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Exact Solution of the Simple Exclusion Process from Boundary Symmetry*

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Abstract. The exact solution of a model of nonequilibrium physics derived from the boundary symmetry in the form of deformed Onzager algebra is presented. The symmetry is generated by nonlocal charges that render the model to be exactly solvable in the steady state and provide a unified description of the dynamics of the boundary driven process.

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The simple exclusion process (SEP) has become a paradigm in nonequilibrium physics [1–3] due to its simplicity, rich behaviour and wide range of applicability. It is an exactly solvable model of an open many-particle stochastic system interacting with hard core exclusion and is considered as fundamental as the Ising model is for the equilibrium statistical mechanics. Introduced originally as a simplified model of one dimensional transport for phenomena like hopping conductivity [4] and kinetics of biopolymerization [5], it has found applications from traffic flow [6] to interface growth [7], shock formation [8], hydrodynamic systems obeying the noisy Burger equation, problems of sequence alignment in biology [9]. The asymmetric process (ASEP) has rich physics. At large time the ASEP exhibits relaxation to a steady state, and even after the relaxation it has a nonvanishing current. An intriguing feature is the occurrence of boundary induced phase transitions [10] and the fact that the stationary bulk properties strongly depend on the boundary rates.

The SEP dynamics is governed by a master equation for the probability distribution $P(s_i, t)$ of a stochastic variable $s_i = 0, 1$, at a site $i = 1, 2, \ldots, L$ of a linear chain. In the set of occupation numbers $(s_1, s_2, \ldots, s_L)$ specifying a configuration of the system $s_i = 0$ if a site i is empty, or $s_i = 1$ if the site i is occupied. On successive sites particles hop with probability $g_{01}dt$ to the left, and $g_{10}dt$ to the right. The event of hopping occurs if out of two adjacent sites

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one is a vacancy and the other is occupied by a particle. The symmetric simple exclusion process is the lattice gas model of particles hopping between nearest-neighbour sites with a constant rate $g_{01} = g_{10} = g$. The asymmetric simple exclusion process with hopping in a preferred direction is the driven diffusive lattice gas of particles moving under the action of an external field. The process is totally asymmetric if all jumps occur in one direction only, and partially asymmetric if there is a different non-zero probability of both left and right hopping. The number of particles in the bulk is conserved and this is the case of periodic boundary conditions. In the case of open systems, the lattice gas is coupled to external reservoirs of particles of fixed density and additional processes can take place at the boundaries. Namely, at the left boundary $i = 1$ a particle can be added with probability αdt and removed with probability γdt, and at the right boundary $i = L$ it can be removed with probability βdt and added with probability δdt. Without loss of generality we can choose the right probability rate $g_{10} = 1$ and the left probability rate $g_{01} = q$. The totally asymmetric process corresponds to $q = 0$ and the symmetric - to $q = 1$. In the quantum picture the master equation $\frac{dP(s,t)}{dt} = \sum_{s'} \Gamma(s,s')P(s',t)$ is related to a Schroedinger equation in imaginary time for a quantum Hamiltonian with nearest-neighbour interaction in the bulk and single-site boundary terms. The equivalence to the integrable spin $1/2$ XXZ quantum spin chain and added most general non diagonal boundary terms [11] allowed for derivation of exact results for the ASEP, like the inferred Bethe ansatz (BA) by de Gier and Essler [13] for even number of lattice sites. Despite this equivalence the two systems describe different physics due to the stochastic nature of the diffusion process. Important distinguishing features [3] are: i) the elements of $P(t)$ are probabilities and must all be positive in contrast to the quantum case where the wave function elements squared are probabilities, ii) the special form of the transition rate matrix where each column sums up to zero due to probability conservation. Therefore one has been looking for an independent treatment of the stochastic dynamics. The matrix product approach (MPA) was developed for description of the stationary stochastic behaviour. We generalize the MPA to a tridiagonal algebra approach which reveals hidden symmetries of the stochastic process and allows for the exact solvability in the stationary state and description of the dynamics. In our study we are led by the major importance of the boundary conditions for a nonequilibrium system in contrast to an equilibrium one. We use the boundary algebra which is the deformed analogue of the Onsager algebra of the Ising model. In this paper we present the diagonalization of the transition rate matrix in a complete set of 2^L states for arbitrary range of the boundary parameters.

