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Kaluza–Klein Universe with Linearly Varying Deceleration Parameter

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Abstract. We have studied Kaluza–Klein space-time containing perfect fluid. The solutions of the field equations are obtained by applying the law of linearly varying deceleration parameter(LVDP) in general theory of relativity. The geometrical and physical aspects of the model are also studied.

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1 Introduction

The SNIa type supernovae observations, the large scale structures and the cosmic microwave background (CBM) radiations [1–7] have confirmed that the present universe is not only expanding but accelerating also. Cunha and Lima [8], Cunha [9] directly evidenced the cause for the present accelerating universe. Analyzing the sample of baryonic acoustic oscillation (BAO) with cosmic microwave background (CMB) radiation, Li *et al.* [10] concluded that such sample of BAO with CMB increases the present cosmic acceleration which has been further explained by plotting graphs for change of deceleration parameter q with red shift $z < 2$.

The study of higher dimensional theories has been revived and generalized after realizing the fact that many interesting theories of particle interactions need more than four dimensions for their consistent formulation. Thus, it is important to generalize the results obtained in four dimensional theory of gravitation in the framework of higher dimensions and look for the effects due to incorporation of extra dimensions in the theory. The concept of higher dimensional space-time is not unphysical. Weinberg has studied the unification of the fundamental forces with gravity which reveals that the space-time should be different from four [11]. The string theories are discussed in ten dimensions or twenty six dimensions of space-time. Because of this, studies in n-dimensions inspired many researchers to enter into such field of study to explore the hidden knowledge about the universe. Many researchers [12–22] have used this concept for studying the multidimensional cosmological models in particle physics and cosmology.

The theory of five dimensions is due to the idea of Kaluza [23] and Klein [24]. A five dimensional [5D] general relativity is the best outcome of an attempt made by these two by using one extra dimension to unify gravity and electromagnetism. Realistic unification through the Kaluza–Klein approach requires $d = 5$ manifold topology and the spatial extra dimension radius to be of Planck length order. As per Witten [25], the classical and quantum motion criteria indicate the presence of the fifth dimension as a compact extra dimension of Planck radius at each point of 4-dimensional space-time, confirming beyond any doubt the physical existence of fifth dimension as postulated by Klein [24]. Hence, the physical structure of the 5-dimensional space-time proposed by Kaluza–Klein unification is not anymore a mere theoretical assumption. According to Wesson [26, 27] and Bellini [28], the matter is induced in $4D$ by $5D$ vacuum theory for studying the cosmology of $5D$ with pure geometry in non-compact Kaluza–Klein theory. The Kaluza–Klein theory is essentially an extension of Einstein's general relativity in five dimensions which is of much interest in particle physics and cosmology.

Akarsu and Dereli [29] proposed a linearly varying deceleration parameter (LVDP) and obtained the accelerating cosmological solutions by considering the spatially homogeneous and isotropic Robertson–Walker (RW) space-time filled with perfect fluid in general relativity. This new LVDP includes the Berman's [30, 31] special law of variation for Hubble parameter which yields constant deceleration parameter (CDP) models of the universe as a special case. This generalization of CDP ansatz to LVDP ansatz found to be more consistent with the recent observations [9, 10]. This LVDP gives the opportunity to generalize most of the cosmological models which are earlier based on CDP, *i.e.*, using LVDP, one can generalize the cosmological solutions that have been obtained earlier *via* CDP. As per Akarsu and Dereli [29], the LVDP law can be used within the framework of spatially homogeneous but anisotropic Bianchi type space-times and Kantowski–Sachs space-time in the presence of isotropic and/or anisotropic fluid. For this, Akarsu and Dereli [29] have defined the mean scale factor, most generally, as

$$V = a^3 = (ABC)^{1/3},$$

where A, B, and C are the directional scale factors and then generalizing LVDP as

$$q = -\frac{\ddot{a}a}{\dot{a}^2} = -kt + m - 1\,, \qquad \text{where } k \geqslant 0,\ m \geqslant 0.$$

This law has been further extended to LRS Bianchi type-I and Bianchi type-V cosmological model containing perfect fluid with linearly varying deceleration parameter by Adhav [32, 33]. Recently Akarsu & Dereli [34] have compared the cosmological kinematics obtained in a law of linearly varying deceleration parameter (LVDP) with the kinematics obtained in the ΛCDM model.

