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On Pseudo Concircularly Symmetric Spacetimes

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Abstract. In the present paper we study pseudo concircularly symmetric spacetimes.

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1 Introduction

The present paper is concerned with certain investigations in general relativity by the coordinate free method of differential geometry. In this method of study spacetime of general relativity is regarded as a connected four dimensional semi-Riemannian manifold (M^4, g) with Lorentzian metric g with signature $(-, +, +, +)$. The geometry of the Lorentzian manifold begins with the study of the causal character of vectors of the manifold. It is due to this causality that the Lorentzian manifold becomes a convenient choice for the study of general relativity. Spacetime of general relativity have been studied by different authors in different ways such as Chaki and Ray [4], De and Mallick [7], Mantica and Suh [8], Özen [10] and many others. The Einstein equations ([9], p. 337), imply that the energy-momentum tensor is of vanishing divergence. This requirement is satisfied if the energy-momentum tensor is covariant-constant [4]. In the paper [4] M.C. Chaki and Sarbari Ray showed that a general relativistic spacetime with covariant-constant energy-momentum tensor is Ricci symmetric, that is, $\nabla S = 0$, where S is the Ricci tensor of the spacetime.

In general, a geodesic circle (a curve whose first curvature is constant and second curvature is identically zero) does not transform into a geodesic circle by the conformal transformation

$$\tilde{g}_{ij} = \psi^2 g_{ij}, \tag{1}$$

of the fundamental tensor g_{ij}. The transformation which preserves geodesic circles was first introduced by Yano [14]. The conformal transformation (1) satisfying the partial differential equation

$$\psi;_{i;j} = \phi g_{ij}, \tag{2}$$

changes a geodesic circle into a geodesic circle. Such a transformation is known as the concircular transformation and the geometry which deals with such transformation is called the concircular geometry [14].

A $(1,3)$ type tensor $\tilde{C}(X,Y)Z$ which remains invariant under concircular transformation, for an n-dimensional Riemannian or semi-Riemannian manifold M^n, is given by Yano and Kon [13, 15]

$$\tilde{C}(X,Y)Z = R(X,Y)Z - \frac{r}{n(n-1)}[g(Y,Z)X - g(X,Z)Y], \quad (3)$$

where R is the Riemannian curvature tensor and r, the scalar curvature.

The importance of concircular transformation and concircular curvature tensor is very well known in the differential geometry of certain F-structure such as complex, almost complex, Kahler, almost Kahler, contact and almost contact structure etc. [3, 12, 15]. In a recent paper Ahsan and Siddiqui [2] studied the application of concircular curvature tensor in fluid spacetime.

In the present paper we consider a special type of spacetime which is called pseudo concircularly symmetric spacetime. The notion of pseudo concircularly symmetric manifold was introduced by De and Tarafdar [6]. A non-flat semi-Riemannian manifold (M^n, g) is called pseudo concircularly symmetric manifold if its concircular curvature tensor satisfies the condition

$$\begin{aligned}(\nabla_X \tilde{C})(Y,Z,W) &= 2A(X)\tilde{C}(Y,Z)W + A(Y)\tilde{C}(X,Z)W \\ &\quad + A(Z)\tilde{C}(Y,X)W + A(W)\tilde{C}(Y,Z)X + g(\tilde{C}(Y,Z)W,X)\rho, \quad (4)\end{aligned}$$

where A is a non-zero 1-form, $g(X,\rho) = A(X)$, for every vector field X and ∇ denotes the operator of covariant differentiation with respect to the metric g. Such an n-dimensional manifold shall be denoted by $(P\tilde{C}S)_n$.

The present paper is organized as follows: After preliminaries, in Section 3, we study perfect fluid $(P\tilde{C}S)_4$. In Section 4 we shown that the spacetimes under consideration do not admit heat flux. Finally, we shown that if in an Einstein $(P\tilde{C}S)_4$ spacetime the vector field ρ is a unit torse-forming vector field, then the integral curves of the vector field ρ are geodesics.

2 Preliminaries

From (3) we have

$$\begin{aligned}\tilde{C}^\star(X,Y,Z,W) &= R^\star(X,Y,Z,W) \\ &\quad - \frac{r}{n(n-1)}[g(X,W)g(Y,Z) - g(X,Z)g(Y,W)], \quad (5)\end{aligned}$$

where

$$\tilde{C}^\star(X,Y,Z,W) = g(\tilde{C}(X,Y)Z,W)] \qquad \text{and}$$
$$R^\star(X,Y,Z,W) = g(R(X,Y)Z,W).$$

Let

$$P(X,W) = \tilde{C}^\star(X,e_i,e_i,W),$$

where $\{e_i\}$, $(i = 1,\ 2,\ 3,\dots,n)$ is an orthonormal basis of the tangent space at each point and i is sumed $1 \le i \le n$.

Then using (5) we have

$$P(X,W) = S(X,W) - \frac{r}{n}g(X,W), \tag{6}$$

where S is the Ricci tensor.

