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# The Invariance of the Schrödinger Equation with Respect to the Transformations Derived from the Special Test

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**Abstract.** The results of the special test reveal the occurrence of two kinds of transformations. The invariance of the Schrödinger equation is expressed with respect to these transformations.

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## 1 Introduction

Recently, the relativistic problem of the absolute observations of light signals has been posed [1]. This is the problem of how the result of the absolute observation of a light signal in an inertial system (an observation such as an inertial observer viewing a light signal emitted from his source, but not viewing a light signal emitted from a source in the other inertial system) is related to the result of the absolute observation of a light signal in the other inertial system. In order to solve this, the test that starts when two inertial systems separate, and ends when they coincide, was given by using four light signals. Consequently, two kinds of transformations [1] are derived from this special test. Then, the invariance of the electromagnetic wave equation [2] based on Maxwell's theory was shown with respect to these transformations [3].

The special test can also be given by using four particles instead of four light signals. This is explained in Section 2. Then, two kinds of transformations are also derived from the special test for particles. The Schrödinger equation can be derived by using the wave equation

$$\nabla^2\psi - \frac{1}{u^2}\frac{\partial^2\psi}{\partial t^2} = 0,$$

where $\psi$ is the wave function. Then, if this equation is invariant with respect to the transformations derived from the special test, the Schrödinger equation

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must also be invariant with respect to these transformations. This is explained in Section 3. This paper deals with the invariance of the Schrödinger equation with respect to the transformations derived from the special test.

## 2 Special Test Given by Using Four Particles

Let us explain the special test for particles using Figure 1. Let us assume that four particles travel at a constant speed $u$ from the points $A'$, $P'$, $A$, and $P$, respectively. Let the length of $A'O'$ equals the length of $P'O'$; let the length of $AO$ equals the length of $PO$. When the origin $O'$ approaches the origin $O$ (or $O$ approaches $O'$) at a constant speed $v$, let us assume that a particle is emitted toward $O'$ from $A'$ at time $t'_1$ and that, after the particle is emitted from $A'$, another particle is emitted toward $O'$ from $P'$ at time $t'_2$. When $O'$ coincides

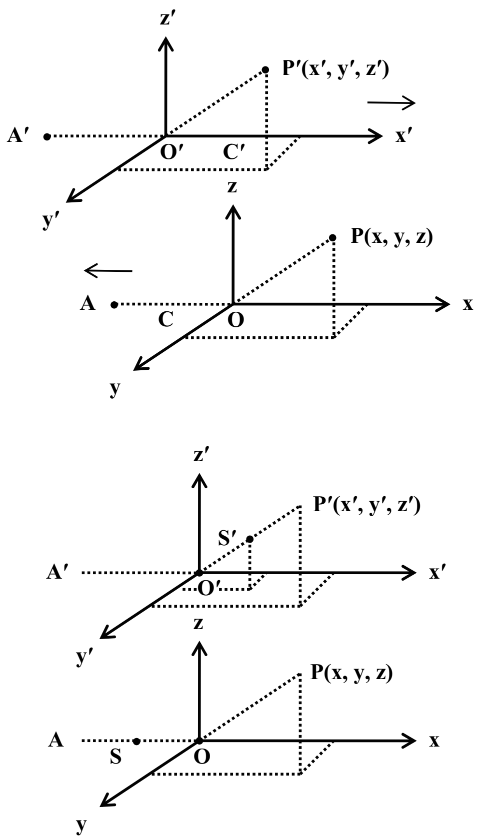


Figure 1. The special test for particles.

