Bulg. J. Phys. **37** (2010) 48–55

Fictitious Roots in the Dispersion Relation

S. Chandra, **G.M. Dak**, **M. Sharma**

School of Physics, Shri Mata Vaishno Devi University, Katra 182 320 (J&K), India

Received *9 January 2010*

Abstract. For application of magnetohydrodynamics (MHD) in solar physics as well as in plasma physics, dispersion relation plays key role. For a common set of equations, some authors have derived the dispersion relation as a sixth degree polynomial in ω, whereas the others have derived a fifth degree polynomial. Both groups are claiming that their dispersion relation is correct and consequently their results for the fast and slow mode waves are correct.

We have shown that for the same set of equations, one can have a fifth degree, sixth degree or even seventh degree polynomial, depending on the way used in solving the set of equations. For these polynomials however the five roots are found to be common and they are the actual roots giving the same results for the fast and slow mode waves. Other roots (one for the sixth degree polynomial and two for the seventh degree polynomial) are fictitious. It explicitly shows that the results for the fast and slow mode waves do not depend on the degree of the polynomial for the dispersion relation.

PACS number: 94.30.cq

1 Introduction

For application of magnetohydrodynamics (MHD) in solar physics as well as in plasma physics, the set of basic equations under investigation can be expressed as [1-5]

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \cdot \vec{v}) = 0, \tag{1}$$

$$\rho \frac{\partial \vec{v}}{\partial t} = -\nabla p + \frac{1}{4\pi}(\nabla \times \vec{B}) \times \vec{B} - \nabla \cdot \Pi, \tag{2}$$

$$\frac{\partial \vec{B}}{\partial t} = \nabla \times (\vec{v} \times \vec{B}), \tag{3}$$

$$\frac{\partial p}{\partial t} + \gamma p (\nabla \cdot \vec{v}) = (\gamma - 1)[\nabla \cdot \kappa \nabla T + Q_{\text{vis}} - Q_{\text{rad}}], \tag{4}$$

$$p = \frac{2\rho k_B T}{m_p}\,. \tag{5}$$

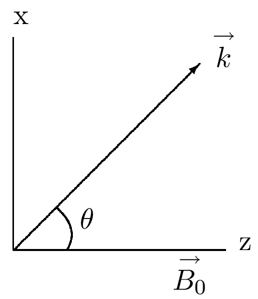

Figure 1. The applied magnetic field is along z-axis whereas the propagation vector $\vec{k}$ lies in the z–x plane.

These are, respectively, the equation of continuity, equation of momentum, induction equation, energy equation and the equation of state. Here, ρ, k_B, m_p, $\vec{v}$, p, $\vec{B}$, γ, T and Π are the total mass density, Boltzmann constant, proton mass, velocity, total pressure, magnetic field, ratio of two specific heats, the temperature and viscous stress tensor [6]. The quantities Q_{th}, Q_{vis} and Q_{rad} are [1-5].

$$Q_{\text{th}} = \kappa_{\|}\left(\frac{\partial T}{\partial z}\right)^2 T^{-1},$$
$$Q_{\text{vis}} = \frac{\eta_0}{3}(\nabla\cdot\vec{v})^2,$$
$$Q_{\text{rad}} = n_e n_H Q(T),$$

where $\kappa_{\|}$ represents the conductivity along the magnetic field and is expressed by $\kappa_{\|} \approx 10^{-6}T^{5/2}$. Q_{vis} is the volumetric heating rate due to viscosity; Q_{th} is the volumetric heating rate due to electron thermal conduction; Q_{rad} stands for the radiative loss rate per unit volume. More precisely, $Q_{\text{vis}} = -\Pi_{\alpha,\beta}\left(\frac{\partial v_\alpha}{\partial x_\beta}\right)$ as expressed by Braginskii [6]. In equation (3), the term $\eta\nabla^2\vec{B}$ is not accounted for, as the value of the magnetic Raynold number is quite large for ($R_m \approx 10^9$) the region considered. For small perturbations from the equilibrium, we have

