Bulg. J. Phys. **45** (2018) 1–6

A Quick and Easy Test for Deciding Entanglement Status of an N-Qubit Pure Quantum State

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Received *October 20, 2017*

Abstract. We develop a simple criterion in terms of a necessary-sufficient condition (NS condition) for deciding separability of an arbitrary n-qubit pure quantum state. This NS condition provides a quick and easy test procedure to determine the entanglement status of a pure quantum state. We normalize the given quantum state and using this normalized state we can easily build a simplest system of equations containing trigonometric functions by making use of the well known Bloch Sphere representation for single qubit states and check whether or not this system of equations is consistent. According to proposed NS condition the given pure quantum state is separable (entangled) if and only if the above mentioned system of equations is consistent (inconsistent). We build this system of equations by equating the coefficients of computational basis states in the superposition representing the given pure quantum state with certain products of trigonometric functions obtained using standard Bloch Sphere representation for single qubit states. To establish separability of given state one requires to find a valid solution of the above mentioned system of equations but entanglement on the other hand follows when any two equations in this system of equations are mutually inconsistent. Thus, entanglement of the state can follow easily if one succeeds in finding any two mutually inconsistent equations in the above mentioned system of equations.

PACS codes: 03.67.Mn, 03.65.Ca, 03.65.Ud

1 Introduction

The important information one is interested to know about a quantum state is whether it is separable or entangled [1–5]. A quick and easy test to decide about the entanglement status of a given multiqubit pure state will be quite useful. In this paper a simple NS condition is proposed for deciding separability of a given multiqubit pure quantum state. This NS condition for separability demands consistency of certain system of equations constructed using the given pure quantum state to be tested for separability. The presence of a pair of mutually inconsistent equations in the associated system of equations immediately implies that the state under consideration is an entangled one. The present result thus could be quite useful for picking out entangled states by locating some inconsistent pair of equations in the associated system of equations.

Let $|\psi\rangle$ be an n-qubit pure quantum state. This n-qubit pure quantum state $|\psi\rangle$ can be expressed in terms of the computational basis as

$$|\psi\rangle = \sum_{i_1,i_2,\ldots i_n} a_{i_1 i_2 \ldots i_n} |i_1 i_2 \ldots i_n\rangle \,, \tag{1}$$

where all the coefficients $a_{i_1 i_2 \ldots i_n}$ belong to $\mathbb{C}$, the field of complex numbers, and where each of $i_1, i_2 \ldots, i_n$ takes values in $\{0,1\}$. We further assume (without loss of generality since we can always normalize a state if and when required) that this n-qubit pure quantum state is normalized, i.e. $\sum_{i_1,i_2,\ldots i_n} |a_{i_1 i_2 \ldots i_n}|^2 = 1$. This expression for $|\psi\rangle$ as a sum over computational basis states can contain in all $N = 2^n$ computational basis states, each of length n, namely, $|00\ldots 0\rangle$, $|00\ldots 1\rangle, \ldots, |i_1 i_2 \ldots i_n\rangle, \ldots, |11\ldots 1\rangle$.

2 The Main Result

We now proceed to state and prove our main result in terms of the following

Theorem: An n-qubit normalized pure quantum state, $|\psi\rangle$, as given above is separable if and only if there exist angles, $0 \le \theta_j \le \pi$ and $0 \le \phi_j < 2\pi$, where $j \in \{1,\ldots,n\}$ such that the following system of equations

$$m_{i_1} \cdot m_{i_2} \cdots m_{i_n} = a_{i_1 i_2 \ldots i_n}$$

is consistent, where

$$m_{i_j} = \begin{cases} \cos\left(\dfrac{\theta_j}{2}\right), & i_j = 0, \\ e^{i\phi_j} \sin\left(\dfrac{\theta_j}{2}\right), & i_j = 1. \end{cases}$$

Proof: Let $|\psi\rangle$ as given in equation (1) be separable. Then we can express this $|\psi\rangle$ as tensor product of n single qubit states, $|\phi_j\rangle = a_j|0\rangle + b_j|1\rangle$, such that $|a_j|^2 + |b_j|^2 = 1$, where $a_j, b_j \in \mathbb{C}$. Thus, $|\psi\rangle = \prod_{j=1}^{\otimes n} |\phi_j\rangle$. Further, it is well known that any single qubit normalized state can always be given the so called Bloch Sphere representation in which we replace a_j, b_j by suitable trigonometric functions. Thus, the above single qubit states $|\phi_j\rangle$ become $|\phi_j\rangle = \cos(\frac{\theta_j}{2})|0\rangle + e^{(i\phi_j)}\sin(\frac{\theta_j}{2})|1\rangle$, where angles, $0 \le \theta_j \le \pi$ and $0 \le \phi_j < 2\pi$, $j \in \{1,\ldots,n\}$ and $i = \sqrt{-1}$. Thus, replacing each single qubit normalized state $|\phi_j\rangle$ by its Block Sphere representation, the state $|\psi\rangle$ given in equation (1) becomes

$$|\psi\rangle = \prod_{j=1}^{\otimes n}[\cos(\frac{\theta_j}{2})|0\rangle + e^{(i\phi_j)}\sin(\frac{\theta_j}{2})|1\rangle]. \tag{2}$$

