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Thermo-Field Dynamics of Higher-Derivative Oscillators*

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Abstract. The higher-derivative oscillator proposed by Pais and Uhlenbeck has a well-defined Hamiltonian formulation in terms of a system of harmonic oscillators, which allows consistent quantum treatment. In this report we consider a system of interacting Pais-Uhlenbeck oscillators and calculate their entanglement entropy using the framework of thermo-field dynamics. We also make connection with information theory via the Fisher information metric and with the AdS/CFT correspondence via the supergravity solution of Pilch and Warner.

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1 Introduction

The interest in higher-derivative theories is motivated by the search for a quantum field theory, which is ultraviolet (UV) finite [1]. This idea is applicable in gravitational context because higher-powers-of-curvature terms in the Einstein action lead to a renormalisable theory [2]. However, the Hamiltonian approach to such theories is limited to the Ostrogradsky's approach [3], which gives unbounded from below Hamiltonian hence negative energies. Alternatively, the last is equivalent to instabilities still at classical level and negative-norm states (or ghosts) after quantisation.

One of the simplest toy models for higher-derivative theories is the Pais-Uhlenbeck oscillator (PUO) [4]. Since the dynamical degrees of freedom obey constraints due to the presence of higher derivatives, the quantisation of the model requires the method of Dirac constraints [5] or path integral formalism [6]. Fortunately, the PUO was recently equipped with a few well-defined Hamiltonian formulations [7–9]. Amongst them, the most convenient for our

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purposes is the representation if terms of harmonic oscillators [9]. Consequently, one may use the framework of thermo-field dynamics (TFD) [10] in order to investigate different quantum properties of systems of interacting PUOs at (non-)equilibrium and at finite temperatures.

The structure of this report is the following. In section 2 we consider a ring of N Pais-Uhlenbeck oscillators and compute the entanglement entropy of one of them in regard to the others. In section 3 we make connection with information geometry and the AdS/CFT correspondence. In the concluding section 4 we briefly discuss our results.

2 Ring of Pais-Uhlenbeck Oscillators

We start our journey by considering a ring of N fourth-order Pais-Uhlenbeck oscillators (Figure 1), each of them interacting only with the nearest neighbours [11]. The generalisation to higher than fourth-order oscillators is straightforward. Since any fourth-order PUO has Hamiltonian representation in terms of two harmonic oscillators, the equivalent system of $2N$ harmonic oscillators contains effective next-to-nearest neighbours interaction (dashed red lines in Figure 1). The system is effectively one-dimensional and its dynamics is governed

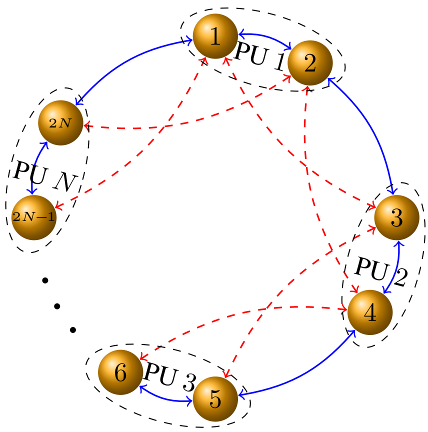

Figure 1. Closed chain of N identical PU oscillators.

by the following Hamiltonian consisting of free and interacting parts:

$$H_N = \frac{1}{2}\sum_{\mu=1}^{N}\sum_{k=0}^{1} \operatorname{sgn}(\alpha_{\mu,k})\left(p_\mu^k p_\mu^k + \omega_{\mu,k}^2 x_\mu^k x_\mu^k\right) + \frac{1}{2}\sum_{\langle\mu,\nu\rangle=1}^{N} c_{\mu\nu} x_\mu x_\nu, \quad (1)$$

where x_μ are the coordinates of the PUOs, and x_μ^k and p_μ^k are the coordinates and momenta of the corresponding harmonic oscillators. The interaction strength between the PUOs is set by the constants $c_{\mu\nu}$ and $\omega_{\mu,k}$ are the frequencies of the harmonic oscillators. The appearance of the arbitrary non-zero constants $\alpha_{\mu,k}$ in front of the free oscillators part of eq. (1) is inherited from the alternative Hamiltonian formulation[2]. From now on, we will consider only positive values of $\alpha_{\mu,k}$, which assures positive definiteness of the Hamiltonian. The diagonalisation of eq. (1) boils down to diagonalisation of certain type of circulant matrices, namely symmetric block circulant matrices with symmetric blocks, which are diagonalised by a discrete Fourier transform. This calculation is performed in detail in [11] and results in a Hamiltonian of $2N$ free harmonic oscillators with frequencies λ_j, $j = 1, \ldots, 2N$, which depend on the parameters $c_{\mu\nu}$, $\omega_{\mu,k}$, and $\alpha_{\mu,k}$ of the original system in a highly non-trivial way. Having computed the frequencies, one can define creation and annihilation operators $\mathfrak{a}_j^\dagger$ and $\mathfrak{a}_j$, $j = 1, \ldots, 2N$, and write the Hamiltonian as

$$H_N = \sum_{j=1}^{2N} \hbar\lambda_j \left(\mathfrak{a}_j^\dagger \mathfrak{a}_j + \frac{1}{2} \right). \tag{2}$$

