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Modeling Soliton Interactions of the Perturbed Vector Nonlinear Schrödinger Equation

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Abstract. The interactions of N solitons of the vector NLS eq. in adiabatic approximation is modeled by a generalized complex Toda chain (GCTC). A comparative analysis with the scalar N-soliton interactions is given. Additional constraints that have to be imposed on the polarization vectors which make the adiabatic approximation applicable are formulated. The effects of two types of external potentials are discussed.

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1 Introduction

The soliton interactions are an important area of today nonlinear physics and especially in nonlinear optics, see [1, 2, 5, 8, 16–18, 20–23].

The idea of the adiabatic approximation to the soliton interactions [15] led to effective modeling of the N-soliton trains of the perturbed NLS equation

$$iu_t + \frac{1}{2}u_{xx} + |u|^2 u(x,t) = iR[u]. \tag{1}$$

By N-soliton train we mean a solution of the NLS eq. with initial condition

$$\begin{aligned} u(x,t=0) &= \sum_{k=1}^{N} \vec{u}_k(x,t=0), & u_k(x,t) &= \frac{2\nu_k e^{i\phi_k}}{\cosh(z_k)}, \\ z_k &= 2\nu_k(x-\xi_k(t)), & \xi_k(t) &= 2\mu_k t + \xi_{k,0}, \\ \phi_k &= \frac{\mu_k}{\nu_k} z_k + \delta_k(t), & \delta_k(t) &= 2(\mu_k^2+\nu_k^2)t + \delta_{k,0}. \end{aligned} \tag{2}$$

The adiabatic approximation holds true if the soliton parameters satisfy [15]

$$|\nu_k - \nu_0| \ll \nu_0, \quad |\mu_k - \mu_0| \ll \mu_0, \quad |\nu_k - \nu_0||\xi_{k+1,0} - \xi_{k,0}| \gg 1, \tag{3}$$

where $\nu_0 = \frac{1}{N}\sum_{k=1}^{N}\nu_k$, and $\mu_0 = \frac{1}{N}\sum_{k=1}^{N}\mu_k$ are the average amplitude and velocity respectively. In fact we have two different scales

$$|\nu_k - \nu_0| \simeq \varepsilon_0^{1/2}, \qquad |\mu_k - \mu_0| \simeq \varepsilon_0^{1/2}, \qquad |\xi_{k+1,0} - \xi_{k,0}| \simeq \varepsilon_0^{-1}.$$

In this approximation the dynamics of the N-soliton train is described by a dynamical system for the $4N$ soliton parameters. In what follows we will consider perturbation by external potentials:

$$iR[u] = (V_2x^2 + V_1x + V_0 + A\cos(\Omega x + \Omega_0))u(x,t), \tag{4}$$

where $V_2 > 0$.

The corresponding model is known as the perturbed CTC model which can be written down in the form [4, 10, 11, 14, 20, 21]

$$\begin{aligned} \frac{d\lambda_k}{dt} &= -4\nu_0\left(e^{Q_{k+1}-Q_k} - e^{Q_k-Q_{k-1}}\right) + M_k + iN_k, \\ \frac{dQ_k}{dt} &= -4\nu_0\lambda_k + 2i(\mu_0 + i\nu_0)\Xi_k - iX_k, \end{aligned} \tag{5}$$

where $\lambda_k = \mu_k + i\nu_k$ and $X_k = 2\mu_k\Xi_k + D_k$ and

$$\begin{aligned} Q_k &= -2\nu_0\xi_k + k\ln 4\nu_0^2 - i(\delta_k + \delta_0 + k\pi - 2\mu_0\xi_k), \\ \nu_0 &= \frac{1}{N}\sum_{s=1}^{N}\nu_s, \qquad \mu_0 = \frac{1}{N}\sum_{s=1}^{N}\mu_s, \qquad \delta_0 = \frac{1}{N}\sum_{s=1}^{N}\delta_s. \end{aligned} \tag{6}$$