$$\rho = \rho_0 + \rho_1, \quad \vec{v} = \vec{v}_1, \quad \vec{B} = \vec{B}_0 + \vec{B}_1,$$
$$p = p_0 + p_1, \quad T = T_0 + T_1, \quad \Pi = \Pi_0 + \Pi_1,$$

where the equilibrium part is denoted by the subscript '0' and the perturbation part by the subscript '1'. For the magnetic field taken along the z-axis, (*i.e.*, $\vec{B}_0 = B_0\hat{z}$) and the propagation vector $\vec{k} = k_\perp\hat{x} + k_\|\hat{z}$, as shown in Figure 1.

The equations (1)–(5) can be linearized in the following form:

$$\frac{\partial\rho_1}{\partial t} + \rho_0(\nabla\cdot\vec{v}_1) = 0, \tag{6}$$

$$\rho_0 \frac{\partial \vec{v}_1}{\partial t} = -\nabla p_1 + \frac{1}{4\pi}(\nabla \times \vec{B}_1) \times \vec{B}_0 - \nabla \cdot \Pi_0, \tag{7}$$

$$\frac{\partial \vec{B}_1}{\partial t} = \nabla \times (\vec{v_1} \times \vec{B_0}), \tag{8}$$

$$\frac{\partial p_1}{\partial t} + \gamma p_0(\nabla \cdot \vec{v}_1) + (\gamma - 1)\kappa_{\|} k_z^2 T_1 = 0, \tag{9}$$

$$\frac{p_1}{p_0} = \frac{\rho_1}{\rho_0} + \frac{T_1}{T_0}\,. \tag{10}$$

For the perturbations that are proportional to $\exp[i(\vec{k} \cdot \vec{r} - \omega t)]$, equations (6)–(10) reduce to the following equations:

$$\omega\rho_1 - \rho_0(k_x v_{1x} + k_z v_{1z}) = 0, \tag{11}$$

$$\omega\rho_0 v_{1x} - k_x p_1 - \frac{B_0}{4\pi}(k_x B_{1z} - k_z B_{1x}) + \frac{i\eta_0}{3}(k_x^2 v_{1x} - 2k_x k_z v_{1z}) = 0, \tag{12}$$

$$\omega\rho_0 v_{1y} + \frac{B_0}{4\pi}(k_z B_{1y}) = 0, \tag{13}$$

$$\omega\rho_0 v_{1z} - k_z p_1 + \frac{i\eta_0}{3}(4k_z^2 v_{1z} - 2k_x k_z v_{1x}) = 0, \tag{14}$$

$$\omega B_{1x} + k_z B_0 v_{1x} = 0, \tag{15}$$

$$\omega B_{1y} + k_z B_0 v_{1y} = 0, \tag{16}$$

$$\omega B_{1z} - k_x B_0 v_{1x} = 0, \tag{17}$$

$$i\omega p_1 - i\rho_0 c_s^2(k_x v_{1x} + k_z v_{1z}) - (\gamma - 1)\kappa_{\|} k_z^2 T_1 = 0, \tag{18}$$

$$\frac{p_1}{p_0} - \frac{\rho_1}{\rho_0} - \frac{T_1}{T_0} = 0. \tag{19}$$

Equations (13) and (16) for the variables v_{1y} and B_{1y} are decoupled from the rest and describe the Alfvén waves. The rest of the equations for p_1, ρ_1, T_1, B_{1x}, B_{1z}, v_{1x} and v_{1z} describe the damped magnetoacoustic waves. Now, on substituting B_{1x} and B_{1z} from equations (15) and (17) in equations (12) and (14), we get

$$\Big(\omega^2\rho_0 + \frac{i\omega\eta_0}{3}k_x^2 - v_A^2\rho_0 k^2\Big)v_{1x} - \frac{2i\omega\eta_0 k_x k_z}{3}v_{1z} - k_x\omega p_1 = 0 \tag{20}$$

and

$$\frac{2i\eta_0 k_x k_z}{3}\, v_{1x} - \Big(\omega\rho_0 + \frac{4i\eta_0}{3}k_z^2\Big)v_{1z} + k_z p_1 = 0. \tag{21}$$