Matrix Product State Ansatz (MPA): In this elegant algebraic approach [8, 14] particle and hole configurations are represented by strings of two elements in terms of which the steady state properties of the SEP can be obtained exactly. For a given configuration $(s_1, s_2, ..., s_L)$ the stationary probability is defined by the expectation value $P(s) = \frac{\langle w|D_{s_1}D_{s_2}\cdots D_{s_L}|v\rangle}{Z_L}$, where $D_{s_i} = D_1$ if a site $i = 1, 2, ..., L$ is occupied and $D_{s_i} = D_0$ if a site i is empty and $Z_L = \langle w|(D_0 + D_1)^L|v\rangle$ is the normalization factor to the stationary probability distribution. In the asymmetric case the operators $D_i, i = 0, 1$ satisfy the quadratic (bulk) algebra

$$D_1D_0 - qD_0D_1 = x_1D_0 - D_1x_0, \qquad x_0 + x_1 = 0 \tag{1}$$

with boundary conditions of the form

$$\begin{aligned} (\beta D_1 - \delta D_0)|v\rangle &= x_0|v\rangle \\ \langle w|(\alpha D_0 - \gamma D_1) &= -\langle w|x_1. \end{aligned} \tag{2}$$

and $\langle w|v\rangle \neq 0$. We stress the one parameter dependence of the MPA due to $x_0 = -x_1 = \zeta$ with $0 < \zeta < \infty$. In most known applications it is restricted to the choice $\zeta = 1$. The relation $x_0 + x_1 = 0$ is due to the boundary rate matrices whose columns sum up to zero. It reflects the stochastic nature and implies an abelian symmetry with a conserved quantity $D_0 + D_1$, following from $D_0 \to D_0 + x_0, D_1 \to D_1 + x_1$.

The MPA is an efficient method to evaluate exactly all the relevant physical quantities, such as the mean density at a site i, the correlation functions, the current J through a bond between sites i and $i + 1$. Exact results for the ASEP with open boundaries were obtained within the MPA through the relation of the stationary state to q-Hermite [15] and Al-Salam-Chihara polynomials [16] in the case $\gamma = \delta = 0$ and to the Askey-Wilson polynomials [17] in the general case. dynamical MPA [18].

The generalization of the MPA to a tridiagonal algebra approach is based on the appearance of the quantum affine $U_q(\hat{su}(2))$ algebra as the hidden bulk symmetry of the ASEP and the interpretation of the boundary operators as covariant elements (coideals) of the bulk symmetry. The boundary symmetry is a tridiagonal algebra [19] in the form of a deformed Dolan-Grady algebra [20]. (Our algebra and method differ from the ones in [21] and [22] for the exact spectrum of the XXZ chain). The irreducible modules of this algebra in the form of infinite-dimensional matrices are important for the exact solution in the stationary state while the nontrivial truncation produces a complete set of eigenvalues diagonalizing the transition rate matrix. The crucial point for the quantum affine symmetry to arise as a hidden symmetry is the continuous parameter in the quadratic algebra and in the boundary conditions. As already pointed out the open ASEP is equivalent through a similarity transformation to the $U_q(su(2))$ invariant XXZ spin chain with added general nondiagonal boundary terms (in the proper parametrization). The affine symmetry is an extension of the quantum $U_q(su(2))$ algebra which introduces a new quantum number called the level with the "value added" [23] that the larger symmetry gives more information about the solution.

We now describe our procedure of obtaining the exact solution of the open driven diffusive system. In the general case of open ASEP with incoming and outgoing particles at both boundaries with all boundary parameters nonzero the four operators $\beta D_1, -\delta D_0, -\gamma D_1, \alpha D_0$ form two linear independent boundary operators D^L and D^R acting on the dual boundary vectors. Our starting point is the bulk quadratic algebra of the MPA. It can be imbedded in a larger algebra which upon the implementation of the affine transformations takes the form

$$
\begin{aligned}
&[D_0[D_0[D_0, D_1]_q]_{q^{-1}} = 0, \\
&[D_1[D_1[D_1, D_0]_q]_{q^{-1}} = 0.
\end{aligned}
\tag{3}
$$

