

# Representations and $q$ -Deformation of Anti de Sitter Symmetry\*

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**Abstract.** We give a brief overview of the representation theory of anti de Sitter symmetry, its invariant differential operators, and character formulae. We also present some interesting facts for the corresponding quantum group deformation.

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## 1 Introduction

Anti de-Sitter symmetry plays an important role in modern mathematics and physics. In mathematics, the  $n$ -dimensional anti de Sitter space  $AdS_n$ , is a maximally symmetric Lorentzian manifold with a constant negative scalar curvature. It is described as submerged in  $(n + 1)$ -dimensional Euclidean space by the formula

$$\sum_{i=1}^{n-1} x_i^2 - x_0^2 - x_n^2 = R^2. \quad (1)$$

In general relativity, anti de Sitter space is a maximally symmetric, vacuum solution of Einstein's field equation with a negative (attractive) cosmological constant.

The same space, though in momentum space, was the founding stepping stone of the quantum field theory with fundamental length developed by Kadyshevsky and Mateev for more than 40 years.

In group theory, the isometry group of  $AdS_n$  is the pseudo-orthogonal group  $SO(n-1,2)$ . This group is also the conformal group of  $(n - 1)$ -dimensional Minkowski space-time.

The latter properties were the group-theoretical foundation of the so-called AdS/CFT correspondence between objects (*e.g.*, strings) in the  $n$ -dimensional

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\*This work is dedicated to the memory of Academician Matey Mateev.

$AdS_n$ , called the bulk, and the  $(n - 1)$ -dimensional Minkowski space-time as a boundary of the bulk.

We discuss the simplest characteristic case  $n = 4$  in the algebraic framework, namely, we discuss mostly the Lie algebra  $so(3, 2)$ . This Lie algebra was the first that was called anti de Sitter algebra.

## 2 Preliminaries

The anti de Sitter algebra  $\mathcal{G} = so(3, 2)$  is ten-dimensional. It is a split form of its complexification  $\mathcal{G}^{\mathbb{C}} = so(5, \mathbb{C})$  and we can use the same basis for both (though over different fields,  $\mathbb{R}, \mathbb{C}$ , resp.).

For  $\mathcal{G}^{\mathbb{C}}$  we use the standard (triangular) decomposition

$$\mathcal{G}^{\mathbb{C}} = \mathcal{G}_- \oplus \mathcal{H} \oplus \mathcal{G}_+, \quad (2)$$

where  $\mathcal{H}$  is the diagonal two-dimensional Cartan subalgebra with generators denoted by  $H_1, H_2$ ,  $\mathcal{G}_+, \mathcal{G}_-$  are four-dimensional subalgebras of raising, lowering, generators  $X_i^+, X_i^-$ , resp.,  $i = 1, 2, 3, 4$ . For simplicity we give the following basis in terms of  $4 \times 4$  matrices:

$$\begin{aligned} H_1 &= \begin{pmatrix} \sigma_3 & 0 \\ 0 & \sigma_3 \end{pmatrix}, & H_2 &= \begin{pmatrix} e_2 & 0 \\ 0 & -e_1 \end{pmatrix}, \\ X_1^+ &= \begin{pmatrix} \sigma_+ & 0 \\ 0 & -\sigma_+ \end{pmatrix}, & X_1^- &= \begin{pmatrix} \sigma_- & 0 \\ 0 & -\sigma_- \end{pmatrix}, \\ X_2^+ &= \begin{pmatrix} 0 & \sigma_- \\ 0 & 0 \end{pmatrix}, & X_2^- &= \begin{pmatrix} 0 & 0 \\ \sigma_+ & 0 \end{pmatrix}, \\ X_3^+ &= \begin{pmatrix} 0 & 1_2 \\ 0 & 0 \end{pmatrix}, & X_3^- &= \begin{pmatrix} 0 & 0 \\ 1_2 & 0 \end{pmatrix}, \\ X_4^+ &= \begin{pmatrix} 0 & \sigma_+ \\ 0 & 0 \end{pmatrix}, & X_4^- &= \begin{pmatrix} 0 & 0 \\ \sigma_- & 0 \end{pmatrix}, \\ e_1 &\equiv \frac{1}{2}(1 + \sigma_3) = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, & e_2 &\equiv \frac{1}{2}(1 - \sigma_3) = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \\ \sigma_+ &\equiv \frac{1}{2}(\sigma_1 + i\sigma_2) = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, & \sigma_- &\equiv {}^t\sigma_+ = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \end{aligned} \quad (3)$$

where  $\sigma_i$  are the standard  $2 \times 2$  Pauli matrices.