The Fock space $|\{n_j\}\rangle = |n_1\rangle \otimes \cdots \otimes |n_{2N}\rangle$ is built from the vacuum $|0\rangle$ by acting with creation operators on it. We are now in a position to apply thermo-field dynamics to our problem. The TFD explores two isomorphic Hilbert spaces—$|n\rangle$ and its identical copy $|\tilde{m}\rangle$—forming the double Hilbert space $|n\rangle \otimes |\tilde{m}\rangle \equiv |n\rangle|\tilde{m}\rangle \equiv |n, \tilde{m}\rangle$. Here the states from the auxiliary Hilbert space play the role of pursuers of the initial quantum states. In this set-up one can calculate the relevant statistical quantities, schematically shown bellow:

$$\begin{aligned} \rho_{\rm eq} = e^{-\beta H_N}/Z \;\longrightarrow\; |\Psi\rangle = \sum_n \sqrt{\rho_{\rm eq}}|n\rangle|\tilde{n}\rangle \;\longrightarrow\; \widehat{\rho} = |\Psi\rangle\langle\Psi| \;\longrightarrow \\ \widehat{\rho}_{1,2} = {\rm Tr}_{3,\ldots,2N}\widehat{\rho} \;\longrightarrow\; \widehat{S}_{1,2} = -k_{\rm B}{\rm Tr}_{1,2}\left[\widehat{\rho}_{1,2}\log\widehat{\rho}_{1,2}\right], \end{aligned} \tag{3}$$

where $Z(K_j) = {\rm Tr}_{\{j\}} e^{-\beta H_N}$ is the statistical sum, $K_j = \hbar\beta\lambda_j$, and β is the inverse temperature. The quantities in eq. (3) are as follows: the standard density matrix in equilibrium $\rho_{\rm eq}(K_j)$, the statistical state $|\Psi(K_j)\rangle$, the extended density matrix $\widehat{\rho}(K_j)$, and the renormalised extended density matrix $\widehat{\rho}_{1,2}(K_1, K_2)$. After some tedious but straightforward calculations we obtain for the extended entanglement entropy the expression [11]:

$$\begin{aligned} \widehat{S}_{1,2}(K_1, K_2) = \frac{k_{\rm B}}{2} \coth\frac{K_1}{4} \coth\frac{K_2}{4} \Bigg[K_1 \left(1 + \coth\frac{K_1}{4} \right) \\ + K_2 \left(1 + \coth\frac{K_2}{4} \right) - 2\log\left[\left(e^{K_1} - 1\right)\left(e^{K_2} - 1\right) \right] \Bigg]. \end{aligned} \tag{4}$$

For large values of the temperature (equivalently small values of K_1 and K_2) the entanglement entropy goes to infinity (corresponding to $K_1 = 0$ and/or

[2]For further details on this issue, please refer to [9].

$K_2 = 0$). Conversely, the EE approaches zero when the temperature approaches zero (corresponding to $K_1 = 0$ and/or $K_2 = 0$ going to infinity). The rate of increasing and decreasing of the EE is specified by the frequencies λ_1 and λ_2. Therefore the renormalised density matrix is thermal state and the extended entanglement entropy satisfies the Nernst heat theorem.

3 Information Geometry and the AdS/CFT Correspondence

In this section we will consider two particular examples of systems appearing in the so-called Pilch-Warner (PW) background [12,13], in which higher-derivative oscillators arise naturally. The PW geometry is a solution to five-dimensional $\mathcal{N} = 8$ supergravity lifted to ten dimensions, whose infrared (IR) point is holographically dual to the large N limit of the $\mathcal{N} = 1$ superconformal theory of Leigh-Strassler [14]. The first example is the Penrose limit of the PW solution in the presence of Kalb-Ramond B-field [15]. The limit is taken along a null geodesic, which corresponds to the moduli space of a probe D3-brane. We will consider only the bosonic part of the world-sheet action including the metric and the NS-NS antisymmetric two-form B_2, since it is relevant for our conclusions. The ansatz $X^i(\tau, \sigma) = e^{i\sigma} x_i(\tau),\ i = 1, \ldots, 8$, for the embedding coordinates splits the variables in the system of partial differential equations describing the dynamics of the world-sheet scalars X^i [15], and leads to the following system of ordinary differential equations [11]:

$$
\begin{aligned}
& x_q^{(4)} + \left(5M^2 + 2\right) x_q^{(2)} + \left(4M^4 + 2M^2 + 1\right) x_q = 0, \\
& x_p^{(2)} + \left(M^2 + 1\right) x_p = 0,
\end{aligned} \tag{5}
$$

where $q = 1, 2$ labels the directions affected by the B-field, $p = 5, 6, 7, 8$ marks the directions unaffected by the B-field, and the quantity $M = E\alpha' p^+$ is expressed in terms of the conserved energy corresponding to the Killing vector $\partial/\partial\tau$. The derivatives in eq. (5) are taken with respect to the world-sheet time τ. Hence, we conclude that the presence of B-field (which is analog of magnetic field) ties the equations in such way that the dynamics is governed by two fourth-order PUO equations.