with $O$ at time $t'$, suppose it turns out that the particle from $A'$ reaches $O'$. When the particle is emitted from $A'$, another particle is not emitted from $P'$. Then, when the particle from $A'$ reaches $O'$, the particle from $P'$ does not reach $O'$ and reaches some point $S'$. Therefore, the length of $A'O'$ is expressed by

$$\overline{A'O'} = \overline{P'S'} + \overline{S'O'}. \tag{1}$$

Let us assume that, when $O'$ passes some point $C$ on the $x$-axis, the particle is emitted from $P'$. Then, when $O'$ coincides with $O$, $c$ is at the position of $-v(t' - t'_2)$ from $O'$. Let the length of $S'O'$ equals the length of $CO'$, i.e.,

$$w(t' - t'_2) = v(t' - t'_2).$$

Therefore, (1) is expressed as

$$u\Delta t'_1 = u\Delta t' + w\Delta t', \tag{2}$$

where $\Delta t'_1 = t' - t'_1$ and $\Delta t' = t' - t'_2$. From (2), the time when the particle is emitted from $P'$ is given by

$$t'_2 = \frac{t'_1 + (w/u)t'}{1 + w/u}$$

and $S'$ is at the position of

$$w\Delta t' = \frac{w(t' - t'_1)}{1 + w/u}$$

from $O'$. Since the length of $S'O'$ is dependent on time $t'_2$, the inverse transformation of (2) is

$$u\Delta t' = u\Delta t'_1 - w\Delta t'. \tag{3}$$

On the other hand, when $O$ approaches $O'$ (or $O'$ approaches $O$), let us assume that a particle is emitted toward $O$ from $P$ at time $t_2$ and that, after the particle is emitted from $P$, another particle is emitted toward $O$ from $A$ at time $t_1$. When $O'$ coincides with $O$ at time $t$, suppose it turns out that the particle from $P$ reaches $O$. When the particle is emitted from $P$, another particle is not emitted from $A$. Then, when the particle from $P$ reaches $O$, the particle from $A$ does not reach $O$ and reaches some point $S$. Therefore, the length of $PO$ is expressed by

$$\overline{PO} = \overline{AS} + \overline{SO}\,. \tag{4}$$

Let us assume that, when $O$ passes some point $C'$ on the $x'$-axis, the particle is emitted from $A$. Then, when $O'$ coincides with $O$, $C'$ is at the position of $v(t - t_1)$ from $O$. Let the length of $SO$ equals the length of $C'O$. Therefore, (4) is expressed as

$$u\Delta t = u\Delta t_1 + v\Delta t_1, \tag{5}$$

where $\Delta t = t - t_2$ and $\Delta t_1 = t - t_1$. From (5), the time when the particle is emitted from $A$ is given by

$$t_1 = \frac{t_2 + (v/u)t}{1 + v/u}$$

and $S$ is at the position of

$$-v\Delta t_1 = \frac{v(t_2 - t)}{1 + v/u}$$

from $O$. Since the length of $SO$ is dependent on time $t_1$, the inverse transformation of (5) is

$$u\Delta t_1 = u\Delta t - v\Delta t_1 . \tag{6}$$

Let us assume that, when $O'$ coincides with $O$, $A'$ coincides with $A$. Then, the length of $A'O'$ equals the length of $AO$. Since the length of $A'O'$ equals the length of $P'O'$ (the length of $AO$ equals the length of $PO$), $P'$ coimcides with $P$. When $O'$ coincides with $O$, the particle emitted from $A'$ at time $t'_1$ reaches $O'$ at time $t'$ and the particle emitted from $P$ at time $t_2$ reaches $O$ at time $t$. Then, the observers will be justified in concluding that the particles are emitted from $A'$ and $P$ at the same time ($t'_1 = t_2$) and reach $O'$ ($O$) at the same time ($t' = t$). Therefore, (2) is expressed as

$$u\Delta t = u\Delta t' + w\Delta t' . \tag{7}$$

(7) is also expressed by $\Delta r = u\Delta t$ and $\Delta r' = u\Delta t'$ as

$$\Delta r = \Delta r' + w\Delta t' = \gamma \Delta r' , \tag{8}$$

where $\gamma = 1 + w/u$. Then, each component can be given by

$$\begin{bmatrix} \Delta x \\ \Delta y \\ \Delta z \\ u\Delta t \end{bmatrix} = \begin{bmatrix} \gamma & 0 & 0 & 0 \\ 0 & \gamma & 0 & 0 \\ 0 & 0 & \gamma & 0 \\ 0 & 0 & 0 & \gamma \end{bmatrix} \begin{bmatrix} \Delta x' \\ \Delta y' \\ \Delta z' \\ u\Delta t' \end{bmatrix} . \tag{9}$$