Using the same basis over  $\mathbb{R}$  the anti de Sitter algebra  $\mathcal{G}$  has the following (Bruhat) decomposition:

$$\mathcal{G} = \mathcal{N}_- \oplus \mathcal{M} \oplus \mathcal{A} \oplus \mathcal{N}_+ \quad (4)$$

in which the four subalgebras have physical meaning related to the fact that  $\mathcal{G}$  is also the conformal algebra of the three-dimensional Minkowski space-time

$M^3$ . Namely, the subalgebra  $\mathcal{M} \cong so(2, 1)$  is the Lorentz algebra of  $M^3$ , the subalgebras  $\mathcal{N}_\pm$  (with basis  $X_k^\pm, k = 2, 3, 4$ ) are abelian and represent the translations of  $M^3$  and special conformal transformations of  $M^3$ , and the algebra  $\mathcal{A}$  (spanned by  $H_3 \equiv H_1 + 2H_2$ ) represents the dilatations of  $M^3$ .

### 3 Representations and Invariant Operators

We work with two kind of representations. Both are characterized by two quantum numbers  $E_0, s_0$  called energy and spin,  $s_0 = 0, \frac{1}{2}, \dots$

The first kind are the so-called elementary representations (ERs), denoted by  $C^\Lambda$ , where  $\Lambda$  is a weight  $\Lambda \in \mathcal{H}^*$  depending on  $E_0, s_0$ . These can be realized as complex-valued  $C^\infty$  functions on  $G^{\mathbb{C}}/B$ , where  $G = SO_0(3, 2)$ ,  $B = \exp(\mathcal{H}) \exp(\mathcal{G}_+)$  is a Borel subgroup of  $G^{\mathbb{C}}$ . Since  $G^{\mathbb{C}}/B$  is a completion of  $G_- = \exp(\mathcal{G}_-)$ , we shall use the four local coordinates of  $G_-$  denoted by:  $z, u, v, x$ . The functions of  $C^\Lambda$ , denoted  $\hat{\varphi}(z, u, v, x)$ , are polynomials in the variable  $z$  of degree  $2s_0$  and smooth functions in the other three variables.

The other kind of representations are highest weight modules, in particular, Verma modules. Verma modules give explicit realization of the so-called positive energy UIRs  $D(E_0, s_0)$  of  $\mathcal{G}$  given as follows [1–4]:

$$\begin{aligned} D(E_0, s_0) &= D(1/2, 0), & D(E_0, s_0) &= D(1, 1/2), \\ D(E_0 > 1/2 + s_0, s_0 = 0, 1/2), & & & \\ D(E_0 \geq s_0 + 1, s_0 \geq 1). & & & \end{aligned} \tag{5}$$

The first two are the singleton representations discovered by Dirac, and called later ‘Rac’, ‘Di’, resp. The last ones for  $E_0 = s_0 + 1$  correspond to the spin- $s_0$  massless representations. Pictorially, the positive energy UIRs are given in Figures 1 and 2

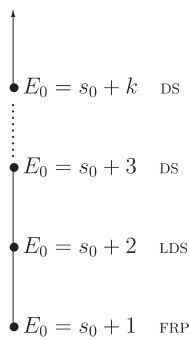


Figure 1.  $s_0 = 1, \frac{3}{2}, \dots$

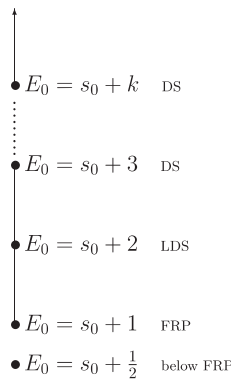


Figure 2.  $s_0 = 0, \frac{1}{2}$ , Rac, Di

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In the figures black dots denote irreducible factors of reducible  $C^\Lambda$  and  $V^\Lambda$ , DS means Discrete Series of unitary representations, LDS means Limit of Discrete Series representations, FRP means First Reduction Point – the end point of the continuous series of positive energy representations.

