Bulg. J. Phys. **37** (2010) 10–15

Thermodynamics of the $(2+1)$-dimensional Black Hole with non linear Electrodynamics and without Cosmological Constant from the Generalized Uncertainty Principle

A. Larrañaga[1,2]

[1]Universidad Nacional de Colombia. Observatorio Astronomico Nacional (OAN)

[2]Universidad Distrital Francisco Jose de Caldas. Facultad de Ingenieria. Proyecto Curricular de Ingenieria Electronica

Received *24 June 2009*

Abstract. In this paper, we study the thermodynamical properties of the $(2+1)$ dimensional black hole with a non-linear electrodynamics and without cosmological constant using the Generalized Uncertainty Principle (GUP). This approach shows that there is a maximum temperature for the black hole depending only on the electric charge and corresponding to the minimum radius of the event horizon, of the order of the Planck scale. Finally, we show that the heat capacity for this black hole has the expected behavior.

PACS number: 04.70.Dy; 04.20.-q; 11.10.Lm

1 Introduction

In $(2+1)$ dimensional gravity, the charged black hole (static BTZ solution) has an electric field that is proportional to the inverse of r, hence its potential is logarithmic. If the source of the Einstein equations is the stress-energy tensor of non-linear electrodynamics, which satisfies the weak energy conditions, one can find a solution with a Coulomb-like electric field (proportional to the inverse of r^2). This kind of solution was reported by Cataldo et. al. [1], and describes charged-AdS space when considering a negative cosmological constant.

The thermodynamical properties of black holes are associated with the presence of the event horizon. As shown recently [2], the three-dimensional black hole with nonlinear electrodynamics satisfies a differential first law with the usual form

$$dM = TdS + \Phi dQ, \tag{1}$$

where T is the Hawking temperature that can be expressed in terms of the surface gravity at the horizon κ by

$$T = \frac{\kappa}{2\pi}. \tag{2}$$

However, in recent years the uncertainty relation that includes gravity effects, known as the Generalized Uncertainty Principle (GUP), has shown interesting results in the context of black hole evaporation [3], extending the relation between temperature and mass to scales of the order of the Planck length, $l_p = 1.61 \times 10^{-33} cm$. This treatment has shown that l_p is the smallest length scale in the theory and it is related to the existence of an extreme mass (the Planck mass $m_p = 1.22 \times 10^{19} GeV$, which becomes the black hole remnant), that corresponds to the maximum possible temperature.

In this paper we investigate the thermodynamics of the three-dimensional black hole with a nonlinear electric field reported in [1], to show how the GUP can be used to calculate the Hawking temperature associated with the black hole. The $T(M)$ equation gives the usual relation for small masses but gets deformed when the mass becomes larger. We also show how there is a maximum temperature for the black hole that corresponds to a horizon with size in the Planck scale. Finally we calculate the heat capacity for this black hole, to show that it has the right physical behavior.

2 The 3-Dimensional Black Hole with Non-Linear Electrodynamics

The metric reported by Cataldo et. al. [1] is a solution of the $(2+1)$ dimensional Einstein's field equations with a negative cosmological constant $\Lambda < 0$,

$$G_{\mu\nu} + \Lambda g_{\mu\nu} = 8\pi G T_{\mu\nu}. \tag{3}$$

To obtain a Coulomb-like electric field, Cataldo *et al*. used a nonlinear electrodynamics. In the non-linear theory, the electromagnetic action I does not depend only on the invariant $F = \frac{1}{4} F_{\mu\nu} F^{\mu\nu}$, but it can be a generalization of it, for example

$$I \propto \int d^3x \sqrt{|g|} \left(F_{\mu\nu} F^{\mu\nu} \right)^p , \tag{4}$$

where p is some constant exponent. If the energy-momentum tensor is restricted to be traceless, the action becomes a function of $F^{3/4}$, and the static circularly symmetric solution obtained has the line element