Now, if we expand the right hand side of above equation by taking the tensor product and compare the coefficients of the computational basis states, $|i_1 i_2 \dots i_n\rangle$, from equations (1), (2) then we get the following consistent system of equations, namely,

$$m_{i_1} \cdot m_{i_2} \cdots m_{i_n} = a_{i_1 i_2 \dots i_n} \, ,$$

where, $m_{i_j} = \cos(\frac{\theta_j}{2})$ if $i_j = 0$, and $m_{i_j} = e^{i\phi_j}\sin(\frac{\theta_j}{2})$ if $i_j = 1$, where $0 \le \theta_j \le \pi$ and $0 \le \phi_j < 2\pi$, $j\epsilon\{1, \dots, n\}$.

Conversely, let the following system of equations, namely,

$$m_{i_1} \cdot m_{i_2} \cdots m_{i_n} = a_{i_1 i_2 \dots i_n}$$

be consistent where, $a_{i_1 i_2 \dots i_n}$ are the coefficients of computational basis states $|i_1 i_2 \dots i_n\rangle$ in equation (1) and where $m_{i_j} = \cos(\frac{\theta_j}{2})$ if $i_j = 0$, and $m_j = e^{i\phi_j}\sin(\frac{\theta_j}{2})$ if $i_j = 1$ and there exist angles as required, namely, $0 \le \theta_j \le \pi$ and $0 \le \phi_j < 2\pi$, $j \in \{1, \dots, n\}$. Therefore, we can replace the coefficients $a_{i_1 i_2 \dots i_n}$ of computational basis states $|i_1 i_2 \dots i_n\rangle$ in equation (1) by the above given corresponding product terms $m_{i_1} \cdot m_{i_2} \cdots m_{i_n}$, where as mentioned above, $m_j = \cos(\frac{\theta_j}{2})$ if $i_j = 0$, and $m_j = e^{i\phi_j}.\sin(\frac{\theta_j}{2})$ if $i_j = 1$.

Therefore, we can rewrite equation (1) as follows:

$$\begin{aligned}|\psi\rangle = {} & \cos(\frac{\theta_1}{2})\cos(\frac{\theta_2}{2})\cdots\cos(\frac{\theta_n}{2})|00\dots0\rangle \\ & + \cos(\frac{\theta_1}{2})\cos(\frac{\theta_2}{2})\cdots e^{i\phi_n}\sin(\frac{\theta_n}{2})|00\dots1\rangle + \dots \\ & + e^{i\phi_1}\sin(\frac{\theta_1}{2})e^{i\phi_2}\sin(\frac{\theta_2}{2})\cdots e^{i\phi_n}\sin(\frac{\theta_n}{2})|11\dots1\rangle \, .\end{aligned} \tag{3}$$

Clearly, we can express equation (3) above in terms of following factors:

$$|\psi\rangle = [\cos(\frac{\theta_1}{2})|0\rangle + e^{i\phi_1}\sin(\frac{\theta_1}{2})|1\rangle] \otimes |\psi_1\rangle \, , \tag{4}$$

where

$$\begin{aligned}|\psi_1\rangle = {} & \cos(\frac{\theta_2}{2})\cos(\frac{\theta_3}{2})\cdots\cos(\frac{\theta_n}{2})|00\dots0\rangle \\ & + \cos(\frac{\theta_2}{2})\cos(\frac{\theta_3}{2})\cdots e^{i\phi_n}\sin(\frac{\theta_n}{2})|00\dots1\rangle + \dots \\ & + e^{i\phi_2}\sin(\frac{\theta_2}{2})e^{i\phi_3}\sin(\frac{\theta_3}{2})\cdots e^{i\phi_n}\sin(\frac{\theta_n}{2})|11\dots1\rangle \, .\end{aligned} \tag{5}$$

