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Quantum Equations of Motion in BF Theory with Sources

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Abstract. The 2-dimensional BF theory is both a gauge theory and a topological Poisson σ-model corresponding to a linear Poisson bracket. In [1], Torossian discovered a connection which governs correlation functions of the BF theory with sources for the B-field. This connection is flat, and it is a close relative of the KZ connection in the WZW model. In this paper, we show that flatness of the Torossian connection follows from (properly regularized) quantum equations of motion of the BF theory.

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1 Introduction

The 2-dimensional BF theory is an interesting example of a model which is at the same time a gauge theory and a (topological) Poisson σ-model corresponding to a linear Poisson bracket. Hence, we have an interesting opportunity to compare two different approaches to quantization of the model.

The universality of deformation quantization was proved by Kontsevich, who showed in a celebrated paper [2], that this construction is possible for any Poisson manifold. The corresponding deformed product (the “star product”) is given as an expansion, resembling the Feynman perturbation series for a two-dimensional field theory on a disc with boundary. A QFT interpretation of the Kontsevich formula was given in [3], linking it with the perturbative expansion of the path integral of a simple topological bosonic open string theory.

As a Poisson σ-model, the BF theory gives rise to a star product on the dual space of a Lie algebra $\mathcal{G}$. In this context, a natural connection which governs the behavior of correlation functions of exponentials of the B-field was descovered [1] and proved to be flat [4]. It is a close relative of the Knizhnik-Zamolodchikov connection [5] in the WZW model.

We shall explain the origin of this connection from the point of view of gauge theory. To this end, we consider the BF theory with source terms for the B-field placed at the points $z_1, \ldots, z_n$, and we study the expectations of the quantum gauge field $\mathcal{A}$ and of the quantum $\mathcal{B}$-field. Since $\mathcal{A}$ diverges at the points where the source terms are located, a proper regularization is required. We shall show that the set of regularized values $\mathcal{A}^{reg}(z_1), \ldots, \mathcal{A}^{reg}(z_n)$ forms a connection $\mathbb{A}$ on the space of configurations of points $z_1, \ldots, z_n$. This connection governs the behavior of correlation functions, and it takes values in the Lie algebra of vector fields on n copies of $\mathcal{G}$. $\mathbb{A}$ is precisely the connection from [1,4]. In this context, its flatness follows from the quantum equations of motion for $\mathcal{A}$ and $\mathcal{B}$.

2 Classical and Quantum BF Theory

2.1 Classical action and equations of motion

Topological field theories (see, *e.g.*, [6] for a review) were introduced some 20 years ago as a novel class of field theories whose partition functions are independent of the metric. Their roots are in the papers of Schwarz [7,8] and Witten [9] devoted to seemingly different problems — the relation of the Ray–Singer torsion with the partition function of some quantum field theory, and the representation of Morse theory in terms of supersymmetric QM. The observables in these theories are topological invariants of the underlying space-time manifold $\mathcal{M}$.

The above two series of papers exhibit the archetypes of all later topological field theories. The BF-theory belongs to the first type. It is a topological gauge theory which can be defined in any dimension. Let G be a connected Lie group, $\mathcal{G}$ – its Lie algebra, and denote by $\mathrm{tr}\,(ab)$ an invariant scalar product on $\mathcal{G}$ (for instance, the Killing form if G is semisimple). For $\mathcal{M}$ an oriented manifold of dimension n (the space-time of the model) and P – a principal G-bundle over $\mathcal{M}$, fields of the BF theory are the gauge field A on the bundle P and the $\mathcal{G}$-valued $(n-2)$-form B. The action is given by

$$S_{BF} = \mathrm{tr} \int BF, \qquad F = dA + \frac{1}{2}[A, A]. \tag{1}$$

Its quadratic part is of the first order in derivatives, so the theory has no physical degrees of freedom. Setting the variation of the action equal to zero, we obtain the field equations

$$dB + [A, B] = D_A B = 0\,, \tag{2}$$

$$dA + \frac{1}{2}[A, A] = F = 0\,. \tag{3}$$

The gauge transformations are of the form

$$A^g = g^{-1}dg + g^{-1}Ag\,, \quad B^g = g^{-1}Bg\,. \tag{4}$$

Since F is the curvature form, Eq. (3) states that the connection A is flat. It is this feature that we shall investigate in the context of quantum gauge theory.