For the class of perturbation (4) we have [7, 8]

$$\begin{aligned} &N_k = 0, \quad M_k = -V_2\xi_k - \frac{V_1}{2} + \frac{\pi A\Omega^2}{8\nu_k\sinh Z_k}\sin(\Omega\xi_k + \Omega_0), \quad \Xi_k = 0, \\ &D_k = V_2\left(\frac{\pi^2}{48\nu_k^2} - \xi_k^2\right) - V_1\xi_k - V_0 - \frac{\pi^2 A\Omega^2}{16\nu_k^2}\frac{\cosh Z_k}{\sinh^2 Z_k}\cos(\Omega\xi_k + \Omega_0), \end{aligned} \tag{7}$$

where $Z_k = \Omega\pi/(4\nu_k)$. Obviously, for $N_k = M_k = \Xi_k = X_k = 0$ the system (5) becomes the complex Toda chain for the complex variables Q_k.

In the present paper we generalize the above results to the perturbed vector NLS, see also [9]

$$i\vec{u}_t + \frac{1}{2}\vec{u}_{xx} + (\vec{u}^\dagger, \vec{u})\vec{u}(x,t) = iR[\vec{u}]. \tag{8}$$

The corresponding vector N-soliton train is determined by the initial condition

$$\begin{aligned} \vec{u}(x,t=0) &= \sum_{k=1}^{N}\vec{u}_k(x,t=0), \qquad & \vec{u}_k(x,t) &= \frac{2\nu_k e^{i\phi_k}}{\cosh(z_k)}\vec{n}_k, \\ z_k &= 2\nu_k(x - \xi_k(t)), & \xi_k(t) &= 2\mu_k t + \xi_{k,0}, \\ \phi_k &= \frac{\mu_k}{\nu_k}z_k + \delta_k(t), & \delta_k(t) &= 2(\mu_k^2 + \nu_k^2)t + \delta_{k,0}. \end{aligned} \tag{9}$$

where the constant polarization vector $\vec{n}_k$ is normalized by

$$(\vec{n}_k^\dagger, \vec{n}_k) = 1, \qquad \sum_{s=1}^{n} \arg \vec{n}_{k;s} = 0.$$

The present paper extends the results of our previous ones [6–9, 12]. More precisely we derive a generalized version of the CTC (GCTC) model (17) which now depends also on the polarization vectors $\vec{n}_k$ and models the behavior of the N-soliton train of the vector NLS. We also analyze how the changes of the polarization vectors influences the soliton interactions. In Section 2 we use the variational approach [3] and derive the GCTC model. We show that like the (unperturbed) CTC, GCTC is a finite dimensional completely integrable model, allowing Lax representation. In Section 3 we analyze the effects of the polarization vectors $\vec{n}_s$ on the soliton interaction. In particular we formulate a condition on $\vec{n}_s$ that is compatible with the adiabatic approximation. We also formulate the conditions on the initial vector soliton parameters responsible for the different asymptotic regimes. The last Section 4 is dedicated to the analysis of the effects of the external potentials (4) on the GCTC. In particular we propose a formula for the critical value of the periodic potential intensity that stabilizes the N-soliton train into a bound state. We end with discussion and conclusions.

2 Variational Approach and PCTC for PVNLS and Generalized CTC

The Lagrangian of the vector NLS perturbed by external potential is

$$\begin{aligned} \mathcal{L}[\vec{u}] &= \int_{-\infty}^{\infty} dt\ \frac{i}{2}\left[(\vec{u}\dagger, \vec{u}_t) - (\vec{u}_t^\dagger, \vec{u}) \right] - H, \\ H[\vec{u}] &= \int_{-\infty}^{\infty} dx\ \left[-\frac{1}{2}(\vec{u}_x^\dagger, \vec{u}_x) + \frac{1}{2}(\vec{u}^\dagger, \vec{u})^2 - (\vec{u}^\dagger, \vec{u}) V(x) \right]. \end{aligned} \tag{10}$$

Then the Lagrange equations of motion

$$\frac{d}{dt}\frac{\delta \mathcal{L}}{\delta \vec{u}_t^\dagger} - \frac{\delta \mathcal{L}}{\delta \vec{u}\dagger} = 0, \tag{11}$$

coincide with the vector NLS with the external potential $V(x)$.