In the present paper, we have considered a Kaluza–Klein cosmological model with linearly varying deceleration parameter in general relativity. The physical and geometrical aspects of the model are also discussed.

2 Metric and Field Equations

The Kaluza-Klein line element can be written as

$$ds^2 = dt^2 - A^2(dx^2 + dy^2 + dz^2) - B^2 d\psi^2 \,, \tag{2.1}$$

where $A(t)$ and $B(t)$ are the scale factors (metric tensor) and functions of the cosmic time t only (non-static case). Here the extra coordinate ψ is taken to be space-like.

The Einstein field equations are ($8\pi G = 1$ and $c = 1$)

$$R_{ij} - \frac{1}{2} g_{ij} R = -{}^{m}T_{ij} \,, \tag{2.2}$$

where ${}^{m}T_{ij}$ is the matter tensor of Einstein theory (perfect fluid) and is given by

$${}^{m}T^{i}_{j} = diag(\rho, -p, -p, -p), \tag{2.3}$$

where the energy density ρ is related to the pressure p by the equation of state $p = \gamma\rho$. Here γ varies between the interval $0 \leqslant \gamma \leqslant 1$, whereas $\gamma = 0$ describes the dust universe, $\gamma = 1/3$ presents radiation universe, $1/3 < \gamma < 1$ ascribes hard universe and $\gamma = 1$ corresponds to the stiff matter.

The Einstein's field equations (2.2) for metric (2.1) with the help of equations (2.3) can be written as

$$3\left(\frac{\dot{A}}{A}\right)^2 + 3\frac{\dot{A}\dot{B}}{AB} = \rho \,, \tag{2.4}$$

$$2\frac{\ddot{A}}{A} + \frac{\ddot{B}}{B} + \left(\frac{\dot{A}}{A}\right)^2 + 2\frac{\dot{A}\dot{B}}{AB} = -p \,, \tag{2.5}$$

$$3\frac{\ddot{A}}{A} + 3\left(\frac{\dot{A}}{A}\right)^2 = -p \,, \tag{2.6}$$

where dot $(\cdot)$ indicates the derivative with respect to t.

3 Solutions of the Field Equations

There are three linearly independent equations (2.4)–(2.6) with four unknowns A, B, ρ, and p. In order to solve the system completely we impose a linearly varying deceleration parameter proposed in [29] as

$$q = -\frac{\ddot{a}a}{\dot{a}^2} = -kt + m - 1\,, \tag{3.1}$$

where a is the mean scale factor of the universe, $k \geqslant 0$ and $m \geqslant 0$ are constants.

After solving equation (3.1) one can obtain the three different forms of the mean scale factor as

$$a = a_0 e^{\frac{2}{\sqrt{m^2-2c_1k}} \operatorname{arctanh}\left(\frac{kt-m}{\sqrt{m^2-2c_1k}}\right)} \qquad \text{for } k > 0 \text{ and } m \geqslant 0 \tag{3.2}$$

$$a = a_0 (mt + c_2)^{1/m} \qquad \text{for } k = 0 \text{ and } m > 0, \tag{3.3}$$

$$a = a_0 e^{c_3 t} \qquad \text{for } k = 0 \text{ and } m = 0. \tag{3.4}$$

The equation (3.3) and equation (3.4) give cosmological models with constant deceleration parameter which has been studied by Adhav *et al*. [35] in Kaluza–Klein space-time for anisotropic fluid.

Hence here we only study the cosmological model for equation (3.2).