These are the defining q-Serre relations of a level zero quantum affine $U_q(\hat{su}(2))$ algebra. It arises as the (hidden) bulk symmetry and one can construct the boundary operators with the properties of covariant objects. In the case of quantum affine symmetries these are the coideal elements. On them the algebra acts by comultiplication whose physical understanding is through the analogy with vector addition of angular momenta with the difference that the addition depends on the order. Hence we can present the boundary operators in the form

$$
\begin{aligned}
D^R &\equiv \beta D_1 - \delta D_0 = A + x_0 \frac{\beta - \delta}{1-q} \\
D^L &\equiv \alpha D_0 - \gamma D_1 = A^* + x_0 \frac{\alpha - \gamma}{1-q} \, ,
\end{aligned}
\tag{4}
$$

where the operators A, A^* are infinite-dimensional matrices. The importance of this presentation is that A, A^* obey the deformed Onsager algebra

$$
\begin{aligned}
[A, [A[A, A^*]_q]_{q^{-1}}] &= \rho [A, A^*] \\
[A^*, [A^*[A^*, A]_q]_{q^{-1}}] &= \rho^* [A^*, A],
\end{aligned}
\tag{5}
$$

where $[A, A^*]_q = q^{1/2} A A^* - q^{-1/2} A^* A$ is the q-commutator and with the structure constants depending on $q, \alpha, \beta, \gamma, \delta$

$$
\begin{aligned}
\rho &= x_0^2 \beta \delta q^{-1} (q^{1/2} + q^{-1/2})^2, \\
\rho^* &= x_0^2 \alpha \gamma q^{-1} (q^{1/2} + q^{-1/2})^2.
\end{aligned}
\tag{6}
$$

The AW algebra is represented by infinite-dimensional matrices in the space of symmetric Laurent polynomials with basis the Askey-Wilson (AW) polynomials, denoted $p_n = p_n(x; a, b, c, d|q)$ [26], depending on four parameters a, b, c, d, with $p_0 = 1$, $x = y + y^{-1}$, $0 < q < 1$ and a three term recurrence relation

$$
x p_n = b_n p_{n+1} + a_n p_n + c_n p_{n-1}, \qquad p_{-1} = 0. \tag{7}
$$

We need only the explicit form of the matrix elements b_n

$$
b_n = \frac{(1 - abq^n)(1 - acq^n)(1 - adq^n)(1 - abcdq^{n-1})}{a(1 - abcdq^{2n-1})(1 - abcdq^{2n})} \, . \tag{8}
$$

Let $\mathcal{A}$ denote the matrix satisfying (10) in the basis $(p_0, p_1, p_2, ...)$. Then in the space of Laurent polynomials with basis $(p_0, p_1, p_2, ...)^t$ the right boundary charge $D^R = (D_1 - \frac{\delta}{\beta} D_0)$ is diagonal with diagonal eigenvalues

$$\lambda_n = \frac{1}{1-q}\left(bq^{-n} + dq^n\right) + \frac{1}{1-q}(1+bd)\,, \tag{9}$$

and the left one $(D_0 - \dfrac{\gamma}{\alpha} D_1)$ is tridiagonal

$$(D_0 - \frac{\gamma}{\alpha} D_1) = \frac{1}{1-q} b\mathcal{A}^t + \frac{1}{1-q}(1+ac).$$

The dual representation has a basis $p_0, p_1, p_2, \ldots$ with respect to which the left charge $D^L = (D_0 - \dfrac{\gamma}{\alpha} D_1)$ is diagonal with eigenvalues

$$\lambda_n^* = \frac{1}{1-q}\left(aq^{-n} + cq^n\right) + \frac{1}{1-q}(1+ac)\,, \tag{10}$$

and the right one $(D_1 - \dfrac{\delta}{\beta} D_0)$ is tridiagonal

$$(D_1 - \frac{\delta}{\beta} D_0) = \frac{1}{1-q} a\mathcal{A} + \frac{1}{1-q}(1+bd)\,.$$