Let l and L be respectively the symmetric endomorphisms of the tangent space at each point corresponding to the tensor P and S. Then $g(lX,Y) = P(X,Y$ and $g(LX,Y) = S(X,Y)$. Hence

$$lX = \tilde{C}(X,e_i,e_i), \tag{7}$$

and

$$LX = R(X,e_i,e_i). \tag{8}$$

Also from (6) we obtain

$$A(lX) = A(LX) - \frac{r}{n}A(X). \tag{9}$$

Now differentiating (3) covariantly we get

$$(\nabla_U\tilde{C})(X,Y,Z) = (\nabla_U R)(X,Y,Z) - \frac{dr(U)}{n(n-1)}[g(Y,Z)X - g(X,Z)Y]. \tag{10}$$

On contraction of (10) gives

$$\begin{aligned}(div\tilde{C})(X,Y,Z) &= (\nabla_X S)(Y,Z) - (\nabla_Y S)(X,Z)\\ &\quad - \frac{1}{n(n-1)}[dr(X)g(Y,Z) - dr(Y)g(X,Z)],\end{aligned} \tag{11}$$

where 'div' denotes divergence.

Again contracting (4) and writing X, Y and Z for Y, Z and W, we obtain

$$\begin{aligned}(div\tilde{C})(X,Y,Z) &= 3A(\tilde{C}(X,Y)Z) + A(X)[S(Y,Z) - \frac{r}{n}g(Y,Z)]\\ &\quad + A(Y)[-S(X,Z) + \frac{r}{n}g(X,Z)].\end{aligned} \tag{12}$$

From (11) and (12) it follows that:

$$\begin{aligned}(\nabla_X S)(Y,Z) - (\nabla_Y S)(X,Z) - \frac{1}{n(n-1)}[dr(X)g(Y,Z) - dr(Y)g(X,Z)] \\ = 3A(\tilde{C}(X,Y)Z) + A(X)[S(Y,Z) - \frac{r}{n}g(Y,Z)] \\ + A(Y)[-S(X,Z) + \frac{r}{n}g(X,Z)]. \quad (13)\end{aligned}$$

Putting $Y = Z = e_i$ in (13) and taking summation over i, $1 \leq i \leq n$ we get

$$\frac{n-2}{2n}dr(X) = 3A(lX) - A(LX) + \frac{r}{n}A(X), \quad (14)$$

where $g(e_i, e_i) = \epsilon_i = \pm 1$. Now equations (9) and (14) together yield

$$dr(X) = \frac{4n}{n-2}A(lX). \quad (15)$$

3 Perfect Fluid $(P\tilde{C}S)_4$ Spacetime

In this section we consider a perfect fluid $(P\tilde{C}S)_4$ spacetime of non-zero constant scalar curvature with the basic vector field ρ of the $(P\tilde{C}S)_4$ as the velocity vector field of the fluid. Soon after obtaining the field equations of general relativity, Einstein applied these equations to find a model of universe. The universe on a large scale shows isotropy and homogeneity and the matter contents of the universe (stars, galaxies, nebulas, etc.) can be assumed to be that of a perfect fluid. We take Einstein's field equation without cosmological constant. Then Einstein's equation can be expressed as

$$S(X,Y) - \frac{r}{2}g(X,Y) = kT(X,Y), \quad (16)$$

where S and r denote the Ricci tensor and the scalar curvature respectively and T is the energy-momentum tensor.

The energy momentum tensor $T(X,Y)$ of a perfect fluid is given by [9]

$$T(X,Y) = (\mu + p)A(X)A(Y) + pg(X,Y), \quad (17)$$

where μ is the energy density, p the isotropic pressure, $g(X,\rho) = A(X)$ and ρ is a unit timelike vector field.

Since the scalar curvature is non-zero constant, from (15) we have $A(lX) = 0$. Hence from (14) we obtain

$$S(X,\rho) = \frac{r}{4}A(X). \quad (18)$$

Since ρ is the flow vector field of the perfect fluid under consideration and $g(X,\rho) = A(X)$, equation (16) can be written as follows:

$$S(X,Y) - \frac{r}{2}g(X,Y) = \kappa[(\mu + p)A(X)A(Y) + pg(X,Y)]. \tag{19}$$

Putting $Y = \rho$ in (19) and using the relation $g(\rho,\rho) = -1$, we obtain

$$S(X,\rho) = (\frac{r}{2} - \kappa\mu)A(X). \tag{20}$$

From equations (18) and (20) we have

$$\mu = \frac{r}{4\kappa}. \tag{21}$$

Since r is constant and κ is also constant, it follows from (21) that μ is constant. Now taking frame field and contracting (19) over X and Y, we obtain

$$r = -\kappa(3p - \mu). \tag{22}$$

Using (21) and (22) we have

$$p + \mu = 0. \tag{23}$$

Since μ is constant, from (23) it follows that p is also constant.

