathbf{R}^3 with the usual cartesian coordinates denoted as $J^i \in Fun(B)$, $i = 1, 2, 3$. The group structure on B is (dually) defined by the coproduct, the antipode and the counit

$$\Delta J^1 = e^{\epsilon J^3} \otimes J^1 + J^1 \otimes e^{-\epsilon J^3}, \quad (2a)$$

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$$\Delta J^2 = e^{\epsilon J^3} \otimes J^2 + J^2 \otimes e^{-\epsilon J^3}, \quad (2b)$$

$$\Delta J^3 = 1 \otimes J^3 + J^3 \otimes 1, \quad (2c)$$

$$S(J^i) = -J^i, \quad \varepsilon(J^i) = 0, \quad i = 1, 2, 3. \quad (2d)$$

The Poisson-Lie bracket on B is given by

$$\{J^3, J^1\}_B = J^2, \quad \{J^3, J^2\}_B = -J^1, \quad \{J^1, J^2\}_B = \frac{\sinh(2\epsilon J^3)}{2\epsilon}. \quad (3)$$

Let us now describe a (G, B) -dynamical system (M, ω_ϵ, v) . For M we take $\mathbf{C}^2 (= \mathbf{R}^4)$ and we parametrize it by two complex coordinates $A, B \in \text{Fun}^{\mathbf{C}}(M)$ given in terms of the cartesian coordinates of \mathbf{R}^4 as $A = x^1 + ix^2$, $B = x^3 + ix^4$. For the (ϵ -independent) evolution vector field v we take

$$v = -\frac{i}{2}(A\bar{A} + B\bar{B}) \left(\bar{A} \frac{\partial}{\partial A} + \bar{B} \frac{\partial}{\partial B} - A \frac{\partial}{\partial \bar{A}} - B \frac{\partial}{\partial \bar{B}} \right)$$

Instead of detailing the symplectic form ω_ϵ , we directly describe the Poisson bracket which it induces. Thus:

$$\{A, \bar{A}\}_M = i\sqrt{1 + \epsilon^2(A\bar{A} + B\bar{B})^2} + i\epsilon B\bar{B},$$

$$\{B, \bar{B}\}_M = i\sqrt{1 + \epsilon^2(A\bar{A} + B\bar{B})^2} - i\epsilon A\bar{A},$$

$$\{A, B\} = -i\epsilon AB, \quad \{\bar{A}, \bar{B}\} = i\epsilon \bar{A}\bar{B}, \quad \{A, \bar{B}\} = 0, \quad \{\bar{A}, B\} = 0.$$

Note that in the limit $\epsilon \rightarrow 0$ the bracket on $M = \mathbf{R}^4$ becomes just the Darboux Poisson bracket.

The action of $G = SU(2)$ on M is given by

$$\begin{pmatrix} A \\ B \end{pmatrix} \rightarrow \begin{pmatrix} \alpha & -\bar{\beta} \\ \beta & \bar{\alpha} \end{pmatrix} \begin{pmatrix} A \\ B \end{pmatrix} \quad (4)$$

The comomentum map μ_ϵ^* reads

$$\mu_\epsilon^*(J^3) = -\frac{1}{2\epsilon} \ln \left(\sqrt{1 + \epsilon^2(A\bar{A} + B\bar{B})^2} + \epsilon(B\bar{B} - A\bar{A}) \right),$$

$$\mu_\epsilon^*(J^1) = \frac{1}{2}(\bar{A}B + A\bar{B}) \exp(\epsilon \mu_\epsilon^*(J^3)),$$

$$\mu_\epsilon^*(J^2) = \frac{i}{2}(\bar{A}B - A\bar{B}) \exp(\epsilon \mu_\epsilon^*(J^3)),$$

It is straightforward to verify that $\text{Im}(\mu_\epsilon^*)$ is the Poisson subalgebra of $\text{Fun}(M)$. Indeed, we have

$$\{J^3, J^1\}_M = J^2, \quad \{J^3, J^2\}_M = -J^1, \quad \{J^1, J^2\}_M = \frac{\sinh(2\epsilon J^3)}{2\epsilon}.$$

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where we have (somewhat abusively) written just J^j instead of $\mu_\epsilon^*(J^j)$. It turns out that the evolution vector field v is Hamiltonian. The Hamiltonian H_ϵ reads

$$H_\epsilon = (J^1)^2 + (J^2)^2 + \frac{\sinh^2(\epsilon J^3)}{\epsilon^2}$$

and fulfils

$$\{H, J^k\}_M = 0, \quad k = 1, 2, 3.$$

Thus $Im(\mu^*)$ is indeed evolution invariant as it should.