The choice of the boundary vectors $\langle w| = h_0^{-1/2}(p_0, 0, 0, \ldots)$, $|v\rangle = h_0^{-1/2}(p_0, 0, 0, ...)^t$ (h_0 is a normalization) as eigenvectors of the diagonal matrices $(D_1 - \dfrac{\delta}{\beta} D_0)$ and $(D_0 - \dfrac{\gamma}{\alpha} D_1)$ uniquely relates a, b, c, d to the boundary parameters α, β, γ, δ in the form $a = \kappa_+(\alpha, \gamma)$, $b = \kappa_+(\beta, \delta)$, $c = \kappa_-(\alpha, \gamma)$, $d = \kappa_-(\beta, \delta)$, with

$$\kappa_\pm(\nu, \tau) = \frac{(1-q) + \tau - \nu \pm \sqrt{(\nu - \tau - (1-q))^2 + 4\nu\tau}}{2\nu}. \tag{11}$$

It is important to emphasize that these are the functions of the parameters that define the phase diagram of the model. In all applications they have been taken for granted. Here only they are uniquely determined by the properties of the boundary algebra. The transfer matrix $D_0 + D_1$ is also determined by the boundary symmetry. Its form of an infinite-dimensional matrix was exploited in [17] for the exact calculation of the current, normalization factor to the stationary probability distribution, the correlation functions. Thus the boundary symmetry is the deep algebraic property behind the exact stationary solution of the ASEP.

To diagonalize the transition rate matrix we need construct an irreducible representation of dimension 2^L. This is achieved by truncation of the three-terms recurrence relation so that $b_n = 0$, *i.e.*, if any of the factors in the numerator in (8) is zero, in particular, with our normalization $abcd = \frac{\gamma\delta}{\alpha\beta}$, if the condition holds $\alpha\beta = q^{L-1}\gamma\delta$ which imposes a constraint on the model parameters and restricts the nonequilibrium behaviour. Indeed, this condition is known as the detailed balance condition and it is a characteristic of equilibrium. We need a nontrivial way to obtain finite-dimensional boundary matrices. Here is the important point where the added value of the quantum affine symmetry comes into play. We implement the affine parameter ζ to rescale the parameters a, b, c, d in the matrix elements of the shifted boundary operators and use the zeros $x_k = y_k + y_k^{-1}$ of the AW polynomial of order L $p_L(y) = \prod_{m=1}^{L}(y - y_m)(y - y_m^{-1})$, which for any finite L obey the Bethe ansatz equation [27, 28]

$$\Phi(y_i, \alpha, \gamma)\Phi(y_i, \beta, \delta) = \prod_{k=1, k\neq i}^{L} \frac{(qy_i - y_k)(qy_i y_k - 1)}{(y_i - qy_k)(y_i y_k - q)}\,, \tag{12}$$

where

$$\Phi(y_i, \alpha, \gamma) = \frac{(y_i - k_+(\alpha, \gamma))(\hat{y}_i - k_-(\alpha, \gamma))}{(k_+(\alpha, \gamma)y_i - 1)(k_-(\alpha, \gamma)y_i - 1)}\,, \tag{13}$$

and $k_\pm(\nu, \tau)$ given by (11). For each zero $x_k, k = 1, ..., L$ the truncation yields finite-dimensional matrices A, A^* which obey the deformed Dolan-Grady relations (12). The next step is to take as a basis for the algebra (12) L two-dimensional modules $p_0(x_k), p_1(x_k)$. On each of them the operator A acts as multiplication by x_k. On a tensor product it acts by means of the comultiplication

$$\Delta(A) = A_{i_1} \otimes I + I \otimes A_{k_2} + A_{i_1} \otimes A_{k_2}\,, \tag{14}$$

which is such that the order of how the operators act is important. We associate further with each lattice site i a basis vector $p_0(x_i)$ if a site is empty (occupation number $s_i = 0$) or $p_1(x_i)$ if there is a particle on the site (occupation number $s_i = 1$). The tensor product of L such modules has dimension 2^L. The iteration of the comultiplication produces $2^L - 1$ distinct eigenvalues of the matrix A in this representation and correspondingly of the right boundary operator. The missing vector is the eigenvector $(1, 0, ..., 0)$ of the matrix A^*, corresponding to the stationary state of the ASEP. The 2^L dimensional matrices A, A^* satisfy the deformed Dolan-Grady relations. The transition rate matrix is represented as $\Gamma = D^L + D^R$ and in the 2^L complete basis has the form

$$\Gamma = \alpha - \gamma + \beta - \delta + (1-q)\sum_{i=1}^{L}\hat{x}_i + (1-q)^2\sum_{i<j}\hat{x}_i\hat{x}_j + \cdots + (1-q)^L\hat{x}_1\hat{x}_2...\hat{x}_L, \tag{15}$$

where $\hat{x}_i^{-1} = y_i + y_i^{-1}$ and y_i satisfy the Bethe-Ansatz equation (12).