Next we insert $\vec{u}(x,t) = \sum_{k=1}^{N} \vec{u}_k(x,t)$ (see Eq. (9)) and integrate over x neglecting all terms of order ϵ and higher. In doing this we assume that $\xi_1 < \xi_2 < \cdots < \xi_N$ at $t = 0$ and use the fact, that only the nearest neighbor solitons will contribute. All integrals of the form

$$\int_{-\infty}^{\infty} dx\ (\vec{u}_{k,x}^\dagger, \vec{u}_{p,x}), \quad \int_{-\infty}^{\infty} dx\ (\vec{u}_k^\dagger, \vec{u}_p), \quad \int_{-\infty}^{\infty} dx\ (\vec{u}_k^\dagger, \vec{u}_p) V(x), \tag{12}$$

with $|p-k| \geq 2$ can be neglected. The same holds true also for the integrals

$$\int_{-\infty}^{\infty} dx\, (\vec{u}_k^\dagger, \vec{u}_p)(\vec{u}_s^\dagger, \vec{u}_l),$$

where at least three of the indices k, p, s, l have different values. In doing this, a key role play the following integrals:

$$\begin{aligned} \mathcal{J}_2(a) &= \int_{-\infty}^{\infty} \frac{dz\, e^{iaz}}{2\cosh^2 z} = \frac{\pi a}{2\sinh\frac{a\pi}{2}}, \\ K(a,\Delta) &\equiv \int_{-\infty}^{\infty} \frac{dz\ e^{iaz}}{2\cosh z \cosh(z+\Delta)} = \frac{\pi(1-e^{-ia\Delta})}{2i\sinh(\Delta)\sinh(\pi a/2)}, \end{aligned} \tag{13}$$

Thus after long calculations we obtain

$$\begin{aligned} &\mathcal{L} = \sum_{k=1}^{N} \mathcal{L}_k + \sum_{k=1}^{N} \sum_{n=k\pm 1} \widetilde{\mathcal{L}}_{k,n}, \qquad \mathcal{L}_{k,n} = 16\nu_0^3 e^{-\Delta_{k,n}} (R_{k,n} + R_{k,n}^*), \\ &R_{k,n} = e^{i(\widetilde{\delta}_n - \widetilde{\delta}_k)} (\vec{n}_k^\dagger \vec{n}_n), \qquad \widetilde{\delta}_k = \delta_k - 2\mu_0 \xi_k, \\ &\Delta_{k,n} = 2 s_{k,n} \nu_0 (\xi_k - \xi_n) \gg 1, \qquad s_{k,k+1} = -1, \qquad s_{k,k-1} = 1. \end{aligned} \tag{14}$$

where

$$\begin{aligned} \mathcal{L}_k &= -2i\nu_k \left((\vec{n}_{k,t}^\dagger, \vec{n}_k) - (\vec{n}_k^\dagger, \vec{n}_{k,t}) \right) + 8\mu_k \nu_k \frac{d\xi_k}{dt} \\ &- 4\nu_k \frac{d\delta_k}{dt} - 8\mu_k^2 \nu_k + \frac{8\nu_k^3}{3} + 2\pi \nu_k V_0 + \frac{\pi^3}{8\nu_k} V_2 + \frac{\pi A \cos(\Omega_0)}{2\cosh(Z_k)} . \end{aligned} \tag{15}$$