2.2 Feynman diagrams

It is convenient to rewrite the classical action in the form

$$S_{BF} = \mathrm{tr} \int \left(BdA + \frac{1}{2} B[A,A] \right) , \qquad (5)$$

where the first term can be viewed as a free part of the action (in fact, it corresponds to an Abelian BF theory) while the second term represents the interaction. The Feynman diagrams in this theory are built of oriented edges pointing from A to B and of trivalent vertices with one incoming B-field and two outgoing A-fields, see Figure 1.

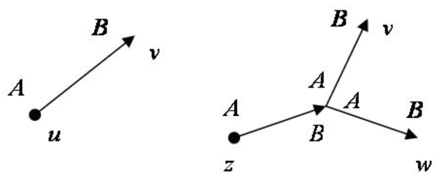

Figure 1. Diagram building blocks: (a) single edge; (b) vertex.

Depending on the choice of $\mathcal{M}$, the propagator corresponding to an oriented edge can be chosen in various ways. For the BF theory on a plane, one can choose

$$\langle A_a(u) B_b(v) \rangle = \frac{\delta_{ab}}{2\pi} \, d \arg(u-v) \, ,$$

where u and v are complex coordinates on the plane, and the right hand side is viewed as a 1-form with respect to u. Note that the choice of propagator corresponds to a particular gauge fixing in the theory. The triple vertex corresponds to structure constants f_{abc} of the Lie algebra $\mathcal{G}$.

Connected Feynman graphs of the BF theory are tree diagrams with one external A-field and an arbitrary number of B-fields and one-loop (or wheel-type) diagrams with only B-fields on the external lines.

2.3 BF theory with sources

We shall be interested in the BF theory with source terms for B-field added. For the classical action, we have

$$S_\eta = \mathrm{tr} \left(\int_{\mathcal{M}} BF + \sum_{i=1}^{n} \eta_i B(z_i) \right) , \qquad (6)$$

The partition function is then given by

$$K_\eta(z_1,\dots,z_n)=\int e^{S_\eta}=\int e^{S_{BF}+\sum_{i=1}^{n}\mathrm{tr}\,(\eta_i B(z_i))}\,, \tag{7}$$

and it can be viewed as a correlation function of the operators $\exp\mathrm{tr}\,(\eta_i B(z_i))$ in the theory without sources,

$$K_\eta(z_1,\dots,z_n)=\left\langle e^{\mathrm{tr}\,(\eta_i B(z_i))}\dots e^{\mathrm{tr}\,(\eta_i B(z_i))}\right\rangle\,. \tag{8}$$

The expectation value of an operator $\mathcal{O}$ will be given by

$$\langle\mathcal{O}\rangle_\eta=\left\langle \mathcal{O}\, e^{\sum_{i=1}^{n}\mathrm{tr}\,\left(\eta_i B(z_i)\right)}\right\rangle\Big/\left\langle e^{\sum_{i=1}^{n}\mathrm{tr}\,\left(\eta_i B(z_i)\right)}\right\rangle\,. \tag{9}$$

In particular, we shall study two cases: when $\mathcal{O}$ is the gauge field $A(u)$ and when $\mathcal{O}$ is the B-field $B(u)$. Note that these are not gauge invariant observables, and that the source terms explicitly break the gauge invariance of the action.

First, we observe that the expectation value of the A-field obtains contributions only from tree-type diagrams. This defines the quantum gauge field $\mathcal{A}$,

$$\mathcal{A}(u)=\langle A(u)\rangle_\eta=\sum_{\textit{all trees}}\Big(\qquad\Big)\,. \tag{10}$$

For a B-field, it is slightly more complicated: we obtain all possible wheel-type diagrams hanging on a branch of a tree-type diagram

$$\mathcal{B}(u)=\langle B(u)\rangle_\eta=\sum_{\textit{all}\ [\mathrm{TW}]\ \textit{compositions}}\Big(\qquad\Big)\,. \tag{11}$$

Note that both trees and wheels may have arbitrary lengths. In particular, among tree diagrams there are short trees (containing only one edge, $[\mathrm{T}(\mathrm{l}=1)]$), see Figure 1. It is convenient to rewrite Eq. (10) as a sum of two terms

$$\mathcal{A}(u)=\sum_{i=1}^{n}\eta_i\, d\,\arg(u-z_i)+a(u;z_1,\dots,z_n), \tag{12}$$

where $a(u;z_1,\dots,z_n)$ is the sum over all trees with length $l>1$, $[\mathrm{T}(\mathrm{l}\!>\!1)]$.