Now we calculate

$$\begin{aligned}\phi(J^1) &= e^{\epsilon J^3} dJ^1 - J^1 d(e^{\epsilon J^3}), \\ \phi(J^2) &= e^{\epsilon J^3} dJ^2 - J^2 d(e^{\epsilon J^3}), \\ \phi(J^3) &= dJ^3.\end{aligned}$$

The map w is indeed the Lie algebra homomorphism (this can be checked by using (3)) and it gives three vector fields acting on $f \in Fun(M)$:

$$\begin{aligned}w(J^1)f &= e^{\epsilon J^3} \{f, J^1\}_M - J^1 \{f, e^{\epsilon J^3}\}_M, \\ w(J^2)f &= e^{\epsilon J^3} \{f, J^2\}_M - J^2 \{f, e^{\epsilon J^3}\}_M, \\ w(J^3)f &= \{f, J^3\}_M.\end{aligned}$$

Whatever is the value of ϵ , the vector fields $w(J^k)$ turn out to be the same:

$$\begin{aligned}w(J^1) &= -\frac{i}{2} \left(\bar{B} \frac{\partial}{\partial \bar{A}} + \bar{A} \frac{\partial}{\partial \bar{B}} - B \frac{\partial}{\partial A} - A \frac{\partial}{\partial B} \right) \\ w(J^2) &= -\frac{1}{2} \left(\bar{B} \frac{\partial}{\partial \bar{A}} - \bar{A} \frac{\partial}{\partial \bar{B}} + B \frac{\partial}{\partial A} - A \frac{\partial}{\partial B} \right) \\ w(J^3) &= -\frac{i}{2} \left(\bar{A} \frac{\partial}{\partial \bar{A}} - \bar{B} \frac{\partial}{\partial \bar{B}} - A \frac{\partial}{\partial A} + B \frac{\partial}{\partial B} \right)\end{aligned}$$

They verify

$$[w(J^j), w(J^k)] = -\epsilon^{jkl} w(J^l),$$

where ϵ^{jkl} is the well-known alternating symbol. For completeness, let us mention that the vector fields $w(J^k)$ indeed encode the infinitesimal version of the action (4) of the group $G = SU(2)$ on $M = \mathbf{C}^2$, as they should.

Deformation program

Let us first clarify the role of the (real) parameter ϵ in the example above. Actually, Eqs. (2abcd) define a one-parameter family B_ϵ of the cosymmetry groups.

Note that for $\epsilon = 0$ the group B_0 is Abelian (the group law becomes just the addition of vectors in \mathbf{R}^3) and the brackets (3) become the Kirillov-Kostant brackets on the dual of the Lie algebra $su(2)(= \mathbf{R}^3)$. Thus, for $\epsilon = 0$, the vector fields $w(J^k)$ become all Hamiltonian (i.e. $w(J^k)f = \{f, J^k\}_M$) and the dynamical system (M, ω_0, v) is symmetric in the ordinary sense.

More generally, having a dynamical system (M, ω, v) possessing an ordinary symmetry with respect to the group G acting on M in the Hamiltonian way, the deformation program consists in finding a one-parameter family of cosymmetry groups B_ϵ and a one parameter family of (G, B_ϵ) -Poisson-Lie symmetric dynamical systems (M, ω_ϵ, v) in such a way that for $\epsilon = 0$ we recover the original system (M, ω, v) . In particular, this means that $\omega = \omega_0$ and B_0 becomes the Abelian group.

Note that in the deformation process the manifold $M = \mathbf{C}^2$ was not deformed, neither the action of the symmetry group $G = SU(2)$ on M and not even the evolution vector field v . Only the symplectic form ω and the cosymmetry group B got deformed. Such situation takes place also for q -WZW model.