The second example is the quadratic fluctuations around classical solutions of rotating strings in PW background considered in [16]. The experience of the previous example hints that the Kalb-Ramond field will play crucial roll in this case too. For this reason we will consider only the Lagrangian of the quadratic fluctuations in the five-sphere part of the geometry wherein the B-field is turned on. Using similar ansatz as before, i.e. $\tilde{\zeta}_{\tilde{i}} = e^{i\sigma} y_i(\tau)$, we reduce the system of five partial differential equations of motion for the quadratic fluctuations obtained in [16] to the following system of ordinary differential equations [11]:

$$
\begin{aligned}
& y_p^{(4)} + \left[5\tilde{M}^2 + 2 + \frac{3}{2}\bar{\rho}'^2\right] y_p^{(2)} \\
& \quad + \left[4\tilde{M}^4 + 2\tilde{M}^2 + 1 + \left(\tilde{M}^2 + 1\right)\frac{3}{2}\bar{\rho}'^2\right] y_p = 0, \\
& y_1^{(2)} + y_1 = 0,
\end{aligned} \tag{6}
$$

where $p = 2, 4$ and the constant $\tilde{M}^2 = \frac{4}{9}(c_\beta + c_\gamma + c_\phi)^2$ is expressed in terms of the angular velocities c_β, c_γ, and c_ϕ of the rotating string. In general $\bar{\rho}'$ in eq. (6) is σ-dependent. However, in the two limiting cases of short and long strings $\bar{\rho}'$ takes particular constant values and we again obtain the familiar two fourth-order PUO equations of motion.

Finally, we would like to relate our results to the contemporary theory of information geometry. The arguments K_1 and K_2 of the entanglement entropy in a natural manner span a parameter space equipped with Riemannian metric. This metric is known as Fisher information metric and can be derived from the entanglement entropy using the formula $g_{\mu\nu}(K_1, K_2) = \partial_\mu \partial_\nu S(K_1, K_2)$, where $\partial_\mu = \partial / \partial K_\mu$ for $\mu = 1, 2$. For the entanglement entropy given by eq. (4) the Fisher metric takes the form:

$$
\begin{aligned}
g_{11} = & \frac{1}{64} k_{\mathrm{B}} \coth\frac{K_2}{4} \operatorname{csch}^2\frac{K_1}{4} \left[K_1 \left(3 + 5\coth^2\frac{K_1}{4} + 7\operatorname{csch}^2\frac{K_1}{4}\right)\right. \\
& +4\tanh\frac{K_1}{4} + 4\coth\frac{K_1}{4}\left(K_1 + K_2 - 5 + K_2\coth\frac{K_2}{4}\right. \\
& \left.\left. -2\log\left[\left(e^{K_1} - 1\right)\left(e^{K_2} - 1\right)\right]\right)\right],
\end{aligned} \tag{7}
$$

$$
\begin{aligned}
g_{12} = & \frac{1}{32} k_{\mathrm{B}} \operatorname{csch}^2\frac{K_1}{4} \operatorname{csch}^2\frac{K_2}{4} \left[-4 + K_1\left(1 + 2\coth\frac{K_1}{4}\right)\right. \\
& \left. +K_2\left(1 + 2\coth\frac{K_2}{4}\right) - 2\log\left[\left(e^{K_1} - 1\right)\left(e^{K_2} - 1\right)\right]\right].
\end{aligned} \tag{8}
$$

The component g_{22} is obtained from g_{11} by exchanging K_1 and K_2, and $g_{21} = g_{12}$. At the end of the day we emphasise the importance of the knowledge we can gain by exploring specific corners of the information space of the Pais-Uhlenbeck oscillators. We hope that we will report on this issue soon.

4 Conclusion

We considered a system (ring) of N fourth-order Pais-Uhlenbeck oscillators, any of which interacts with the nearest neighbours. The existence of an alternative Hamiltonian overcomes the problem of Ostrogradsky's instabilities and the consequent ghost problem occurring after quantisation. The diagonalised Hamiltonian is equivalent to the one of a system of $2N$ harmonic oscillators with modified frequencies depending on the parameters of the initial system of PUOs and can be quantised canonically. Then we introduced thermo-field dynamics and calculated the extended entanglement entropy in this framework. In section 3 we showed the appearance of the fourth-order PUO in the holographic Pilch-Warner solution for two particular cases—the Penrose limit of the geometry and the quadratic fluctuations around classical solutions of rotating strings. Our conclusion is that the TFD is a convenient tool for studying quantum entanglement of higher-derivative oscillators and that the Fisher information metric could provide useful knowledge about the parametric space structure.

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