We define the spatial volume of the universe as

$$V = A^3 B. \tag{3.5}$$

We define $a = (A^3 B)^{1/4}$ as the average scale factor so that the mean Hubble parameter in the anisotropic model may be defined as

$$H = \frac{\dot{a}}{a} = \frac{1}{4}\sum_{i=1}^{4} H_i = \frac{1}{4}\Big(3\frac{\dot{A}}{A} + \frac{\dot{B}}{B}\Big), \tag{3.6}$$

where the directional Hubble parameters H_i $(i = 1, 2, 3, 4)$ in the directions x, y, z and ψ respectively are defined as

$$H_x = H_y = H_z = \frac{\dot{A}}{A} \qquad \text{and} \qquad H_\psi = \frac{\dot{B}}{B}. \tag{3.7}$$

Subtracting equation (2.5) from equation (2.6), we get

$$\frac{d}{dt}\Big(\frac{\dot{A}}{A} - \frac{\dot{B}}{B}\Big) + \Big(\frac{\dot{A}}{A} - \frac{\dot{B}}{B}\Big)\Big(3\frac{\dot{A}}{A} + \frac{\dot{B}}{B}\Big) = 0\,. \tag{3.8}$$

Now, from equations (3.5) and (3.8), we get

$$\frac{d}{dt}\Big(\frac{\dot{A}}{A} - \frac{\dot{B}}{B}\Big) + \Big(\frac{\dot{A}}{A} - \frac{\dot{B}}{B}\Big)\frac{\dot{V}}{V} = 0,.$$

On integrating the above equation, we get

$$\frac{A}{B} = d \exp\Big(x \int \frac{dt}{V}\Big), \qquad d = \text{const}, \;\; x = \text{cons}. \tag{3.9}$$

Using equation (3.9), the values of scale factors A and B can be written explicitly as

$$A = D_1 V^{1/4} \exp\Big(X_1 \int \frac{dt}{V}\Big), \tag{3.10}$$

$$B = D_2 V^{1/4} \exp\Big(X_2 \int \frac{dt}{V}\Big), \tag{3.11}$$

where the relation $D_1^3 D_2 = 1$ and $3X_1 + X_2 = 0$ are satisfied by D_1, D_2 and X_1, X_2 and $D_1 = d^{1/4}$, $D_2 = d^{-3/4}$, $X_1 = x/4$, $X_2 = -3x/4$.

Using equation (3.2) for $c_1 = 0$ in equation (3.5), we get

$$V = a^4 = a_0^4 \exp\left[\frac{8}{m}\text{arctanh}\left(\frac{kt}{m} - 1\right)\right]. \tag{3.12}$$

Using equation (3.12) in equations (3.10) and (3.11), we obtain the exact value of the scale factor as

$$\begin{aligned} A(t) = D_1 a_0 e^{\frac{2}{m}\text{arctanh}(\frac{kt}{m}-1)} \exp \frac{X_1}{a_0^4}\Bigg\{ & \frac{-1}{k(m+4)} e^{\frac{8}{m}\text{arctanh}(1-\frac{kt}{m})} \\ & \times \Big[m(m+4)\,{}_2F_1\left(1; \frac{4}{m}; \frac{m+4}{m}; -e^{2\text{arctanh}(1-\frac{kt}{m})}\right) \\ & - 4me^{2\text{arctanh}(1-\frac{kt}{m})}\,{}_2F_1\left(1; \frac{m+4}{m}; \frac{4}{m}+2; -e^{2\,\text{arctan}\,\text{h}(1-\frac{kt}{m})}\right) \\ & + (m+4)(m-kt)\Big]\Bigg\}, \end{aligned} \tag{3.13}$$

$$\begin{aligned} B(t) = D_2 a_0 e^{\frac{2}{m}\text{arctanh}(\frac{kt}{m}-1)} \exp \frac{X_2}{a_0^4}\Bigg\{ & \frac{-1}{k(m+4)} e^{\frac{8}{m}\text{arctanh}(1-\frac{kt}{m})} \\ & \times \Big[m(m+4)\,{}_2F_1\left(1; \frac{4}{m}; \frac{m+4}{m}; -e^{2\,\text{arctan}\,\text{h}(1-\frac{kt}{m})}\right) \\ & - 4me^{2\text{arctanh}(1-\frac{kt}{m})}\,{}_2F_1\left(1; \frac{m+4}{m}; \frac{4}{m}+2; -e^{2\,\text{arctan}\,\text{h}(1-\frac{kt}{m})}\right) \\ & + (m+4)(m-kt)\Big]\Bigg\}, \end{aligned} \tag{3.14}$$

where ${}_2F_1(a, b; c; t)$ is the hypergeometric function.