4 Twisted Heisenberg Double

The basic observation which triggered our work is as follows: The ordinary WZW model is a particular example of a general construction of (anomalous) Poisson-Lie symmetric systems known under the name of twisted Heisenberg double [2]. Let us describe this construction in more detail by stating two lemmas:

Lemma 1:

Consider a Lie group D equipped with a non-degenerated bi-invariant metric and possessing two maximally isotropic subgroups G and B . Let κ be a metric preserving automorphism of D and let T^i and t_i be the respective basis of $\mathcal{G} = Lie(G)$ and $\mathcal{B} = Lie(B)$ such that $(T^i, t_j)_D = \delta_j^i$. Then the (basis independent) expression

$$\{f_1, f_2\}_D \equiv \nabla_{T^i}^R f_1 \nabla_{t_i}^R f_2 - \nabla_{\kappa(t_i)}^L f_1 \nabla_{\kappa(T^i)}^L f_2, \quad f_1, f_2 \in Fun(D) \quad (5)$$

is a non-degenerated Poisson bracket making D a symplectic manifold.

Remarks and explanations:

a) The group D equipped with the Poisson bracket $\{.,.\}_D$ is called the twisted

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Heisenberg double of G . We denote as ω_D the symplectic form corresponding to $\{\cdot, \cdot\}_D$.

b) Bi-invariant means both left- and right-invariant. The non-degenerated bi-invariant metric on D obviously induces an Ad -invariant non-degenerate bilinear form $(\cdot, \cdot)_D$ on $\mathcal{D} = Lie(D)$. An isotropic submanifold of D is such that the induced metric on it vanishes. Maximally isotropic means that it is not contained in any bigger isotropic submanifold.

c) The vector fields $\nabla_T^{L,R}$ are defined as

$$\nabla_T^L f(K) = \left(\frac{d}{ds} \right)_{s=0} f(e^{sT} K), \quad \nabla_T^R f(K) = \left(\frac{d}{ds} \right)_{s=0} f(K e^{sT}),$$

where $K \in D, T \in \mathcal{D}$.

Lemma 2:

Suppose that the automorphism κ preserves the subgroup B and two global unambiguous decompositions holds: $D = BG$ and $D = \kappa(G)B$. Consider (surjective) maps $\Lambda_L, \Lambda_R : D \rightarrow B$ induced by the decompositions $D = BG$ and $D = \kappa(G)B$, respectively. Let v be any vector field leaving invariant the images of their dual maps $\Lambda_L^*, \Lambda_R^* : Fun(B) \rightarrow Fun(D)$. Then (D, ω_D, v) is the Poisson-Lie symmetric dynamical system with the symmetry group $G \times G$ and the cosymmetry group $B \times B$. The symplectic form ω_D reads

$$\omega_D = \frac{1}{2}(\Lambda_L^* \rho_B \hat{\wedge} \rho_D)_{\mathcal{D}} - \frac{1}{2}(\Lambda_R^* \rho_B \hat{\wedge} \lambda_D)_{\mathcal{D}}, \quad (6)$$

the group $G \times G$ acts as

$$(h_L, h_R) \triangleright l = \kappa(h_L) l h_R^{-1}, \quad h_L, h_R \in G, \quad l \in D$$

and the momentum map $\mu : D \rightarrow B \times B$ is given by $\mu = \Lambda^L \times \Lambda^R$.

Remarks and explanations:

a) Global unambiguous decomposition $D = \kappa(G)B$ means that for every element $l \in D$ it exists a unique $g \in G$ and a unique $b \in B$ such that $l = \kappa(g)b^{-1}$. Similarly for $D = BG$: it exists a unique $\tilde{g} \in G$ and a unique $\tilde{b} \in B$ such that $l = \tilde{b}\tilde{g}^{-1}$.

b) ρ_D (λ_D) is right(left)-invariant Maurer-Cartan form on the group D , $\Lambda_{L(R)}^* \rho_B$ is the pull-back of the right-invariant Maurer-Cartan form on B by the map $\Lambda_{L(R)}^*$.

Example: The standard WZW model.