On similar lines we can factorise $|\psi_1\rangle$ as follows:

$$|\psi_1\rangle = [\cos(\frac{\theta_2}{2})|0\rangle + e^{i\phi_2}\sin(\frac{\theta_2}{2})|1\rangle] \otimes |\psi_2\rangle\,, \tag{6}$$

where

$$\begin{aligned}|\psi_2\rangle = \cos(\frac{\theta_3}{2})\cos(\frac{\theta_4}{2})\cdots\cos(\frac{\theta_n}{2})|00\ldots0\rangle \\ + \cos(\frac{\theta_3}{2})\cos(\frac{\theta_4}{2})\cdots e^{i\phi_n}\sin(\frac{\theta_n}{2})|00\ldots1\rangle + \ldots \\ + e^{i\phi_3}\sin(\frac{\theta_3}{2})e^{i\phi_4}\sin(\frac{\theta_4}{2})\cdots e^{i\phi_n}\sin(\frac{\theta_n}{2})|11\ldots1\rangle\,,\end{aligned} \tag{7}$$

and so on. Thus, we can continue on these lines till we reach the complete factorisation of given state $|\psi\rangle$ into n single qubit factors as follows:

$$|\psi\rangle = \prod_{j=1}^{\otimes n}[\cos(\frac{\theta_j}{2})|0\rangle + e^{i\phi_j}\sin(\frac{\theta_j}{2})|1\rangle]. \tag{8}$$

3 Examples

We now consider a few examples to demonstrate the utility of the above obtained criterion to test whether given pure states are entangled or separable. For brevity we denote hereafter the terms $\cos(\frac{\theta_j}{2})$ by C_j, $\sin(\frac{\theta_j}{2})$ by S_j.

(i) Consider the following two qubit state:

$$|\psi\rangle = \frac{1}{\sqrt{3}}|00\rangle - \frac{1}{\sqrt{3}}|01\rangle + \frac{1}{\sqrt{6}}|10\rangle - \frac{1}{\sqrt{6}}|11\rangle\,.$$

For this we write down the system of equations as follows and check their consistency.

$$\begin{aligned}C_1C_2 &= \frac{1}{\sqrt{3}}\,, \\ C_1S_2e^{i\phi_2} &= -\frac{1}{\sqrt{3}}\,, \\ S_1C_2e^{i\phi_1} &= \frac{1}{\sqrt{6}}\,, \\ S_1S_2e^{i(\phi_1+\phi_2)} &= -\frac{1}{\sqrt{6}}\,.\end{aligned}$$

Since the coefficients in $|\psi\rangle$ are all real, we have $\phi_1 = 0$, $\phi_2 = 0$. Then, the first and the third equations give $\tan(\frac{\theta_1}{2}) = \frac{1}{\sqrt{2}}$ and the third and fourth give

$\tan(\frac{\theta_2}{2}) = -1$. These give $C_1 = \sqrt{2}/\sqrt{3}$, $S_1 = 1/\sqrt{3}$, $C_2 = 1/\sqrt{2}$, $S_2 = -1/\sqrt{2}$. These values clearly satisfy the above system of equations. Therefore, the state $|\psi\rangle$ is separable.

(ii) Consider the following two qubit state:

$$|\psi\rangle = \frac{1}{\sqrt{30}}[|00\rangle + 2i|01\rangle - 3|10\rangle - 4i|11\rangle \, .$$

As before we build the following system of equations:

$$\begin{aligned} C_1 C_2 &= \frac{1}{\sqrt{30}} \, , \\ C_1 S_2 e^{i\phi_2} &= \frac{2i}{\sqrt{30}} \, , \\ S_1 C_2 e^{i\phi_1} &= \frac{-3}{\sqrt{30}} \, , \\ S_1 S_2 e^{i(\phi_1+\phi_2)} &= \frac{-4i}{\sqrt{30}} \, . \end{aligned}$$

The second equation gives $\phi_2 = \frac{1}{2}\pi$ and the third gives $\phi_1 = 0$. Then, the first two equations give

$$\tan\left(\frac{\theta_2}{2}\right) = 2$$

while the next two equations give

$$\tan\left(\frac{\theta_2}{2}\right) = \frac{4}{3} \, .$$

Thus the system of equations formed as per the theorem given above leads to two different values for $\tan(\frac{\theta_2}{2})$ which cannot be true. Thus, the system of equations formed as per the theorem given above is clearly inconsistent and so the state $|\psi\rangle$ under test is entangled.

(iii) Consider the following four qubit state:

$$|\psi\rangle = \frac{1}{2}\left[|0001\rangle + |0010\rangle + |1101\rangle + |1110\rangle\right] .$$

We have $C_1C_2C_3C_4 = 0$, $C_1C_2C_3S_4e^{i\phi_4} = \frac{1}{2}$, therefore none of C_1, C_2, C_3 can be equal to zero and so $C_4 = 0$. But also the condition $C_1C_2S_3C_4e^{i\phi_3} = \frac{1}{2}$ is required to be fulfilled, implying C_4 cannot be equal to zero. Thus, the system of equations formed as per the theorem given above is inconsistent, so the state $|\psi\rangle$ under test is entangled.

Acknowledgments

One of us (DPM) thanks Dr. M. R. Modak, S. P. College, Pune-411030, India, for carefully reading the mannuscript and making useful comments and suggestions.

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