2.4 Quantum equations of motion

The canonical way to obtain the quantum equations of motion — the Batalin–Vilkovisky method, implies introducting ghosts and anti-fields with complimentary ghost numbers and degrees. We shall instead make use of the graphical

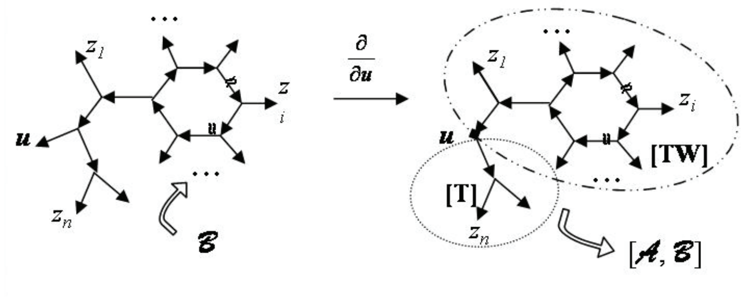

Figure 2. Equation of motion for $\mathcal{B}$-field.

representation of the quantum fields where all terms in the field expansions are present, thus the equations obtained should account for all quantum corrections, including those coming from the gauge-fixing terms.

In Figure 2 we show the differential of the quantum $\mathcal{B}$-field. By taking the derivative with respect to the root-point u, the diagram splits into a wheel-type diagram, and a tree. The two subgraphs are related by a Lie bracket corresponding to the vertex where they meet. Thus, the quantum equation of motion for $\mathcal{B}$ reads

$$d\mathcal{B} = -[\mathcal{A}, \mathcal{B}] \,. \tag{13}$$

In fact, it coincides with the classical equation of motion, Eq. (2).

For the differential of the quantum gauge field $\mathcal{A}$, we use the splitting (12) to obtain the singular and the regular parts of the result. The singular part (one-edge graphs) generates a sum-over-sources term, Figure 3. As seen from Figure 4, the derivative of the regular part, similarly to the case of the B-field, splits into two tree-type subgraphs rooted at u.

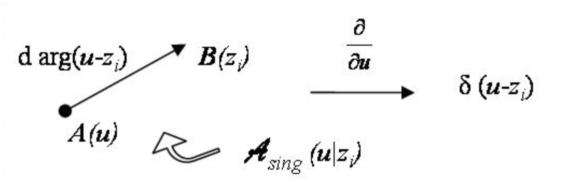

Figure 3. Equation of motion for $\mathcal{A}$: singular terms.

Thus, the quantum equation for $\mathcal{A}$ takes the form

$$d\mathcal{A} = -\frac{1}{2}\,[\mathcal{A},\, \mathcal{A}] + \sum_{i=1}^{n} \eta_i\, \delta(u - z_i) \,, \tag{14}$$

which is again of the same form as the corresponding classical equation.

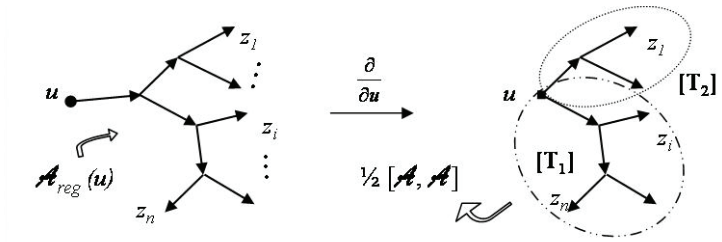

Figure 4. Equation of motion for $\mathcal{A}$: regular terms.

3 Equations for Correlators and Quantum Flat Connection

We are interested in the dependence of the generating functional of the B-field correlators $K_\eta(z_1, \ldots, z_n)$ on the positions of the sources, so we shall study the derivatives of the quantum fields $\mathcal{A}$ and $\mathcal{B}$ with respect to coordinates z_i.