Since the length of $S'O'$ is dependent on time $t'_2$, the inverse transformation of (8) is expressed by

$$\Delta r' = \Delta r - w\Delta t' = \frac{\Delta r}{\gamma} . \tag{10}$$

Then, each component can be given by

$$\begin{bmatrix} \Delta x' \\ \Delta y' \\ \Delta z' \\ u\Delta t' \end{bmatrix} = \begin{bmatrix} 1/\gamma & 0 & 0 & 0 \\ 0 & 1/\gamma & 0 & 0 \\ 0 & 0 & 1/\gamma & 0 \\ 0 & 0 & 0 & 1/\gamma \end{bmatrix} \begin{bmatrix} \Delta x \\ \Delta y \\ \Delta z \\ u\Delta t \end{bmatrix} . \tag{11}$$

From the point of view of $K'$, the particle emitted from $P$ travels at a faster speed $u_k$ than $u$. Then, $\Delta r = u\Delta t = u_k \Delta t'$. Therefore, $u_k$ is expressed from (8) by

$$u_k = u + w = \gamma u, \tag{12}$$

and $u$ is expressed from (10) by

$$u = u_k - w = \gamma' u_k, \tag{13}$$

where $\gamma' = 1 - w/u_k = 1/\gamma$. Therefore, if the travel of the particles is examined from the point of view of $K'$, the transformations that correspond to (9) and (11) are expressed by

$$\begin{bmatrix} \Delta x \\ \Delta y \\ \Delta z \\ u_k \Delta t' \end{bmatrix} = \begin{bmatrix} \gamma & 0 & 0 & 0 \\ 0 & \gamma & 0 & 0 \\ 0 & 0 & \gamma & 0 \\ 0 & 0 & 0 & \gamma \end{bmatrix} \begin{bmatrix} \Delta x' \\ \Delta y' \\ \Delta z' \\ u \Delta t' \end{bmatrix} . \tag{14}$$

and

$$\begin{bmatrix} \Delta x' \\ \Delta y' \\ \Delta z' \\ u \Delta t' \end{bmatrix} = \begin{bmatrix} 1/\gamma & 0 & 0 & 0 \\ 0 & 1/\gamma & 0 & 0 \\ 0 & 0 & 1/\gamma & 0 \\ 0 & 0 & 0 & 1/\gamma \end{bmatrix} \begin{bmatrix} \Delta x \\ \Delta y \\ \Delta z \\ u_k \Delta t' \end{bmatrix} . \tag{15}$$

## 3 Invariance of the Schrödinger Equation

In (8), let us express $\Delta r$ by $r$ and $\Delta r'$ by $r'$. Then, (9) and (11) are also expressed as

$$\begin{bmatrix} x \\ y \\ z \\ ut \end{bmatrix} = \begin{bmatrix} \gamma & 0 & 0 & 0 \\ 0 & \gamma & 0 & 0 \\ 0 & 0 & \gamma & 0 \\ 0 & 0 & 0 & \gamma \end{bmatrix} \begin{bmatrix} x' \\ y' \\ z' \\ ut' \end{bmatrix} . \tag{16}$$

and

$$\begin{bmatrix} x' \\ y' \\ z' \\ ut' \end{bmatrix} = \begin{bmatrix} 1/\gamma & 0 & 0 & 0 \\ 0 & 1/\gamma & 0 & 0 \\ 0 & 0 & 1/\gamma & 0 \\ 0 & 0 & 0 & 1/\gamma \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ ut \end{bmatrix} . \tag{17}$$