Using equations (3.13) and (3.14) the directional Hubble parameters are found as

$$H_x = H_y = H_z = \frac{-2}{(kt^2 - 2mt)} + \frac{X_1}{[a_0^4 e^{\frac{8}{m}\text{arctanh}(\frac{kt}{m}-1)}]}, \tag{3.15}$$

$$H_\psi = \frac{-2}{(kt^2 - 2mt)} + \frac{X_2}{[a_0^4 e^{\frac{8}{m}\text{arctanh}(\frac{kt}{m}-1)}]}. \tag{3.16}$$

Using equations (3.13) (3.14) and (2.4), we obtain the energy density as

$$\rho = 3\Big\{\frac{8}{(kt^2-2mt)^2} + \frac{X}{[a_0^8 e^{\frac{16}{m}\text{arctanh}(\frac{kt}{m}-1)}]}\Big\}, \tag{3.17}$$

where $X = X_1^2 + X_1X_2$.

Adding three times equation (2.5), (2.6) and 4 times equation (2.4), we get

$$3\frac{\ddot{A}}{A} + \frac{\ddot{B}}{B} + 6\Big(\frac{\dot{A}}{A}\Big)^2 + 6\frac{\dot{A}\dot{B}}{AB} = \frac{4}{3}(\rho - p). \tag{3.18}$$

From equation (3.5), we have

$$\frac{\ddot{V}}{V} = 3\frac{\ddot{A}}{A} + \frac{\ddot{B}}{B} + 6\Big(\frac{\dot{A}}{A}\Big)^2 + 6\frac{\dot{A}\dot{B}}{AB}. \tag{3.19}$$

From equations (3.18) and (3.19), we get

$$\frac{\ddot{V}}{V} = \frac{4}{3}(\rho - p). \tag{3.20}$$

Solving equation (3.20) with the use of equation (3.17), the isotropic pressure is found as

$$p = 3\Big\{\frac{4(-kt+m-2)}{(kt^2-2mt)^2} + \frac{X}{[a_0^8 e^{\frac{16}{m}\text{arctanh}(\frac{kt}{m}-1)}]}\Big\}. \tag{3.21}$$

The barotropic parameter $\gamma = p/\rho$ is given by

$$\gamma = \frac{\Big[4a_0^8 e^{\frac{16}{m}\text{arctanh}(\frac{kt}{m}-1)}(-kt+m-2) + Xt^2(kt-2m)^2\Big]}{\Big[8a_0^8 e^{\frac{16}{m}\text{arctanh}(\frac{kt}{m}-1)} + Xt^2(kt-2m)^2\Big]}. \tag{3.22}$$

The anisotropy parameter of the expansion

$$\Delta = \frac{1}{4}\sum_{i=1}^{4}\left(\frac{H_i - H}{H}\right)^2$$

is obtained as

$$\Delta = \frac{(3X_1^2 + X_2^2).t^2(kt-2m)^2}{16a_0^8 e^{\frac{16}{m}\text{arctanh}(\frac{kt}{m}-1)}}. \tag{3.23}$$

The shear scalar σ^2, defined by

$$\sigma^2 = \frac{1}{2}\left(\sum_{i=1}^{4} H_i^2 - 4H^2\right) = \frac{4}{2}\Delta H^2,$$

is obtained as

$$\sigma^2 = \frac{(3X_1^2 + X_2^2)t^2}{2a_0^8 e^{\frac{16}{m}\text{arctanh}(\frac{kt}{m}-1)}}. \tag{3.24}$$

4 Conclusion

The Kaluza–Klein cosmological model with linearly varying deceleration parameter has been studied. Here the deceleration parameter is linear in time with a negative slope. In this model, the universe has finite lifetime. It starts with a big bang at initial time $t_i = 0$ and ends at end time $t_{\text{end}} = 2m/k$. The energy density ρ, the pressure p and the scale factors A, B diverge in finite time as $t \to t_{\text{end}}$ $(= 2m/k)$. This is called as big rip [36]. From equation (3.1) we get that the universe begins with $q_i = m - 1$, enters into the accelerating phase ($q < 0$) at $t > (m-1)/k$ ($m > 1$), further enters into super-exponential expansion phase $q < -1$ at $t > m/k$ and ends with $q_{\text{end}} = -m - 1$.

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