In order to regularize the singularity at the points where the sources are placed, it is convenient to introduce for each i a new splitting of $\mathcal{A}(u)$ in the form

$$\mathcal{A}_{(i)}(u) = \frac{\eta_i}{2\pi}\, d\, \arg(u - z_i) + \mathcal{A}^{reg}_{(i)}(u), \tag{15}$$

where all the unit-length trees but one (connecting the points u and z_i) are now kept in the regular part

$$\mathcal{A}^{reg}_{(i)}(u) = \sum_{j \neq i} \frac{\eta_j}{2\pi}\, d\, \arg(u - z_j) + a(u; z_1, \ldots, z_n)\,. \tag{16}$$

Observe that $\mathcal{A}^{reg}_{(i)}(u)$ has no singularity at $u = z_i$. Let us denote its value by

$$a_i := \mathcal{A}^{reg}_{(i)}(u; z_1, \ldots, z_i, \ldots, z_n)\Big|_{u=z_i}\,. \tag{17}$$

The quantum equation of motion for the $\mathcal{B}$-field leads to the following relation:

$$d\,\mathrm{tr}\left(\eta\mathcal{B}(u)\right) = -\mathrm{tr}\,[\eta, \mathcal{A}(u)]\,\frac{\partial}{\partial \eta}\,\mathrm{tr}\left(\eta\mathcal{B}(u)\right)\,.$$

Naively, we should expect the following equation for $K_\eta(z_1, \ldots, z_n)$ to hold

$$d_{z_i} K_\eta(z_1, \ldots, z_n) + \mathrm{tr}\,[\eta_i, \mathcal{A}(z_i)]\,\frac{\partial}{\partial \eta_i}\,K_\eta(z_1, \ldots, z_n) = 0\,. \tag{18}$$

Here d_{z_i} stands for the de Rham differential with respect to the coordinate z_i (note that it includes both holomorphic and anti-holomorphic differentials).

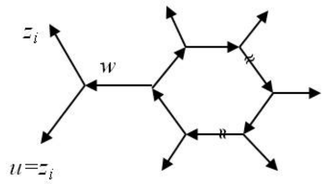

Figure 5. Vanishing B-field diagram.

Since $\mathcal{A}(z_i)$ is ill-defined, we need to re-examine the Feynman graphs which contribute to the right hand side of Eq. (18).

The only interesting (different from the naive approach) case is the diagram shown in Figure 5. Its contribution vanishes because of the factor $(d\arg(w - z_i))^2 = 0$ in the integrand of the corresponding Feynman integral. Hence, the one-edge tree connecting w and z_i does not contribute in the derivative of K_η, and the renormalized quantum formula replacing Eq.(18) is

$$d_{z_i} K_\eta + \mathrm{tr}\,[\eta_i, a_i]\,\frac{\partial}{\partial \eta_i}\,K_\eta = 0\,. \tag{19}$$

Equations (19) for different i can be put together in one equation

$$dK_\eta + \mathrm{tr}\,\sum_{i=1}^{n}[\eta_i, a_i]\,\frac{\partial}{\partial \eta_i}\,K_\eta = 0\,, \tag{20}$$

where d is the total de Rham differential for all variables $z_1, \ldots, z_n$. For functions $\alpha_i(\eta_1, \ldots, \eta_n) \in \mathcal{G}, i = 1, \ldots, n$, operators

$$D_\alpha = \mathrm{tr}\,\sum_{i=1}^{n}[\eta_i, \alpha_i]\,\frac{\partial}{\partial \eta_i} \tag{21}$$

form an interesting Lie algebra

$$[D_\alpha, D_\beta] = D_{\{\alpha,\beta\}}\,, \qquad \{\alpha, \beta\}_i = D_\alpha \beta_i - D_\beta \alpha_i + [\alpha_i, \beta_i]. \tag{22}$$

One can view the collection of 1-forms $(a_1, \ldots, a_n)$ as components of a connection $\mathbb{A} = (a_1, \ldots, a_n)$ with values in this Lie algebra. Then, equation (20) for correlation functions simply reads

$$dK_\eta + D_{\mathbb{A}} K_\eta = 0\,.$$

Similarly, for the differential of gauge field $\mathcal{A}(u)$ with respect to the source positions, we obtain

$$d_{z_i}\mathcal{A}(u) = -\mathrm{tr}\,[\eta_i, a_i]\,\frac{\partial}{\partial \eta_i}\,\mathcal{A}(u) = -D_{a_i}\mathcal{A}(u). \tag{23}$$