Let K be a simple connected and simply connected compact Lie group and LK the group of smooth maps from a circle S^1 into K (the group law is given by pointwise multiplication). It is important for us that there exists a natural non-degenerate invariant bilinear form $(\cdot|\cdot)$ on $LK \equiv Lie(LK)$ given by formula

$$(\alpha|\beta) = \frac{1}{2\pi} \int d\sigma (\alpha(\sigma), \beta(\sigma))_{\mathcal{K}}, \quad (7)$$

where the angle variable $\sigma \in [-\pi, \pi]$ parametrizes the circle S^1 and $(\cdot, \cdot)_{\mathcal{K}}$ is the (appropriately normalized) Killing-Cartan form on $\mathcal{K} \equiv Lie(K)$.

For the twisted Heisenberg double D we take the semidirect product of LK with its Lie algebra $L\mathcal{K}$. Thus the group multiplication law on D reads

$$(g, \chi) \cdot (\tilde{g}, \tilde{\chi}) = (g\tilde{g}, \chi + Ad_g \tilde{\chi}), \quad g \in LK, \chi \in L\mathcal{K}.$$

Lie algebra \mathcal{D} of D has the structure of semidirect sum $\mathcal{D} = L\mathcal{K} \overset{\rightarrow}{\oplus} L\mathcal{K}$ where the second component of the semidirect sum has trivial zero bracket. Thus

$$[\chi \oplus \alpha, \xi \oplus \beta] = [\chi, \xi] \oplus ([\chi, \beta] - [\xi, \alpha]),$$

where $\chi, \xi \in L\mathcal{K}$ are in the first and $\alpha, \beta \in L\mathcal{K}$ in the second component of the semidirect sum. The bi-invariant metric on D comes from Ad -invariant bilinear form $(\cdot, \cdot)_{\mathcal{D}}$ on \mathcal{D} defined with the help of (7):

$$(\chi \oplus \alpha, \xi \oplus \beta)_{\mathcal{D}} = (\chi|\beta) + (\xi|\alpha).$$

The metric preserving automorphism κ of the group D reads

$$\kappa(g, \chi) = (g, \chi + k\partial_{\sigma} g g^{-1})$$

where k is an (integer) parameter. The maximally isotropic subgroups are $G = LK$ and $B = L\mathcal{K}$. Note that B is Abelian since the group law is given just by the addition of vectors in $L\mathcal{K}$. It is simple to establish the decompositions $D = \kappa(G)B$ and $D = BG$. Indeed, we have for every $g \in LK, \chi \in L\mathcal{K}$

$$(g, \chi) = (g, k\partial_{\sigma} g g^{-1})(e, Ad_{g^{-1}}\chi - kg^{-1}\partial_{\sigma} g) = (e, \chi) \cdot (g, 0),$$

where e is the unit element of LK .

The symplectic form ω_D on D is given by (6) and it reads (cf. [3,4])

$$\omega_D = d(J_L|\rho) + \frac{1}{2}k(\rho|\partial_{\sigma}\rho),$$

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where ρ is a $L\mathcal{K}$ -valued right-invariant Maurer-Cartan form on LK often written as $\rho = dg g^{-1}$ and J_L is $L\mathcal{K}$ -valued function on D defined by

$$J_L(g, \chi) = \chi.$$

In order to define the Hamiltonian H we need one more $L\mathcal{K}$ -valued function on D denoted J_R . Thus

$$J_R(g, \chi) = -Ad_{g^{-1}}\chi + kg^{-1}\partial_\sigma g$$

and

$$H = -\frac{1}{2k}(J_L|J_L) - \frac{1}{2k}(J_R|J_R).$$

Let us study the symmetry structure of our dynamical system (D, ω_D, H) . First of all, the symmetry group $G \times G = LK \times LK$ acts on D as follows

$$(h_L, h_R) \triangleright (g, \chi) = (h_L g h_R^{-1}, k\partial_\sigma h_L h_L^{-1} + h_L \chi h_L^{-1}), \quad h_L, h_R, g \in LK, \chi \in L\mathcal{K}. \quad (8)$$