The other property of Verma modules that we use, is the fact that it is easy to find when they are reducible [5]. In our situation this happens every time when at least one of the following four numbers is a positive integer:

$$\begin{aligned} m_1 &= \Lambda(H_1) + 1 = 2s_0 + 1, \\ m_2 &= \Lambda(H_2) + 1 = 1 - E_0 - s_0, \\ m_3 &= m_1 + 2m_2, \\ m_4 &= m_1 + m_2. \end{aligned} \tag{6}$$

An important application of the above reducibility is that when the Verma module  $V^\Lambda$  is reducible, then also the ER  $C^\Lambda$  is reducible. Furthermore, in these cases there exist invariant differential operators between the ERs, and these operators are determined in a straightforward way from special objects in the corresponding Verma modules, called singular vectors [6].

In our case all singular vectors and the corresponding invariant differential operators and equations are known explicitly [7], but because of lack of space we shall show only some operators that are related to the positive energy cases.

For instance, since  $m_1$  is always a positive integer, there is always the following invariant operator:  $(\partial_z)^{m_1}$ , and in order to obtain irreducibility w.r.t. to the variable  $z$ , (recall that our functions  $\hat{\varphi}$  are polynomials in  $z$  of degree  $2s_0 = m_1 - 1$ ), there is always the invariant equation

$$(\partial_z)^{m_1} \hat{\varphi}(z, u, v, x) = 0. \tag{7}$$

**Rac:**  $E_0 = \frac{1}{2}, s_0 = 0$ . There are two equations

$$\begin{aligned} \partial_z \hat{\varphi} &= 0, & m_1 &= 1, \\ (\partial_x^2 - 4\partial_u \partial_v) \hat{\varphi} &= (\partial_0^2 - \partial_1^2 - \partial_2^2) \hat{\varphi} = \square \hat{\varphi} = \hat{\varphi}', & m_3 &= 2, \end{aligned} \tag{8}$$

where we have introduced new variables  $y_0 = x, y_1 = \frac{1}{2}(u + v), y_2 = i\frac{1}{2}(u - v)$ , in terms of which we get the d'Alembert operator  $\square$  in  $M^3$ .

The functions  $\hat{\varphi}'$  belong to the target space  $C^{\Lambda'}$ , also unitary of scalar functions (since  $E'_0 = \frac{3}{2}, s'_0 = 0$ ), but fulfilling only the first equation in (8).

**Di:**  $E_0 = 1, s_0 = \frac{1}{2}$ . The two equations are

$$\begin{aligned} \partial_z^2 \hat{\varphi} &= 0, & m_1 &= 2; \\ \left\{ \frac{1}{2}(\partial_x - 2z\partial_u) + (\partial_v - z\partial_x + z^2\partial_u) \partial_z \right\} \hat{\varphi} &= \hat{\varphi}', & m_3 &= 1. \end{aligned} \tag{9}$$

**Massless representations:**  $E_0 = s_0 + 1, s_0 = 1, \frac{3}{2}, \dots$ . The equations are

$$\begin{aligned} \partial_z^{p+1} \hat{\varphi} &= 0, \quad p = 2s_0 = 2, 3, \dots, \\ \{ p(p-1) \partial_u + (p-1)(\partial_x - 2z\partial_u) \partial_z + \\ &+ (\partial_v - z\partial_x + z^2\partial_u) \partial_z^2 \} \hat{\varphi} = \hat{\varphi}', \quad m_4 = 1. \end{aligned} \quad (10)$$

It is useful to write out (10) in components using:  $\hat{\varphi} = \sum_{j=0}^p z^j \hat{\varphi}_j, \hat{\varphi}' = \sum_{j=0}^{p-2} z^j \hat{\varphi}'_j$ . The resulting equations are

$$\begin{aligned} (p-j)(p-j-1) \partial_u \hat{\varphi}_j + (j+1)(p-j-1) \partial_x \hat{\varphi}_{j+1} + \\ + (j+1)(j+2) \partial_v \hat{\varphi}_{j+2} = \hat{\varphi}'_j, \quad j = 0, 1, \dots, p-2. \end{aligned}$$