$$ds^2 = -f(r)\, dt^2 + \frac{dr^2}{f(r)} + r^2 d\varphi^2, \tag{5}$$

where

$$f(r) = -M - \Lambda r^2 + \frac{4GQ^2}{3r}. \tag{6}$$

The electric field for this solution is

$$E(r) = \frac{Q}{r^2}, \tag{7}$$

which is the standard Coulomb field for a point charge. The metric depends on two parameters Q and M, that are identified as the electric charge and the mass, respectively. The horizons of this solution are defined by the condition

$$f(r) = 0 \tag{8}$$

or

$$-M - \Lambda r^2 + \frac{4GQ^2}{3r} = 0. \tag{9}$$

However, if we consider a zero cosmological constant, $\Lambda = 0$, the resulting black hole has interesting properties. The line element becomes

$$ds^2 = -\left(-M + \frac{4GQ^2}{3r}\right) dt^2 + \frac{dr^2}{\left(-M + \frac{4GQ^2}{3r}\right)} + r^2 d\varphi^2, \tag{10}$$

that shows how this spacetime is asymptotically flat. Note that this charged black hole has just one horizon, located at

$$r_H = \frac{4GQ^2}{3M}. \tag{11}$$

The Hawking temperature for this black hole is given by the usual definition (2), where the surface gravity can be calculated as

$$\kappa = \chi(x^\mu)\, a, \tag{12}$$

with a, the magnitude of the four-acceleration and χ, the red-shift factor. In order to calculate χ, we will consider a static observer, for whom the red-shift factor is just the proportionality factor between the timelike Killing vector K^μ and the four-velocity V^μ, i.e.

$$K^\mu = \chi V^\mu. \tag{13}$$

The metric (5) has the Killing vector

$$K^\mu = (1, 0, 0) \tag{14}$$

while the four-velocity is calculated as

$$V^\mu = \frac{dx^\mu}{d\tau} = \left(\frac{dt}{d\tau}, 0, 0\right). \tag{15}$$

This gives *

$$V^{\mu} = \left(f^{-1}\left(r\right), 0, 0\right) = \left(\frac{1}{\sqrt{-M + \frac{4GQ^2}{3r}}}, 0, 0\right), \tag{16}$$

and therefore, the red-shift factor is

$$\chi\left(r\right) = \sqrt{-M + \frac{4GQ^2}{3r}}. \tag{17}$$

On the other hand, the four-acceleration is given by

$$a^{\mu} = \frac{dV^{\mu}}{d\tau}, \tag{18}$$

that has components

$$a^0 = a^{\varphi} = 0 \tag{19}$$

$$a^r = -\frac{2}{3}\frac{GQ^2}{r^2}. \tag{20}$$

and therefore, the magnitude of the four-acceleration is

$$a = \sqrt{g_{\mu\nu}a^{\mu}a^{\nu}} = \frac{-\frac{2}{3}\frac{GQ^2}{r^2}}{\sqrt{-M + \frac{4GQ^2}{3r}}}. \tag{21}$$

Then, the surface gravity at the event horizon is given by the absolute value

$$\kappa = \left|-\frac{2}{3}\frac{GQ^2}{r^2}\right|_{r=r_H} \tag{22}$$

$$\kappa = \frac{2GQ^2}{3r_H^2} = \frac{3M^2}{8GQ^2} \tag{23}$$

and the Hawking temperature is,

$$T = \frac{GQ^2}{3\pi r_H^2} = \frac{3M^2}{16\pi GQ^2}. \tag{24}$$

3 Hawking Radiation and the Generalized Uncertainty Principle

Now we will consider the GUP and the Hawking radiation derived from it. We will show how the GUP will produce a deformation in the temperature-mass relation when considered close to the Planck lenght. As stated by Adler and Santiago [4] the GUP is given, in units with $c = 1$, by the relation

$$\Delta x \Delta p \gtrsim \hbar + G\hbar \left(\Delta p\right)^2. \tag{25}$$

where G is the gravitational constant and $\hbar$ is the Planck constant. Since the Planck lenght l_p can be written as