The cosymmetry group $B \times B$ is $L\mathcal{K} \oplus L\mathcal{K}$ (with Abelian group law given by the addition of vectors). The (commutative) algebra $Fun(L\mathcal{K})$ is generated by linear functions F_χ of the form

$$F_\chi(\xi) = (\chi|\xi), \quad \chi, \xi \in L\mathcal{K},$$

hence the algebra $Fun(B \times B) = Fun(L\mathcal{K}) \otimes Fun(L\mathcal{K})$ is generated by linear functions F_χ^L, F_χ^R . The Abelian group law on $B \times B$ is then encoded in the coproduct, antipode and counit on $Fun(B \times B)$:

$$\Delta F_\chi^{L(R)} = 1 \otimes F_\chi^{(L)R} + F_\chi^{(L)R} \otimes 1, \quad S(F_\chi^{L(R)}) = F_{-\chi}^{L(R)}, \quad \varepsilon(F_\chi^{L(R)}) = 0.$$

The Poisson-Lie bracket on $B \times B$ is given by

$$\{F_\chi \otimes 1, F_\xi \otimes 1\}_{B \times B} = F_{[\chi, \xi]} \otimes 1, \quad (9a)$$

$$\{1 \otimes F_\chi, 1 \otimes F_\xi\}_{B \times B} = 1 \otimes F_{[\chi, \xi]}, \quad (9a)$$

$$\{F_\chi \otimes 1, 1 \otimes F_\xi\}_{B \times B} = 0, \quad (9c)$$

The (dual) momentum map $\mu^* : Fun(B \times B) \rightarrow Fun(D)$ is simply

$$\mu^*(F_\chi^L) = \mu^*(F_\chi \otimes 1) = (J_L|\chi), \quad \mu^*(F_\chi^R) = \mu^*(1 \otimes F_\chi) = (J_R|\chi), \quad \chi \in L\mathcal{K}.$$

In order to illustrate that the WZW model (D, ω_D, H) is indeed $(G \times G, B \times B)$ -Poisson-Lie symmetric, we have to verify the items i) and ii) of the Basic definition of Section 3. First we notice that $Im(\mu^*)$ is the Poisson subalgebra of $Fun(D)$. Indeed, it is not difficult to calculate

$$\{(J_L|\chi), (J_L|\xi)\}_D = (J_L|([\chi, \xi]) + k(\chi|\partial_\sigma \xi)), \quad (10a)$$

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$$\{(J_R|\chi), (J_R|\xi)\}_D = (J_R|([\chi, \xi]) - k(\chi|\partial_\sigma\xi), \quad (10b)$$

$$\{(J_L|\chi), (J_R|\xi)\}_D = 0. \quad (10c)$$

(By comparing (9abc) with (10abc) we see that μ^* is not the Poisson morphism hence the symmetry is anomalous). Then we verify, that $Im(\mu^*)$ is stable with respect to the time evolution. Indeed, we have

$$\{(J_L|\chi), H\}_D = (J_L|\partial_\sigma\chi).$$

$$\{(J_R|\chi), H\}_D = -(J_R|\partial_\sigma\chi).$$

Verifying the item ii) of the basic definition is also easy. First we identify $Fun(D) = Fun(LK) \otimes Fun(L\mathcal{G})$ and, by inverting the symplectic form ω_D , we find

$$\{f_1 \otimes 1, f_2 \otimes 1\}_D = 0, \quad \{f \otimes 1, 1 \otimes F_\chi\}_D = \nabla_\chi^L f \otimes 1,$$

$$\{1 \otimes F_\chi, 1 \otimes F_\xi\}_D = 1 \otimes F_{[\chi, \xi]} + k(\chi|\partial_\sigma\xi)1 \otimes 1.$$

Then it is easy to calculate

$$w((J_L|\chi))(f \otimes 1) = \{f \otimes 1, (J_L|\chi)\}_M = (\nabla_\chi^L f \otimes 1), \quad f \in Fun(LK).$$

$$w((J_L|\chi))(1 \otimes F_\xi) = \{1 \otimes F_\xi, (J_L|\chi)\}_D = 1 \otimes F_{[\xi, \chi]} + k(\xi|\partial_\sigma\chi)1 \otimes 1,$$

$$w((J_R|\chi))(f \otimes 1) = \{f \otimes 1, (J_R|\chi)\}_D = -(\nabla_\chi^R f \otimes 1), \quad f \in Fun(LK).$$

$$w((J_R|\chi))(1 \otimes F_\xi) = 0.$$

We thus see that the vector fields $w((J_L|\chi))$ and $w((J_R|\chi))$ indeed generate the infinitesimal version of the action (8) of the symmetry group $LK \times LK$ on the algebra of observables $Fun(D)$.