It is known that the equation $divT = 0$ implies the following for a perfect fluid [9]:

$$\rho\mu = -(\mu + p)div\rho. \ (energy\ equation) \tag{24}$$

$$(\mu + p)\nabla_\rho\rho = -grad\, p - (\rho p)\rho. \ (force\ equation) \tag{25}$$

Since in this case both μ and p are constant, it follows from (24) and (25) that $div\rho = 0$ and $\nabla_\rho\rho = 0$. But $div\rho$ represents the expansion scalar and $\nabla_\rho\rho$ represents the acceleration vector. Thus in this case both the expansion scalar and the acceleration vector are zero.

Also $\mu + p = 0$ means the fluid behave as a cosmological constant [11]. This is also turmed as Phantom Barrier [5]. Now in a cosmology we know such a choice $\mu = -p$ leads to rapid expansion of the spacetime which is now termed as inflation [1].

Thus from the above discussion we can state the following theorem:

Theorem 1. *If in a $(P\tilde{C}S)_4$ spacetime of non-zero constant scalar curvature the matter content is a perfect fluid whose velocity vector field is the basic vector field of the $(P\tilde{C}S)_4$, then the spacetime represents inflation. In this case the spacetime has vanishing acceleration vector and expansion scalar and the fluid behaves as a cosmological constant. This is also termed as a Phantom Barrier.*

4 Possibility of a Fluid $(P\tilde{C}S)_4$ Spacetime to Admit Heat Flux

In this section we shall give an answer to the following question: If in a $(P\tilde{C}S)_4$ spacetime of non-zero constant scalar curvature the matter distribution is a fluid with the basic vector field of $(P\tilde{C}S)_4$ as the velocity vector field of the fluid, can this distribution be described by the following form of the energy momentum tensor?

$$T(X,Y) = (\mu + p)A(X)A(Y) + pg(X,Y) \\ + A(X)B(Y) + B(X)A(Y), \quad (26)$$

where $g(X,\zeta) = B(X)$ for all X, ζ being the heat flux vector field.

Then $g(\rho,\zeta) = 0$ that is, $B(\rho) = 0$.

If possible let $T(X,Y)$ be of the form (26). Then Einstein's equation can be written as follows:

$$S(X,Y) - \frac{r}{2}g(X,Y) = \kappa\Big[(\mu + p)A(X)A(Y) + pg(X,Y) \\ + A(X)B(Y) + B(X)A(Y)\Big]. \quad (27)$$

Putting $Y = \rho$ in (27) and using (18) we get

$$\kappa B(X) = (\frac{r}{4} - \kappa\mu)A(X). \quad (28)$$

Since $\kappa \neq 0$, it follows from (28) and (21) that $B(X) = 0$. Therefore, the answer to the question raised in the beginning of the section is negative. This negative answer is expressed in the following form:

Theorem 2. *If in a $(P\tilde{C}S)_4$ spacetime of non-zero constant scalar curvature the matter distribution is a fluid with the basic vector field of $P\tilde{C}S)_4$ as the velocity vector field of the fluid, then such a fluid can not admit heat flux.*

5 The Basic Vector Field ρ as a Torse-Forming Vector Field

In this section we consider a Ricci symmetric $(P\tilde{C}S)_4$ spacetime. Then its Ricci tensor S satisfies

$$\nabla S = 0, \quad (29)$$

which implies $r =$ constant.

We suppose that ρ is a unit torse-forming vector field [16] defined by

$$\nabla_X \rho = \lambda X + \omega(X)\rho, \quad (30)$$

where λ is a non-zero scalar and ω is a non-zero 1-form, called the scalar and 1-form of the vector field ρ respectively.

Now due to (29) we get

$$(\nabla_X S)(Z, \rho) = 0. \tag{31}$$

Remembering that

$$(\nabla_X S)(Z, \rho) = \nabla_X S(Z, \rho) - S(\nabla_X Z, \rho) - S(Z, \nabla_X \rho)$$

and using (18) we have

$$\frac{r}{4}(\nabla_X A)(Z) - S(Z, \nabla_X \rho) = 0. \tag{32}$$

Substituting (30) in (32) we obtain

$$\frac{r}{4}(\nabla_X A)(Z) - \lambda S(Z, X) - \frac{r}{4}\omega(X)A(Z) = 0. \tag{33}$$

Putting $Z = \rho$ in (33) and using (18) we get

$$(\nabla_X A)(\rho) = \lambda A(X) - \omega(X). \tag{34}$$

Since ρ is unit, $\lambda A(X) = \omega(X)$. It means that

$$\lambda = -\omega(\rho). \tag{35}$$

Substituting (35) in (30) we obtain

$$\nabla_X \rho = -\omega(\rho)X + \omega(X)\rho.$$

Hence it follows that $\nabla_\rho \rho = 0$. Therefore, we can state the following theorem:
Theorem 3. *If in an Einstein $(P\tilde{C}S)_4$ spacetime the vector field ρ is a unit torse-forming vector field, then the integral curves of the vector field ρ are geodesics.*

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