$$l_p^2 = G\hbar = \frac{\hbar^2}{m_p^2}, \tag{26}$$

where m_p is the mass of Planck, the GUP can be written as

$$\Delta x \Delta p \gtrsim 1 + l_p^2 \left(\Delta p\right)^2, \tag{27}$$

or

$$\Delta x \Delta p \gtrsim 1 + \frac{\left(\Delta p\right)^2}{m_p^2}, \tag{28}$$

in units with $\hbar = 1$. To apply this uncertainty principle to the black hole evaporation process consist in identifying the Δx with the event horizon radius r_H and the momentum Δp with the Hawking temperature up to a 2π factor [5]. Therefore, we can write the GUP as a quadratic equation for the temperature,

$$r_H 2\pi T = 1 + \frac{4\pi^2 T^2}{m_p^2} \tag{29}$$

$$T^2 - \frac{2Q^2}{3\pi M} T + \frac{m_p^2}{4\pi^2} = 0. \tag{30}$$

From which it follows that the temperature-mass relation is

$$T\left(M\right) = \frac{Q^2}{3\pi M} \left[1 + \sqrt{1 - \frac{9M^2 m_p^2}{4Q^4}}\right], \tag{31}$$

where we have chosen the plus sign of the root in order to obtain the black hole temperature (24) in the limit of small M. Note that the argument in the square root defines a maximum mass for the black hole,

$$M^{max} = \frac{2Q^2}{3m_p}. \tag{32}$$

This mass gives, using equation (24), the maximum temperature permited to the black hole,

$$T^{max} = \frac{Q^2}{12\pi}. \tag{33}$$

Equation (11) implies that the black hole with the maximum temperature has an event horizon with a minimum radius

$$r_H^{min} = \frac{2}{m_p} = 2l_p. \tag{34}$$

Therefore, we conclude that r_H can not be smaller than twice the Planck lenght. On the other hand, the heat capacity for the $\Lambda = 0$ black hole can be calculated using equation (31),

$$C_Q = \left(\frac{\partial M}{\partial T}\right)_Q \tag{35}$$

$$= -\frac{3\pi M}{Q^2}\left[\frac{1}{M}\left(1+\sqrt{1-\frac{9M^2m_p^2}{4Q^4}}\right)+\frac{9M^2m_p^2}{4Q^4}\left(1-\frac{9M^2m_p^2}{4Q^4}\right)^{-1}\right]^{-1} \tag{36}$$

Since the heat capacity of this black hole is always negative, $\frac{\partial T}{\partial M} < 0$, the behavior of T is the expected, as M increases, the temperature decreaces.

4 Conclusion

We have studied the thermodynamics of the $(2+1)$ dimensional black hole with non-linear electrodynamics and without cosmological constant using the Generalized Uncertainty Principle. This gives a maximum mass for the black hole, that corresponds to a maximum Hawking temperature depending only on the electric charge Q. The solution with the maximum temperature is obtained when the black hole has a size of the order of the Planck scale (minimum horizon).

Equation (31) gives the temperature-mass relation, and as is shown, it gives the standard Hawking temperature (24) for small masses, but gets deformed for masses close to M^{max}. Finally, the heat capacity of this black hole is negative, giving the right physical behavior.

This analysis confirms that Planck lenght seems to be the smallest lenght in nature, even in $(2+1)$ dimensions. In a forthcoming paper, consequences of the GUP in the $(2+1)$ dimensional black hole with non-lineal electrodynamics and non-zero cosmological constant will be discussed.

References

[1] M. Cataldo, N. Cruz, S. del Campo and A. Garcia (2000) *Phys. Lett. B* **484**, 154.
[2] A. Larranaga and L.A. Garcia. arXiv: 0811.3368 [gr-qc]
[3] I. Arraut, D. Batic and M. Nowakowski. arXiv: 0810.5156 [gr-qc]
[4] R.J. Adler and D.I. Santiago (1999) *Mod. Phys. Lett*. **A14** 1371.
[5] Y.S. Myung, Y.W. Kim, Y.J. Park (2007) *Phys. Lett*. **B645** 393-397.