When we eliminate ρ_1 and T_1 from equations (11), (18) and (19), we get

$$(c_0 p_0 k_x - i\rho_0 c_s^2 k_x\omega)v_{1x} + (c_0 p_0 k_z - i\rho_0 c_s^2 k_z\omega)v_{1z} - (c_0\omega - i\omega^2)p_1 = 0, \tag{22}$$

where $c_0 = (\gamma - 1)\kappa_{\|} k_z^2 T_0/p_0$; $c_s^2 = \gamma p_0/\rho_0$; and $v_A^2 = B_0^2/4\pi\rho_0$. Equations (20)–(22) now form the set of basic equations to be solved. Up to this stage, all the group agree. For deriving various degrees of polynomial for dispersion relation, they have used different paths for solving the set of equations (20)–(22).

2 Dispersion Relations

The dispersion relations obtained from equations depend on the procedure for solving this set of equations. For convenience, let us express equations (20)–(22) as

$$a_{11} v_{1x} + a_{12} v_{1z} + a_{13} p_1 = 0, \tag{23}$$
$$a_{21} v_{1x} + a_{22} v_{1z} + a_{23} p_1 = 0, \tag{24}$$
$$a_{31} v_{1x} + a_{32} v_{1z} + a_{33} p_1 = 0, \tag{25}$$

where the coefficients a_{ij} are

$$\begin{aligned}
a_{11} &= \left(\omega^2 \rho_0 + \frac{i\omega\eta_0}{3} k_x^2 - v_A^2 \rho_0 k^2\right); \\
a_{12} &= -\frac{2i\omega\eta_0}{3} k_x k_z; \\
a_{13} &= -k_x \omega \\
a_{21} &= \frac{2i\eta_0}{3} k_x k_z; \\
a_{22} &= -\omega\rho_0 - \frac{4i\eta_0}{3} k_z^2; \\
a_{23} &= k_z \\
a_{31} &= c_0 p_0 k_x - i\rho_0 c_s^2 k_x \omega; \\
a_{32} &= c_0 p_0 k_z - i\rho_0 c_s^2 k_z \omega; \\
a_{33} &= i\omega^2 - c_0 \omega
\end{aligned}$$

2.1 First Elimination of p_1

Let us first eliminate p_1 from equations (23)–(25). On eliminating p_1 from equations (23) and (24), we get

$$(a_{11}a_{23} - a_{21}a_{13}) v_{1x} + (a_{12}a_{23} - a_{22}a_{13}) v_{1z} = 0. \tag{26}$$

On eliminating p_1 from equations (23) and (25), we get

$$(a_{11}a_{33} - a_{31}a_{13}) v_{1x} + (a_{12}a_{33} - a_{32}a_{13}) v_{1z} = 0. \tag{27}$$

On eliminating p_1 from equations (24) and (25), we get

$$(a_{21}a_{33} - a_{31}a_{23})v_{1x} + (a_{22}a_{33} - a_{32}a_{23})v_{1z} = 0. \tag{28}$$

With the help of equations (26) and (27), we get a sixth degree polynomial where ω can be taken out and cancelled. Thus, we have

$$\omega^5 + iA\omega^4 - B\omega^3 - iC\omega^2 + D\omega + iE = 0, \tag{29}$$

where

$$\begin{aligned}
A &= c_0 + \frac{\eta_0}{3\rho_0}(k_x^2 + 4k_z^2),\\
B &= \frac{c_0\eta_0}{3\rho_0}(k_x^2 + 4k_z^2) + (c_s^2 + v_A^2)k^2,\\
C &= \frac{3\eta_0}{\rho_0}c_s^2k_x^2k_z^2 + \frac{c_0p_0k^2}{\rho_0} + v_A^2c_0k^2 + \frac{4\eta_0v_A^2k_z^2k^2}{3\rho_0},\\
D &= \frac{3c_0p_0\eta_0k_x^2k_z^2}{\rho_0^2} + \frac{4\eta_0c_0v_A^2k_z^2k^2}{3\rho_0} + v_A^2c_s^2k_z^2k^2,\\
E &= \frac{v_A^2c_0p_0k_z^2k^2}{\rho_0}.
\end{aligned}$$