Furthermore, (14) and (15) are also expressed as

$$\begin{bmatrix} x \\ y \\ z \\ u_k t' \end{bmatrix} = \begin{bmatrix} \gamma & 0 & 0 & 0 \\ 0 & \gamma & 0 & 0 \\ 0 & 0 & \gamma & 0 \\ 0 & 0 & 0 & \gamma \end{bmatrix} \begin{bmatrix} x' \\ y' \\ z' \\ ut' \end{bmatrix} . \tag{18}$$

and

$$\begin{bmatrix} x' \\ y' \\ z' \\ ut' \end{bmatrix} = \begin{bmatrix} 1/\gamma & 0 & 0 & 0 \\ 0 & 1/\gamma & 0 & 0 \\ 0 & 0 & 1/\gamma & 0 \\ 0 & 0 & 0 & 1/\gamma \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ u_k t' \end{bmatrix} . \tag{19}$$

In $K$, let us express the wave equation as

$$\nabla^2 \psi - \frac{1}{u^2} \frac{\partial^2 \psi}{\partial t^2} = 0. \tag{20}$$

where $\psi = \phi \exp(-2\pi i \nu t)$. The same form as (20) can also be expressed in $K'$ by using (17):

$$\nabla'^2 \psi' - \frac{1}{u^2} \frac{\partial^2 \psi'}{\partial t'^2} = 0. \tag{21}$$

where $\psi'(=\psi) = \phi' \exp(-2\pi i \nu' t')$ and $\nu' t' = \nu t$. Then, the frequency $\nu$ is related to $\nu'$ by using $t = \gamma t'$ in (16), as follows:

$$\nu = \frac{\nu'}{\gamma} . \tag{22}$$

Therefore, when the energy is expressed in $K$ as $\varepsilon = h\nu$ and in $K'$ as $\varepsilon' = h\nu'$, $\varepsilon$ is related to $\varepsilon'$ by

$$\varepsilon = \frac{\varepsilon'}{\gamma} . \tag{23}$$

Since

$$u = \lambda \nu = \lambda' \nu',$$

$\lambda$ is related to $\lambda'$ by

$$\lambda = \gamma \lambda' . \tag{24}$$

Then, when the momentum is expressed in $K$ as $p = h/\lambda$ and in $K'$ as $p' = h/\lambda'$, $p$ is related to $p'$ by

$$p = \frac{p'}{\gamma} . \tag{25}$$

Therefore, when $p = mu$ and $p' = m'u$, $m$ is related to $m'$ by

$$m = \frac{m'}{\gamma} . \tag{26}$$

Then, $m$ and $m'$ must be prepared in consideration of $\gamma$. From (25) and (26), the kinetic energies are related by

$$\frac{p^2}{2m} = \frac{p'^2}{2\gamma m'} . \tag{27}$$

Let us express the energies $\varepsilon$ and $\varepsilon'$ as

$$\varepsilon = \frac{p^2}{2m} + V \tag{28}$$

and

$$\varepsilon' = \frac{p'^2}{2m'} + V'. \tag{29}$$

From (23) and (27), therefore, the potential energy $V$ is related to $V'$ by

$$V = \frac{V'}{\gamma}\,. \tag{30}$$

Therefore, when the Schrödinger equation is expressed in $K$ by

$$\Big(-\frac{\hbar^2}{2m}\nabla^2 + V\Big)\psi = i\hbar\frac{\partial\psi}{\partial t}\,, \tag{31}$$

the same form can also be expressed in $K'$ by using (17):

$$\Big(-\frac{\hbar^2}{2m'}\nabla'^2 + V'\Big)\psi' = i\hbar\frac{\partial\psi'}{\partial t'}\,. \tag{32}$$