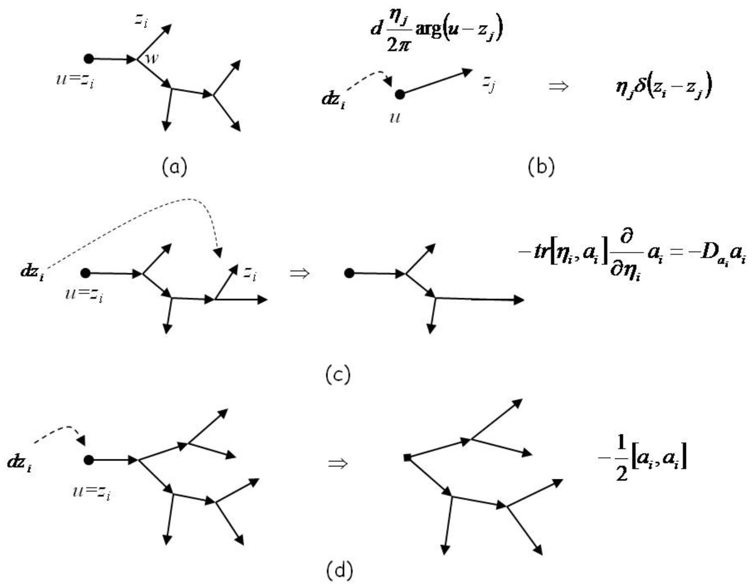

Figure 6. Graphic z_i differentiation of a_i.

Note that for $j \neq i$ we can replace $\mathcal{A}(u)$ by $\mathcal{A}^{reg}_{(j)}$. Then, putting $u = z_j$ yields

$$d_{z_i} a_j = -D_{a_i} a_j . \tag{24}$$

The curvature $\mathbb{F}$ of $\mathbb{A}$ is defined as

$$\mathbb{F} = d\mathbb{A} + \frac{1}{2}\{\mathbb{A}, \mathbb{A}\} . \tag{25}$$

The curvature has holomorphic, anti-holomorphic and mixed components). For $z_i \neq z_j$, $\mathbb{F}_{ij}$ has n components $(\mathbb{F}_{ij})_k$ for $k = 1, \ldots, n$. The components with $k \neq i, j$ vanish identically. For the remaining components, we have

$$(\mathbb{F}_{ij})_i = d_{z_j} a_i + D_{a_j} a_i = 0 , \qquad (\mathbb{F}_{ij})_j = d_{z_i} a_j + D_{a_i} a_j = 0. \tag{26}$$

The curvature F_{ii} has only one nonvanishing component

$$(\mathbb{F}_{ii})_i = d_{z_i} a_i + D_{a_i} a_i + \frac{1}{2}[a_i, a_i] . \tag{27}$$

In Figure 6 the diagrams which contribute to the differential $d_{z_i} a_i$ are shown. Graphs of type (a) vanish, as in the derivative of K_η. Graphs of types (b) and (c) generate source terms and covariant derivative terms. Graphs of type (d) account for an extra z_i dependence due to the root of the tree. The result is [12]

$$d_{z_i} a_i(z_i) + D_{a_i} a_i + \frac{1}{2}\,[a_i, a_i] = \sum_{j\neq i} \eta_j\, \delta(z_i - z_j)\,. \tag{28}$$

That is, away from the sources positions, the connection is flat

$$d\mathbb{A} + \frac{1}{2}\,\{\mathbb{A}, \mathbb{A}\} = 0. \tag{29}$$

With sources taken into account, we have $\mathbb{F} = (\mathbb{F}_1, \ldots, \mathbb{F}_n)$, where

$$\mathbb{F}_i = \sum_{j\neq i} \eta_j \delta(z_i - z_j). \tag{30}$$

4 Outlook

The Torossian connection discussed in Section 3 is a close relative of the Knizhnik-Zamolodchikov (KZ) connection in the WZW theory. Recall that the KZ connection describes correlators of primary fields, and that it has the form

$$d\Psi + \mathbb{A}_{KZ}\,\Psi = 0\,, \qquad \mathbb{A}_{KZ} = \frac{1}{2\pi i} \sum_{i,j} t_{i,j}\, d\ln(z_i - z_j), \tag{31}$$

where $t_{i,j} = \sum_a e_a^i \otimes e_a^j$ are operators acting on the product of irreducible representation of $\mathcal{G}$ carried by primary fields placed at the points $z_1, \ldots, z_n$. Note that operators $t_{i,j}$ play the role of one-edge trees, and the propagator has the form $d\ln(z_i - z_j)/2\pi i$.

The KZ connection admits the second interesting interpretation: one can view it as an equation on the wave function of the Chern-Simons topological field theory with n time-like Wilson lines [13]. From this perspective, holonomy matrices of the flat connection $\mathbb{A}_{KZ}$ correspond to braiding of Wilson lines in the Chern-Simons theory. Therefore it would be interesting to find a three-dimensional topological field theory which has the Torossian connection as an equation on the wave function.

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