This dispersion relation is the same as obtained by Kumar *et al.* [2] and Chandra & Kumthekar [3]. Now, with the help of equations (27) and (28), we get a seventh degree polynomial, where ω can be taken out and cancelled. Thus, we have

$$(\omega + ic_0)(\omega^5 + iA\omega^4 - B\omega^3 - iC\omega^2 + D\omega + iE) = 0. \tag{30}$$

This dispersion relation is the same as obtained by Dwivedi & Pandey [1] and Pandey & Dwivedi [4]. With the help of equations (26) and (28), we again get the dispersion relation (29).

2.2 First Elimination of v_{1x}

The polynomials obtained in this section have been discussed by Chandra *et al.* [7]. On eliminating v_{1x} from equations (23) and (24), we get

$$(a_{12}a_{21} - a_{22}a_{11})v_{1z} + (a_{13}a_{21} - a_{23}a_{11})p_1 = 0. \tag{31}$$

On eliminating v_{1x} from equations (23) and (25), we get

$$(a_{12}a_{31} - a_{32}a_{11})v_{1z} + (a_{13}a_{31} - a_{33}a_{11})p_1 = 0. \tag{32}$$

On eliminating v_{1x} from equations (24) and (25), we get

$$(a_{22}a_{31} - a_{32}a_{21})v_{1z} + (a_{23}a_{31} - a_{33}a_{21})p_1 = 0. \tag{33}$$

With the help of equations (31) and (32), we get

$$\left(\omega^2 + \frac{i\eta_0 k_x^2}{3\rho_0}\omega - v_A^2 k^2\right)(\omega^5 + iA\omega^4 - B\omega^3 - iC\omega^2 + D\omega + iE) = 0. \quad (34)$$

It is a seventh degree polynomial. With the help of equations (31) and (33), we get the dispersion relation (29). With the help of equations (32) and (33), we get

$$\left(\omega + \frac{ic_0}{\gamma}\right)(\omega^5 + iA\omega^4 - B\omega^3 - iC\omega^2 + D\omega + iE) = 0. \quad (35)$$

This is again a sixth degree polynomial. However, different from the equation (30).

2.3 First Elimination of v_{1z}

On eliminating v_{1z} from equations (23) and (24), we get

$$(a_{11}a_{22} - a_{21}a_{12})v_{1x} + (a_{13}a_{22} - a_{23}a_{12})p_1 = 0. \quad (36)$$

On eliminating v_{1z} from equations (23) and (25), we get

$$(a_{11}a_{32} - a_{31}a_{12})v_{1x} + (a_{13}a_{32} - a_{33}a_{12})p_1 = 0. \quad (37)$$

On eliminating v_{1z} from equations (24) and (25), we get

$$(a_{21}a_{32} - a_{31}a_{22})v_{1x} + (a_{23}a_{32} - a_{33}a_{22})p_1 = 0. \quad (38)$$

With the help of equations (36) and (37), we get a sixth degree polynomial where ω can be taken out and cancelled. Thus, we get equation (29). With the help of equations (36) and (38), we get

$$\left(\omega + \frac{4i\eta_0}{3\rho_0}k_z^2\right)(\omega^5 + iA\omega^4 - B\omega^3 - iC\omega^2 + D\omega + iE) = 0. \quad (39)$$

This is again a sixth degree polynomial. However, different from the equation (30) as well as (34). With the help of equations (37) and (38), we get

$$\left(\omega + \frac{ic_0}{\gamma}\right)(\omega^5 + iA\omega^4 - B\omega^3 - iC\omega^2 + D\omega + iE) = 0. \quad (40)$$

This is again a sixth degree polynomial. However, different from the equation (30) and (39).