The equations of motion are given by

$$\frac{d}{dt} \frac{\delta \mathcal{L}}{\delta p_{k,t}} - \frac{\delta \mathcal{L}}{\delta p_k} = 0, \tag{16}$$

where p_k stands for one of the soliton parameters: δ_k, ξ_k, μ_k, ν_k and $\vec{n}_k^\dagger$. The corresponding system is a generalization of CTC

$$\begin{aligned} \frac{d\lambda_k}{dt} &= -4\nu_0 \left(e^{Q_{k+1} - Q_k} (\vec{n}_{k+1}^\dagger, \vec{n}_k) - e^{Q_k - Q_{k-1}} (\vec{n}_k^\dagger, \vec{n}_{k-1}) \right) + M_k + iN_k, \\ \frac{dQ_k}{dt} &= -4\nu_0 \lambda_k + 2i(\mu_0 + i\nu_0)\Xi_k - iX_k, \qquad \frac{d\vec{n}_k}{dt} = \mathcal{O}(\epsilon), \end{aligned} \tag{17}$$

where again $\lambda_k = \mu_k + i\nu_k$ and the other variables are given by (6). Now we have additional equations describing the evolution of the polarization vectors. But note, that their evolution is slow, and in addition the products $(\vec{n}_{k+1}^{\dagger}, \vec{n}_k)$ multiply the exponents $e^{Q_{k+1}-Q_k}$ which are also of the order of ϵ. Since we are keeping only terms of the order of ϵ we can replace $(\vec{n}^{\dagger}_{k+1}, \vec{n}_k)$ by their initial values

$$(\vec{n}^{\dagger}_{k+1}, \vec{n}_k)\Big|_{t=0} = m_{0k}^2 e^{2i\phi_{0k}}, \qquad k = 1, \dots, N-1. \tag{18}$$

The system (17) was derived for the Manakov system $n = 2$ by other methods in [12]. There the GCTC model was tested numerically and found to give very good agreement with the numerical solution of the Manakov model.

3 Effects of the Polarization Vectors on the Soliton Interaction

We first briefly remind the main results concerning the CTC model [10, 11, 13, 14]. The CTC is completely integrable model; it allows Lax representation $L_t = [A.L]$, where

$$\begin{aligned} L &= \sum_{s=1}^{N} \left(b_s E_{ss} + a_s (E_{s,s+1} + E_{s+1,s}) \right), \\ A &= \sum_{s=1}^{N} \left(a_s (E_{s,s+1} - E_{s+1,s}) \right), \end{aligned} \tag{19}$$

where $a_s = \exp((Q_{s+1} - Q_s)/2)$, $b_s = \mu_{s,t} + i\nu_{s,t}$ and the matrices E_{ks} are determined by $(E_{ks})_{pj} = \delta_{kp}\delta_{sj}$. The eigenvalues of L are integrals of motion and determine the asymptotic velocities.

The GCTC derived in [6, 9, 12] is also a completely integrable model; it allows Lax representation $\tilde{L}_t = [\tilde{A}.\tilde{L}]$, where

$$\begin{aligned} \tilde{L} &= \sum_{s=1}^{N} \left(\tilde{b}_s E_{ss} + \tilde{a}_s (E_{s,s+1} + E_{s+1,s}) \right), \\ \tilde{A} &= \sum_{s=1}^{N} \left(\tilde{a}_s (E_{s,s+1} - E_{s+1,s}) \right), \end{aligned} \tag{20}$$

where $\tilde{a}_s = m_{0k}^2 e^{2i\phi_{0k}} a_s$, $b_s = \mu_{s,t} + i\nu_{s,t}$. Like for the scalar case, the eigenvalues of $\tilde{L}$ are integrals of motion. If we denote by $\zeta_s = \kappa_s + i\eta_s$ (resp. $\tilde{\zeta}_s = \tilde{\kappa} + i\tilde{\eta}_s$) the set of eigenvalues of L (resp. $\tilde{L}$), then their real parts κ_s (resp. $\tilde{\kappa}_s$) determine the asymptotic velocities for the soliton train described by CTC (resp. GCTC). Thus, starting from the set of initial soliton parameters we can calculate $L|_{t=0}$ (resp. $\tilde{L}|_{t=0}$), evaluate the real parts of their eigenvalues and thus determine the asymptotic regime of the soliton train.