In $K'$, let us express the wave function $\psi$ as $\psi_k$ and the frequency $\nu$ as $\nu_k$. Then, the wave equation can be expressed in $K$ by

$$\nabla^2\psi_k - \frac{1}{u_k^2}\frac{\partial^2\psi_k}{\partial t'^2} = 0, \tag{33}$$

where

$$\psi_k = \varphi_k \exp(-2\pi i\nu_l t').$$

The same form as (33) can also be expressed in $K'$ by $\psi_k = \psi'$ and $\nu_k = \nu'$, as is in (21). In $K'$, let us express the energy $\varepsilon$ as $\varepsilon_k$ $(= h\nu_k)$. Therefore,

$$\varepsilon_k = \varepsilon'. \tag{34}$$

Since

$$u_k = \gamma u(\lambda_k\nu_k = \gamma\lambda'\nu'),$$

the wavelength $\lambda_k$ is related to $\lambda'$ by

$$\lambda_k = \gamma\lambda'. \tag{35}$$

In $K'$, let us express $p$ as $p_k$ $(= h/\lambda_k)$. Then, $p_k$ is related to $p'$ by

$$p_k = \frac{p'}{\gamma}\,. \tag{36}$$

In $K'$, let us express $m$ as $m_k$. In $p_k = m_k u_k$, the mass $m_k$ is expressed from (12) and (36) by

$$m_k = \frac{{m'}^2}{\gamma} \ . \tag{37}$$

From (36) and (37), the kinetic energies are related by

$$\frac{p_k^2}{2m_k} = \frac{{p'}^2}{2m'} \ . \tag{38}$$

Let us express the energy $\varepsilon_k$ as

$$\varepsilon_k = \frac{p_k^2}{2m_k} + V_k. \tag{39}$$

From (34) and (38), therefore, the potential energy $V_k$ is related to $V'$ by

$$V_k = V'. \tag{40}$$

Therefore, when the Schrödinger equation is expressed in $K$ by

$$\Big( - \frac{\hbar^2}{2m_k} \nabla^2 + V_k \Big) \psi_k = i\hbar \frac{\partial \psi_k}{\partial t'} \ , \tag{41}$$

the same form can also be expressed in $K'$ by using (19), as is in (32).

## 4 Discussion and Conclusion

From (16) and (19), $K$ is related to $K'$ by

$$\begin{bmatrix} x \\ y \\ z \\ ut \end{bmatrix} = \begin{bmatrix} x \\ y \\ z \\ u_k t' \end{bmatrix} \ , \tag{42}$$

where $u_k = \gamma u$ and $t = \gamma t'$. From (23) and (34), $\varepsilon$ is related to $\varepsilon_k$ by

$$\varepsilon = \frac{varepsilon_k}{\gamma} \ . \tag{43}$$

From (24) and (35), $\lambda$ is related to $\lambda_k$ by

$$\lambda = \lambda_k. \tag{44}$$

From (25) and (36), $p$ is related to $p_k$ by

$$p = p_k. \tag{45}$$

From (26) and (37), $m$ is related to $m_k$ by

$$m = \gamma m_k. \tag{46}$$

From (27) and (38), $p^2/(2m)$ is related to $p_k^2/(2m_k)$ by

$$\frac{p^2}{2m} = \frac{p_k^2}{2\gamma m_k}\,. \tag{47}$$

From (30) and (40), $V$ is related to $V_k$ by

$$V = \frac{V_k}{\gamma}\,. \tag{48}$$

Therefore, when the Schrödinger equation is expressed in $K$ by (31), the same form is also expressed in $K'$ by using (42), as is in (41). Equations (42)–(48) are expressed considering one of four particles. This suggests that, when the test that starts when the inertial systems coincide is given by using one particle, the invariance of the Schrödinger equation is also expressed by this general test.

## References


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