3 Results and Discussion

We have shown that depending on the method used in solving the equations (23)–(25), we can get fifth degree polynomial, different sixth degree polynomials and different seventh degree polynomials. It is interesting to note that in all polynomials, the equation (29) is common. Hence, the five roots of all the polynomials are common. Other (sixth root for the sixth degree polynomial, and sixth and seventh roots for the seventh degree polynomial) depend on the method used in solving the equations (23)–(25). Hence, except the five roots, the others are inconsistent. Therefore, these other roots are fictitious. This statement can also be corroborated from the following fact. The equations (23)–(25) for a homogeneous set and therefore one should solve the determinant

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = 0.$$

Solution of this determinant obviously gives equation (29).

In the present investigation, we have derived the seventh degree polynomials as well. So far only the fifth and sixth degree polynomials have been derived. Kumar *et al.* [2] derived the fifth degree polynomial and claimed that since the degree of the polynomial derived by Dwivedi & Pandey [1] is sixth, hence their results are erroneous. As Dwivedi & Pandey [1] did not make calculations, it was not possible to compare the results. Chandra and Kumthekar [3] also found that the dispersion relation should be a fifth degree polynomial. Pandey & Dwivedi [4,5] advocated that the dispersion relation must be a sixth degree polynomial and therefore the work of Chandra and Kumthekar [3] also is wrong.

The roots of the fifth degree polynomial (29) are: $-i\alpha_{1r}$; $\pm\alpha_{2i} - i\alpha_{2r}$; $\pm\alpha_{3i} - i\alpha_{3r}$, satisfying the relations

$$\begin{aligned}
&\alpha_{1r} + 2\alpha_{2r} + 2\alpha_{3r} = A \\
&\alpha_{3i}^2 + \alpha_{3r}^2 + \alpha_{2i}^2 + \alpha_{2r}^2 + 2\alpha_{1r}\alpha_{2r} + 2\alpha_{1r}\alpha_{3r} + 4\alpha_{2r}\alpha_{3r} = B \\
&\alpha_{1r}\alpha_{2i}^2 + \alpha_{1r}\alpha_{2r}^2 + 4\alpha_{1r}\alpha_{2r}\alpha_{3r} + 2\alpha_{2i}^2\alpha_{3r} \\
&\qquad + 2\alpha_{2r}^2\alpha_{3r} + 2\alpha_{2r}\alpha_{3i}^2 + 2\alpha_{3r}^2\alpha_{2r} + \alpha_{3i}^2\alpha_{1r} + \alpha_{3r}^2\alpha_{1r} = C \\
&2\alpha_{2i}^2\alpha_{1r}\alpha_{3r} + \alpha_{2r}^2\alpha_{1r}\alpha_{3r} + \alpha_{2r}^2\alpha_{1r}\alpha_{3r} + \alpha_{2i}^2\alpha_{3i}^2 + \alpha_{2i}^2\alpha_{3r}^2 \\
&\qquad + \alpha_{2r}^2\alpha_{3i}^2 + \alpha_{2r}^2\alpha_{3r}^2 = D \\
&\alpha_{1r}\alpha_{2i}^2\alpha_{3i}^2 + \alpha_{1r}\alpha_{2r}^2\alpha_{3i}^2 + \alpha_{1r}\alpha_{2i}^2\alpha_{3r}^2 + \alpha_{1r}\alpha_{2r}^2\alpha_{3r}^2 = E
\end{aligned}$$

For the given value of parameters, $n_0 = 10^{10}$ cm^{-3}, $B_0 = 100$ G, $T_0 = 2 \times 10^6$ K, $\theta = 45°$, the values of α_{1r}, α_{2i}, α_{2r}, α_{3i} and α_{3r} as a function of the wave number k are given in Table 1. Finally, it can be concluded that there are