5 q -WZW Model

The deformation program outlined in Section 3 can be applied to the standard WZW model (cf. [5, 6]). The resulting theory is called a quasitriangular WZW model and it can be described in rather explicit fashion by performing the so-called chiral decomposition. The technical presentation of the construction of the q -WZW is however lengthy and it cannot be presented on a small space of few pages. Therefore we shall restrict ourselves to the very beginning and the very end of the story:

The twisted Heisenberg double underlying the q -WZW model is the loop group $LK^{\mathbb{C}}$ consisting of smooth maps from the circle S^1 into the complexified group $K^{\mathbb{C}}$. (For example, the complexification of $SU(2)$ is $SL(2, \mathbb{C})$). It is important

to stress that $LK^{\mathbb{C}}$ is viewed as real group. The invariant nondegenerate bilinear form $(\cdot, \cdot)_{\mathcal{D}}$ on $\mathcal{D} = Lie(LK^{\mathbb{C}})$ is then defined as

$$(x, y)_{\mathcal{D}} = \frac{1}{\epsilon} Im(x|y), \quad x, y \in \mathcal{D},$$

where $(\cdot | \cdot)$ is just the bilinear form (7) naturally extended to $LK^{\mathbb{C}}$, Im stands for the imaginary part and ϵ is the deformation parameter. The metric preserving automorphism κ of $LK^{\mathbb{C}}$ is defined most easily if we view the group $LK^{\mathbb{C}}$ as a group of holomorphic maps from a Riemann sphere without poles into the complex group $K^{\mathbb{C}}$. (Clearly, the loop circle S^1 is identified with the equator). Then

$$\kappa(l)(z) = l(e^{\epsilon k} z), \quad l \in LK^{\mathbb{C}},$$

where z is the usual complex coordinate on the Riemann sphere and k is the same integer parameter which appears also in the standard WZW model. The maximally isotropic subgroup G is nothing but LK (the isotropy is a direct consequence of the fact that $(\cdot | \cdot)$ is real when restricted to LK) and $B = L_+ K^{\mathbb{C}}$ is defined as the group of holomorphic maps from the Riemann sphere without the north pole into the complex group $K^{\mathbb{C}}$. We require moreover, that the value of this holomorphic map at the south pole is an element of the group AN defined by the Iwasawa decomposition $K^{\mathbb{C}} = KAN$. It is not difficult to establish the isotropy of $L_+ K^{\mathbb{C}}$ (cf. Sec 4.4.1 of [5]). Finally, the existence of the global decomposition $D = LK^{\mathbb{C}} = \kappa(LK)(L_+ K^{\mathbb{C}}) = \kappa(G)B = BG$ was proved in [7].

The symplectic structure on $D = LK^{\mathbb{C}}$ is defined by the formula (6). It turns out that the symplectic manifold D has the structure of the (reduced) product $P \times P$ of two copies of a simpler symplectic manifold P called chiral q -WZW model. The points of the manifold P are maps $m : \mathbf{R} \rightarrow K$, fulfilling the monodromy condition

$$m(\sigma + 2\pi) = m(\sigma)e^{-2\pi i x},$$

where x is an element of the Weyl alcove of the Cartan subalgebra of $Lie(K)$. The symplectic structure on P is completely characterized by the following matrix Poisson bracket:

$$\{m(\sigma) \otimes m(\sigma')\}_P = (m(\sigma) \otimes m(\sigma'))B_{\epsilon}(x, \sigma - \sigma') + \epsilon \hat{r}(\sigma - \sigma')(m(\sigma) \otimes m(\sigma')), \quad (11)$$

where $\hat{r}(\sigma)$ is a trigonometric solution of the ordinary Yang-Baxter equation with spectral parameter

$$\hat{r}(\sigma) = r + C \cotg \frac{1}{2} \sigma,$$

C is the Casimir element defined by

$$C = \sum_{\mu} H^{\mu} \otimes H^{\mu} + \sum_{\alpha > 0} \frac{|\alpha|^2}{2} (E^{-\alpha} \otimes E^{\alpha} + E^{\alpha} \otimes E^{-\alpha}).$$