If we restrict to the kernel then the last equations may be rewritten as equations for  $p-1$  independent conserved currents  $J^{p,j}$  in  $M^3$ :

$$\partial_0 J_0^{p,j} - \partial_1 J_1^{p,j} - \partial_2 J_2^{p,j} = 0, \quad j = 0, \dots, p-2, \quad (11)$$

where the components are given as follows:

$$\begin{aligned} J_0^{p,j} &= -(j+1)(p-j-1) \hat{\varphi}_{j+1}, \\ J_1^{p,j} &= \frac{1}{2} \{ (j+1)(j+2) \hat{\varphi}_{j+2} + (p-j)(p-j-1) \hat{\varphi}_j \}, \\ J_2^{p,j} &= \frac{i}{2} \{ (j+1)(j+2) \hat{\varphi}_{j+2} - (p-j)(p-j-1) \hat{\varphi}_j \}. \end{aligned} \quad (12)$$

#### 4 Character Formulae

Let  $\mathcal{G}$  be any simple Lie algebra. Let  $\Gamma_+$  be the set of all integral dominant elements of  $\mathcal{H}^*$ , i.e.,  $\lambda \in \mathcal{H}^*$ , such that  $\lambda(H_i) \in \mathbb{Z}$ , (resp.  $\mathbb{Z}_+$ ), for all  $H_i$  of the basis of  $\mathcal{H}$ . We recall that for each invariant subspace  $V$  of a Verma module  $V^\Lambda$  we have the following decomposition

$$V = \bigoplus_{\mu \in \Gamma_+} V_\mu, \quad V_\mu = \{ u \in V \mid H_k u = (\Lambda + \mu)(H_k) u, \forall H_k \}.$$

The character of  $V$  is defined by [8]

$$ch V = \sum_{\mu \in \Gamma_+} (\dim V_\mu) e(\Lambda + \mu)$$

where the formal exponents  $e(\mu)$  have the properties  $e(0) = 1, e(\mu)e(\nu) = e(\mu + \nu)$ . Furthermore for the character of the Verma module we have [8]

$$ch V^\Lambda = e(\Lambda) \prod_{\alpha \in \Delta^+} (1 - e(\alpha))^{-1}, \quad (13)$$

where  $\Delta^+ \subset \mathcal{H}^*$  is the positive root system of  $\mathcal{G}$ .

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The Weyl character formula for the finite-dimensional irreducible representations has the form [8]:

$$ch L_\Lambda = \sum_{w \in W} (-1)^{\ell(w)} ch V^{w \cdot \Lambda}, \quad (14)$$

where  $W$  is the Weyl group of  $\mathcal{G}$  and we use the fact that  $W$  acts on  $\Lambda$ .

Let  $\mathcal{G} = so(3, 2)$ . The positive root system  $\Delta^+$  has four roots:  $\alpha_i, i = 1, 2, 3, 4$ , so that  $\alpha_3 = \alpha_1 + \alpha_2, \alpha_4 = 2\alpha_1 + \alpha_2$ . Denote  $t_i \equiv e(\alpha_i), i = 1, 2$ , then  $e(\alpha_3) = t_1 t_2, e(\alpha_4) = (t_1)^2 t_2$ . Then (13) and (14) can be rewritten, respectively, as

$$\begin{aligned} ch V^\Lambda &= \frac{e(\Lambda)}{(1-t_1)(1-t_2)(1-t_1 t_2)(1-(t_1)^2 t_2)}, \\ ch L_\Lambda &= ch V^\Lambda (1 - t_1^{m_1} - t_2^{m_2} + t_1^{m_1} t_2^{m_1+m_2} + t_1^{m_1+2m_2} t_2^{m_2} \\ &\quad - t_1^{2(m_1+m_2)} t_2^{m_1+m_2} - (t_1 t_2)^{m_1+2m_2} + t_1^{2(m_1+m_2)} t_2^{m_1+2m_2}). \end{aligned} \quad (15)$$

In the last formula we have used the fact that the Weyl group for  $\mathcal{G}$  has eight elements given explicitly in terms of the generating elements  $w_1, w_2$  as follows:

$$\begin{aligned} W &= \{1, w_1, w_2, w_1 w_2, w_2 w_1, w_1 w_2 w_1, w_2 w_1 w_2, \\ &\quad w_1 w_2 w_1 w_2 = w_2 w_1 w_2 w_1\}. \end{aligned} \quad (16)$$