Table 1. Values of Parameters

$\log k$ cm^{-1}	α_{1r} s^{-1}	α_{2i} s^{-1}	α_{2r} s^{-1}	α_{3i} s^{-1}	α_{3r} s^{-1}
– 10.0	4.099×10^{-6}	1.654×10^{-3}	1.472×10^{-6}	2.188×10^{-2}	3.548×10^{-8}
– 9.8	1.030×10^{-5}	2.621×10^{-3}	3.697×10^{-6}	3.467×10^{-2}	8.912×10^{-8}
– 9.6	2.586×10^{-5}	4.154×10^{-3}	9.286×10^{-6}	5.495×10^{-2}	2.239×10^{-7}
– 9.4	6.497×10^{-5}	6.583×10^{-3}	2.332×10^{-5}	8.709×10^{-2}	5.623×10^{-7}
– 9.2	1.632×10^{-4}	1.043×10^{-2}	5.858×10^{-5}	1.380×10^{-1}	1.412×10^{-6}
– 9.0	4.101×10^{-4}	1.653×10^{-2}	1.471×10^{-4}	2.188×10^{-1}	3.548×10^{-6}
– 8.8	1.031×10^{-3}	2.619×10^{-2}	3.692×10^{-4}	3.467×10^{-1}	8.912×10^{-6}
– 8.6	2.593×10^{-3}	4.147×10^{-2}	9.255×10^{-4}	5.495×10^{-1}	2.238×10^{-5}
– 8.4	6.536×10^{-3}	6.559×10^{-2}	2.313×10^{-3}	8.709×10^{-1}	5.623×10^{-5}
– 8.2	1.657×10^{-2}	1.034×10^{-1}	5.733×10^{-3}	1.380	1.412×10^{-4}
– 8.0	4.260×10^{-2}	1.616×10^{-1}	1.391×10^{-2}	2.188	3.547×10^{-4}
– 7.8	1.132×10^{-1}	2.479×10^{-1}	3.183×10^{-2}	3.467	8.907×10^{-4}
– 7.6	3.209×10^{-1}	3.677×10^{-1}	6.173×10^{-2}	5.495	2.236×10^{-3}
– 7.4	9.336×10^{-1}	5.415×10^{-1}	9.133×10^{-2}	8.709	5.604×10^{-3}
– 7.2	2.554	8.246×10^{-1}	1.248×10^{-1}	13.80	1.401×10^{-2}
– 7.0	6.655	1.283	1.948×10^{-1}	21.87	3.480×10^{-2}

fictitious roots; one in the sixth degree polynomial and two in the seventh degree polynomial. Since the complex roots for all the polynomials are the same, the results for the fast and slow magnetoacoustic waves do not depend on the degree of the polynomial.

Acknowledgments

We are thankful to the DST, New Delhi; the ISRO, Bangalore and the UGC, New Delhi for financial support in the form of research projects. A part of this work was done during the visit to the IUCAA, Pune. Financial support from the IUCAA, Pune is thankfully acknowledged.

References

[1] B.N. Dwivedi and V.S. Pandey (2003) *So;. Phys*. **216** 59.
[2] N. Kumar, P. Kumar, and S. Singh (2006) *Astron. Astrophys*. **453**, 1067.
[3] S. Chandra and B.K. Kumthekar (2007) arXiv:0706.1121.
[4] V.S. Pandey and B.N. Dwivedi (2007) *Bull. Astron. Soc. India* **35** 465.
[5] V.S. Pandey and B.N. Dwivedi (2007) arXiv:astro-ph/07090040.
[6] S.I. Braginskii (1965) *Rev. Plasma Phys*. **1** 205.
[7] S. Chandra, G.M. Dak, and M. Sharma (2009) *Bulg. J. Phys*. **36** 294.