Regime (i) $\kappa_k \neq \kappa_j$ (resp. $\tilde{\kappa}_k \neq \tilde{\kappa}_j$) for $k \neq j$, *i.e.* the asymptotic velocities are all different. Then we have asymptotically separating, free solitons, see also [4, 11, 13]

Regime (ii) $\kappa_1 = \kappa_2 = \cdots = \kappa_N = 0$ (resp. $\tilde{\kappa}_1 = \tilde{\kappa}_2 = \cdots = \tilde{\kappa}_N = 0$), *i.e.* all N solitons move with the same mean asymptotic velocity, and form a "bound state".

Regime (iii) a variety of intermediate situations when one group (or several groups) of particles move with the same mean asymptotic velocity; then they would form one (or several) bound state(s) and the rest of the particles will have free asymptotic motion.

Remark 1 *The sets of eigenvalues of L and $\tilde{L}$ are generically different. Thus varying only the polarization vectors one can change the asymptotic regime of the soliton train.*

Let us consider several particular cases.

Case 1 $\vec{n}_1 = \cdots = \vec{n}_N$. Since the vector $\vec{n}_1$ is normalized, then all coefficients $m_{0k} = 1$ and $\phi_{0k} = 0$. Then the interactions of the vector and scalar solitons are identical.

Case 2 $(\vec{n}^\dagger_{s+1}, \vec{n}_s) = 0$. Then the GCTC splits into two unrelated GCTC: one for the solitons $\{1, 2, \dots, s\}$ and another for $\{s+1, s+2, \dots .N\}$. If the two sets of soliton parameters are such that both groups of solitons are in bound state regimes, then these two bound states move independently.

Case 3 $\langle n^\dagger_{k+1} | \vec{n}_k \rangle = m_0 e^{i\varphi_0}$ – effective change of distance and phases of solitons. In this case we can rewrite $\tilde{a}_s = \exp((\tilde{Q}_{s+1} - \tilde{Q}_s)/2)$. where

$$\tilde{Q}_{s+1} - \tilde{Q}_s = Q_{s+1} - Q_s + \ln m_0 + i\varphi_0, \tag{21}$$

i.e. the distance between any two neighboring vector solitons has changed by $\ln m_0/(2\nu_0)$; the phases change similarly.

In what follows we will plots the coordinates $\xi_s(t)$ of some soliton trains. Most of them will be done using the following initial parameters of the solitons:

$$\nu_k(0) = 0.5, \qquad \phi_k(0) = k\pi, \qquad \xi_{k+1}(0) - \xi_k(0) = r_0, \qquad \mu_k = 0. \tag{22}$$

Let us show the first effect of the polarization vectors on the soliton interactions. To make it simple we assume that all $\vec{n}_s$ have only real components and that all m_{0s} are equal to 0.7, see Figure 1. The left panel shows the soliton train of scalar solitons; the right panel obviously shows that the repulsion of the vector solitons is diminished. In this special case both the scalar and the vector soliton trains are in the same asymptotically free regime.

This is rather natural, since $m_{0s} < 1$ and its effect according to eq. (21) will be as if the distance between the solitons has increased by $-\ln m_{0s}$.