We note that for the ASEP there is only one complete set of eigenvectors diagonalizing the transition rate matrix in contrast to the XXZ, where there are two systems. This solution of the ASEP is valid for any finite L. We can compare this solution with the result in [13], for even number of sites and in a different basis, for the energy eigenvalues in the one spin sector, with $(\hat{y}_i + \hat{y}_i^{-1}) \rightarrow \dfrac{1}{1-Q^2}(-Q(z_i + z_i^{-1}) + Q^2 + 1)$, $Q^2 = q$. Due to the ultimate relation of the ASEP exact solution in the stationary state to the AW polynomials they are the most natural basis for the exact description of the dynamics. The tridiagonal method provides an unified description of the three versions of the simple exclusion process by taking the proper limits in Eqs. (13) and (16). *Symmetric exclusion process*: It is known that the $q = 1$ limit of the AW polynomials are the Wilson polynomials. There is an established procedure how to obtain the Bethe equations and they read

$$\frac{\chi_k + iw_1}{\chi_k - iw_1}\frac{\chi_k + iw_2}{\chi_k - iw_2} = \prod_{l=1, l\neq k}^{L} \frac{(\chi_k - \chi_l - i)(\chi_k + \chi_l - i)}{(\chi_k - \chi_l + i)(\chi_k + \chi_l + i)}, \tag{16}$$

where $w_{1,2} = \frac{x_{1,2}-2}{2x_{1,2}}$ and $x_1 = \gamma - \alpha, x_2 = \delta - \beta$. We obtain $2^L - 1$ eigenvalues of Γ as the one zero eigenvalue is the SSEP stationary state. We note that there is one set of complete eigenvectors with the corresponding eigenvalues. This is one of the Bethe systems derived in [13], namely the one which corresponds to the Bethe ansatz equations for the Heisenberg chain in magnetic field obtained in earlier works [18] for the SSEP for any number of sites. The nonzero eigenvalues of the transition rate matrix are

$$\Gamma = \alpha - \gamma + \beta - \delta + \sum_{i=1}^{L} \chi_i^{-2} + \sum_{i<j} \chi_i^{-2}\chi_j^{-2} + ... + \chi_1^{-2}...\chi_L^{-2}. \tag{17}$$

Totally asymmetric exclusion process: We take the limit $q = 0$ in the Bethe ansatz equations and in (15). For the purpose we rescale $z \rightarrow q^{1/2}z$ and for the proper limit of the equations one has to multiply the LHS of (12) by y_k^{-2L-2} to obtain

$$y_k^{-L} = (ay_k - 1)(by_k - 1) \prod_{l=1, l\neq k}^{L} (y_k^{-1} - y_l), \tag{18}$$

where $a = (1-\alpha)/\alpha, b = (1-\beta)/\beta$ and

$$\Gamma = \alpha - \gamma + \beta - \delta + \sum_{i=1}^{L} x_i + \sum_{i<j} x_i x_j + \cdots + x_1 \ldots x_L. \tag{19}$$

It should be noted that there is no factor of degree $2L$ on the LHS of the Bethe-equations. This is different from case of the XXZ chain and the SEP solution in [13]. Bethe-equations of the type (12) have been used in [29, 30] for the solutions to the problem for Bloch electrons in a magnetic field and the Azbel-Hofstadter problem.

The advantages of our method are: i) exact solvability in the stationary state and exact description of the dynamics ii) unified description of the symmetric, the partially and the totally asymmetric processes.
Conclusion: The matrix product ansatz is generalized to a tridiagonal algebra approach based on deformed Dolan-Grady relations which define the boundary operators of the ASEP as nonlocal charges. The quantum affine symmetry arises as a hidden bulk symmetry of the process whose consequences are that the boundary symmetry allows for the exact solvability of the process.

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