The character formulae for the infinite-dimensional irreducible highest weight representations over  $so(3, 2)$  involve less terms than in (15) since the maximal invariant submodules  $I^\Lambda$  of  $V^\Lambda$  are smaller. These character formulae can be given in terms of reduced Weyl groups [6]. This means that the character formulae for the infinite-dimensional irreducible highest weight representations over  $so(3, 2)$  will look like the Weyl character formula (14), however with  $W$  replaced by certain subgroups of  $W$ , called reduced Weyl groups.

For the singletons the reduced Weyl group  $W_R = W^s$  is given by

$$W^s = \{1, w_1, w_2 w_1 w_2, w_1 w_2 w_1 w_2\}. \quad (17)$$

Then the character formula for the singletons is

$$\begin{aligned} ch L^s &= ch V^\Lambda \sum_{w \in W^s} (-1)^{\ell(w)} e(w \cdot \Lambda - \Lambda) \\ &= ch V^\Lambda (1 - t_1^{m_1} - (t_1 t_2)^{m_1+2m_2} + t_1^{2(m_1+m_2)} t_2^{m_1+2m_2}). \end{aligned} \quad (18)$$

More explicitly for the Rac we have  $m_1 = 1$ ,  $m_2 = 1/2$ , and we obtain

$$\begin{aligned}
 ch L_{\text{Rac}} &= ch V^\Lambda (1 - t_1 - (t_1 t_2)^2 + t_1^3 t_2^2) \\
 &= \frac{e(\Lambda) (1 + t_1 t_2)}{(1 - t_2)(1 - (t_1)^2 t_2)} \\
 &= e(\Lambda) \sum_{n=0}^{\infty} (t_1 t_2)^n \sum_{p=-n}^n t_1^p \\
 &= e(\Lambda) \sum_{n=0}^{\infty} \sum_{p=-n}^n (t_1 t_2)^{n-|p|} t'^{|p|}, \tag{19}
 \end{aligned}$$

where

$$t' = \begin{cases} t_1^2 t_2 = e(\alpha_4) & \text{for } p > 0, \\ t_2 = e(\alpha_2) & \text{for } p < 0. \end{cases}$$

Analogously for the Di we have  $m_1 = 2$ ,  $m_2 = -1/2$  and we obtain

$$\begin{aligned}
 ch L_{\text{Di}} &= ch V^\Lambda (1 - t_1^2 - t_1 t_2 + t_1^3 t_2) \\
 &= \frac{e(\Lambda) (1 + t_1)}{(1 - t_2)(1 - (t_1)^2 t_2)} \\
 &= e(\Lambda) \sum_{n=0}^{\infty} (t_1 t_2)^n \sum_{p=-n}^{n+1} t_1^p \\
 &= e(\Lambda) \sum_{n=0}^{\infty} t_2^n \sum_{r=0}^n (t_1^{2r} + t_1^{2r+1}). \tag{20}
 \end{aligned}$$

Clearly, the terms in (19) have different weights, *i.e.*, each weight is represented only once. The same is true for (20). That is why these representations are called singletons. These formulae were known, but this derivation is original.

Finally we consider the massless representations with  $E_0 = s_0 + 1$ ,  $s_0 \geq 1$ . Let us define  $W' \equiv \{1, w_1, w_1 w_2 w_1, w_2 w_1\}$ . Note that the set  $W'$  is not a subgroup of  $W$ , but nevertheless, the character formula is valid

$$\begin{aligned}
 ch L_{\text{massless}} &= ch V^\Lambda \sum_{w \in W'} (-1)^{\ell(w)} e(w \cdot \Lambda - \Lambda) \\
 &= ch V^\Lambda (1 - t_1^{m_1} - t_1^2 t_2 + t_1^{m_1} t_2), \tag{21}
 \end{aligned}$$

$$m_1 = 1 + 2s_0 \geq 3.$$

### 5 Quantum Group Case

The quantum group associated to a Lie algebra  $\mathcal{G}$  is given by a deformation  $U_q(\mathcal{G})$  of the universal enveloping algebra  $U(\mathcal{G})$  depending on a parameter  $q$