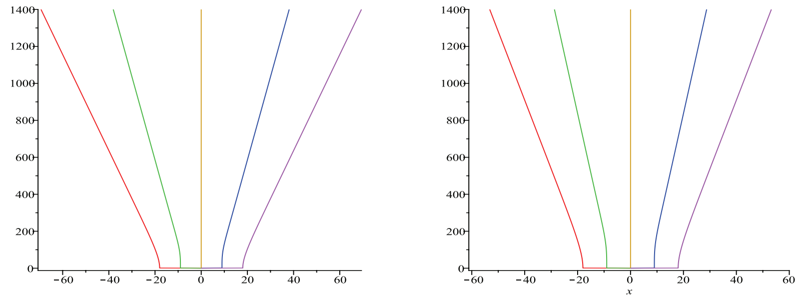

Figure 1. The initial soliton parameters as like in (22) with $r_0 = 9$. Left panel: scalar soliton train; Right panel: vector soliton train with $r_0 = 9$ and $m_{0s} = 0.7$.

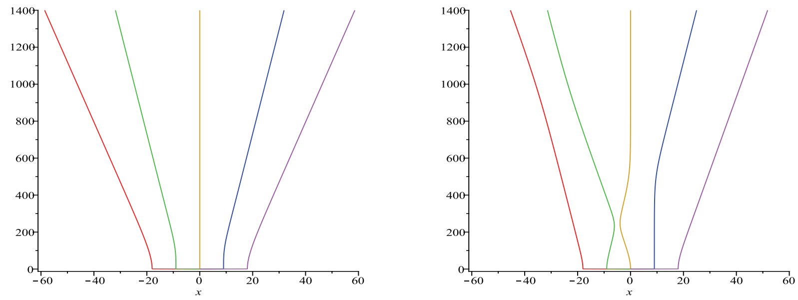

Figure 2. Left panel: vector soliton train with $m_{0s} = 0.8$; Right panel: vector soliton train with $m_{01} = m_{03} = m_{04} = 0.8$ and $m_{02} = 0.031$.

However, if we choose m_{0s} to be different we may encounter additional problems, see Figure 2. From this figure it is obvious that taking substantially different values for m_{0s} we may encounter violation of adiabaticity. This means that our approximation is valid only for such configurations of polarization vectors for which m_{0s} are of the same order of magnitude.

4 Effects of External Potentials

Let us choose the initial positions of the solitons to coincide with the minima of the periodic potential $V(x) = A\cos(\Omega x + \Omega_0)$; *i.e.* $r_0 = 2\pi/\Omega$. Then each soliton of the train experiences confining force of the periodic potential and repulsive force of neighboring solitons. Solitons placed initially at minima of the periodic potential perform small amplitude oscillations around these minima, provided that the strength of the potential is big enough to keep solitons confined, see Figure 3.

In contrast to the quadratic potentials, the weak periodic potential is unable to confine solitons, and repulsive forces between neighboring solitons (at

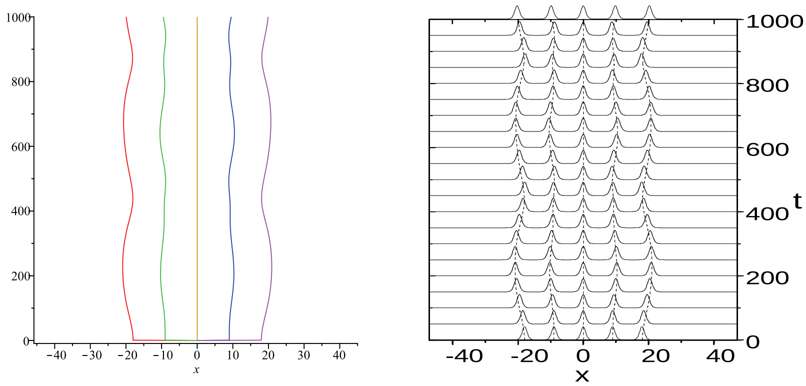

Figure 3. Oscillations of the 5-soliton train (see (22) in a moderately weak periodic potential, $A = 0.0005$, $\Omega = 2\pi/9$, $r_0 = 9$. Left panel: the trajectories as described by the CTC. Right panel: the numerical solution of the NLS eq.