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and

$$r = \sum_{\alpha > 0} \frac{i|\alpha|^2}{2} (E^{-\alpha} \otimes E^{\alpha} - E^{\alpha} \otimes E^{-\alpha}).$$

The matrix $B_{\varepsilon}(x, \sigma)$ is the so called Felder's elliptic r-matrix [8] and it solves the dynamical Yang-Baxter equations with spectral parameter.

$$B_{\varepsilon}(x, \sigma) = -\frac{i}{\kappa} \rho\left(\frac{i\sigma}{2\kappa\varepsilon}, \frac{i\pi}{\kappa\varepsilon}\right) H^{\mu} \otimes H^{\mu} - \frac{i}{\kappa} \sum_{\alpha} \frac{|\alpha|^2}{2} \sigma_{\langle \alpha, x \rangle} \left(\frac{i\sigma}{2\kappa\varepsilon}, \frac{i\pi}{\kappa\varepsilon}\right) E^{\alpha} \otimes E^{-\alpha}.$$

The elliptic functions $\rho(z, \tau), \sigma_w(z, \tau)$ are defined as (cf. [8])

$$\sigma_w(z, \tau) = \frac{\theta_1(w - z, \tau) \theta_1'(0, \tau)}{\theta_1(w, \tau) \theta_1(z, \tau)}, \quad \rho(z, \tau) = \frac{\theta_1'(z, \tau)}{\theta_1(z, \tau)}.$$

Note that $\theta_1(z, \tau)$ is the Jacobi theta function¹

$$\theta_1(z, \tau) = - \sum_{j=-\infty}^{\infty} e^{\pi i(j+\frac{1}{2})^2 \tau + 2\pi i(j+\frac{1}{2})(z+\frac{1}{2})},$$

the prime ' means the derivative with respect to the first argument z and the argument τ (the modular parameter) is a nonzero complex number such that $\text{Im } \tau > 0$.

We note that in the limit $\varepsilon \rightarrow 0$ the symplectic structure (11) becomes that of the ordinary chiral WZW model (cf. [5]). The time evolution in q -WZW model is the same as in the non-deformed case, i.e.

$$[m(\sigma)](\tau) = m(\sigma - \tau),$$

where τ is the evolution parameter (time).

The chiral model P turns out to be Poisson-Lie symmetric with respect to the natural left action of LK on P . For a detailed exposition of this fact see [5], here we just mention what is the ε -deformation of the standard Kac-Moody commutation relation (10a). Thus we set

$$L(\sigma) = m(\sigma + i\kappa\varepsilon) m^{-1}(\sigma - i\kappa\varepsilon)$$

and we use the fundamental Poisson brackets (11) to find well-known relations of deformed current algebra [10]:

$$\begin{aligned} \{L(\sigma) \otimes L(\sigma')\} &= (L(\sigma) \otimes L(\sigma')) \varepsilon \hat{r}(\sigma - \sigma') + \varepsilon \hat{r}(\sigma - \sigma') (L(\sigma) \otimes L(\sigma')) \\ &- (1 \otimes L(\sigma')) \varepsilon \hat{r}(\sigma - \sigma' + 2i\varepsilon\kappa) (L(\sigma) \otimes 1) - (L(\sigma) \otimes 1) \varepsilon \hat{r}(\sigma - \sigma' - 2i\varepsilon\kappa) (1 \otimes L(\sigma')). \end{aligned} \tag{12}$$

¹We have $\theta_1(z, \tau) = \vartheta_1(\pi z, \tau)$ with ϑ_1 in [9].

6 Concluding Remarks

Perhaps the most remarkable result of our deformation program is the fact that the structure of the deformed chiral WZW model is essentially characterized by the Felder elliptic dynamical r -matrix [8]. In particular, we find quite amazing that from so small input (the group structure of $LK^{\mathbb{C}}$) such a nice mathematical object like Felder's matrix naturally emerges. We have shown another interesting thing, namely, the trigonometric deformed current algebra (12) is underlied by a more fundamental elliptic structure (11).

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