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(there are also multiparameter deformations!). Explicitly, the deformation occurs by replacing some commutators by their  $q$ -analogs, *e.g.*, in our case

$$[X^+, X^-] = H \tag{22}$$

is replaced by

$$[X^+, X^-] = [H]_q \equiv \frac{q^H - q^{-H}}{q - q^{-1}}. \tag{23}$$

Matters are arranged so that the undeformed case is obtained from the deformed by setting  $q = 1$ . Indeed, setting  $q = e^h$  and taking the limit  $h \rightarrow 0$ , we see that (23) passes to (22).

The quantum group deformation  $U_q(\mathcal{G}^C)$  is given explicitly by the following commutation relations ( $j, k = 1, 2$ ):

$$\begin{aligned} [H_j, H_k] &= 0, \quad [H_j, X_k^\pm] = \pm a_{jk} X_k^\pm, \\ [X_j^+, X_k^-] &= d_{jk} [H_j]_{q_j} = d_{jk} \frac{q_j^{H_j/2} - q_j^{-H_j/2}}{q_j^{1/2} - q_j^{-1/2}}, \\ q_1 &= q, \quad q_2 = q^2, \\ [x]_q &\equiv (q^{x/2} - q^{-x/2}) / (q^{1/2} - q^{-1/2}), \end{aligned} \tag{24}$$

$(a_{jk})$  is the Cartan matrix of  $\mathcal{G}^C$

$$(a_{jk}) = \begin{pmatrix} 2 & -2 \\ -1 & 2 \end{pmatrix} \tag{25}$$

The commutation relations for  $X_j^\pm, j = 3, 4$ , follow from

$$X_3^\pm = \pm [X_1^\pm, X_2^\pm], \quad X_4^\pm = \pm \frac{1}{2} [X_1^\pm, X_3^\pm]. \tag{26}$$

Generically, the main features of the representations of the quantum group deformation are the same as for the undeformed case, the only difference being that in matrix elements numbers, *e.g.*,  $n$ , are replaced by  $q$ -numbers  $[n]_q$ , defined as above.

There is one interesting situation, where things differ drastically. First, we restrict the parameter  $q$  on the unit circle, namely, we set  $q = e^{i\tau}, \tau \in \mathbb{R}, (|q| = 1)$ . Then the quantum number  $[n]_q$  becomes

$$[n]_q = \frac{q^{n/2} - q^{-n/2}}{q^{1/2} - q^{-1/2}} = \frac{\sin(\tau n/2)}{\sin(\tau/2)}, \tag{27}$$

changing to a more refined definition. The interesting situation is when  $q$  is a root of unity:  $q^N = 1, q = e^{2\pi i/N}$ , where  $N = 2, 3, \dots$ . Then the quantum number becomes

$$[n]_q = \frac{\sin(\pi n/N)}{\sin(\pi/N)}. \tag{28}$$



Obviously,

$$[N]_q = 0.$$

This means that many matrix elements become zero. Furthermore, elements in the enveloping algebra like  $(X^\pm)^N$  become central, and in fact *all representations become finite-dimensional*, including the unitary ones!

For instance, the singleton representations become finite-dimensional and we have found the formula for their dimensions [9]. Our result is

$$\dim \text{Rac}_N = \begin{cases} \frac{N^2 + 1}{2}, & \text{for } N \text{ odd} \\ \frac{N^2}{2}, & \text{for } N \text{ even} \end{cases} \quad (29)$$

$$\dim \text{Di}_N = \begin{cases} \frac{N^2 - 1}{2}, & \text{for } N \text{ odd} \\ \frac{N^2}{2}, & \text{for } N \text{ even} \end{cases} \quad (30)$$

One may ask whether these finite-dimensional representations coincide with some classical finite-dimensional representations of the anti de Sitter algebra  $so(3, 2)$ . The answer is that with two exceptions there are no coincidences. The exceptions happen for the third root of unity,  $N = 3$ , then

$$\dim \text{Rac}_{N=3} = 5, \quad (31)$$

$$\dim \text{Di}_{N=3} = 4, \quad (32)$$

and these are the dimensions of the two fundamental irreps of  $so(3, 2)$ .

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