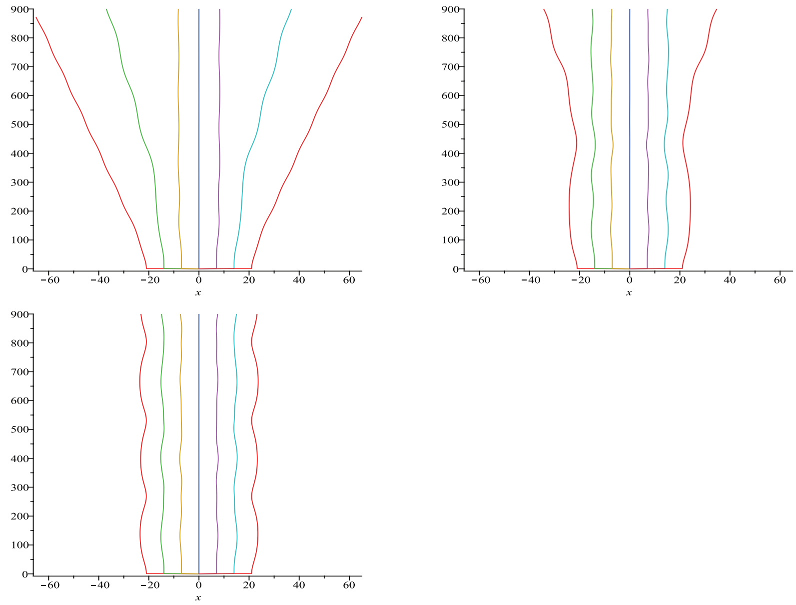

Figure 4. The effect of the periodic potential on 7-soliton trains (22) with $r_0 = 7$ and subcritical intensities. Upper left panel: $V_2 = -0.00075$; Upper right panel: $V_2 = -0.0012$; Below: periodic potential on 7-soliton trains (22) with critical intensity: $V_2 = -0.0013$.

$\nu_k(0) = 1/2$, $\delta_k(0) = k\pi$) induces unbounded expansion of the train similar to what has been shown in the left panel of Figure 4.

The same type of effects take place also for vector soliton interactions. Below we demonstrate the two subcritical regimes of a seven soliton system and one critical regime.

Assume we have only periodic potential present and the initial soliton configuration is (22). For $A = 0$ the solitons will go into asymptotically free regime. Switching on the self-consistent periodic potentials (such that the solitons initially are located at its minima) it is natural to expect that for $A > A_{\rm cr}$ the solitons will be stabilized into a bound state. In [8] for the scalar NLS solitons we derived that

$$A_{\rm cr} = -\left(1 - \frac{1}{N}\right)\frac{64\nu_0^4}{\pi\Omega} m_{00}^2 e^{-2\nu_0 r_0} \sinh\frac{\pi\Omega}{4\nu_0}, \quad m_{00} = \left(\prod_{s=1}^{N} m_{0s}\right)^{1/N}. \quad (23)$$

Note that the critical values generically should depend not only on the number of solitons N, but also on the initial configuration. The approach we used is not very sensitive to this. It cannot provide us with the intermediary critical values when the soliton train is stabilized after emmiting two or more solitons.

In other words the critical value $A_{\rm cr}$ is the same as for the scalar case, provided r_0 is replaced by the average effective distance evaluated on the basis of eq. (21).

5 Discussion and Conclusions

Like any other model, the predictions of the CTC and GCTC should be compared with the numerical solutions of the corresponding NLEE. Such comparison between the CTC and the NLS has been done thoroughly in [7, 8, 10, 11] and excellent match has been found for all dynamical regimes. Similar checks were done for the GCTC corresponding to the Manakov model $n = 2$. Thus both models may be viewed as an *universal model* for the adiabatic N-soliton interactions for several types of NLS.

For the vector N-soliton trains it is important to make a deeper investigation of the constraints that have to be imposed on the polarization vectors which make the adiabatic approximation applicable. Obviously configurations like the one in Case 2 above with $m_{0s} = 0$ should be avoided.

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