Bulg. J. Phys. **45** (2018) 85–92

Bethe-Salpeter Equation with Massive Vector Particle Constituents

David A. Owen
Ben-Gurion University of the Negev, Israel

Received: *January 16, 2018*

Abstract. In the following article, it shall be demonstrated how the quantum field theory can be used to describe the two particle bound system, viz. by the Bethe-Salpeter equation. It is demonstrated that the formalism can be generalized to accommodate a bound state comprised of a scalar meson and a vector meson. The Bethe-Salpeter equation, in its original formulation was only able to accommodate a system of two fermions as it relied on the linearity of the momentum of the inverse propagators to obtain the equation with a perturbation expansion. This is crucial as there are no known exact solutions to this equation. It is shown that by introducing a two-component formalism, the inverse of both the scalar and vector meson propagators are linear in momentum and their bound state can be described adequately by the Bethe-Salpeter formalism.

PACS codes: 11.10.St

1 Introduction

The vector meson W, a massive spin-1 particle is the transmitter of the weak interaction. Due to its very short life time realistically one cannot believe it can be part of any bound state unless a mechanism can be found that could increase its life time. A W-anti W bound state would be very attractive having such a heavy entity which interacts weakly with matter but at the moment, this is a hopeless speculation. Just due to the knowledge that there can exist fundamental particles of spin-1, we show that the Bethe-Salpeter quantum field description of bound states can be generalized to accommodate massive spin-1 particles. In another work, we shall considered how the formalism needs to be modified to accomodate the case where the masses of the particles are generated by spontaneous symmetry breaking, but not here as that takes us to far from our immediate objective.

2 Bethe-Salpeter Equation

In 1971, independently, the two-body relativistic, quantum field theory equation was independently obtained by H.A. Bethe and E.E. Salpeter [1] and Julain Schwinger [2]. Although the derivations were quite different the equation obtained were equivalent. Its use in assessing the agreement between quantum electrodynamics cannot be over stated. It allowed the test of QED on electrons, muons etc. particles lacking internal structure that could obfuscate the results. The predictions from the the work of C. Sommerfled [3], T. Fulton, D. Owen and W. Repko [4] based on the Bethe-Sapeter Equation were borne out by experimental results.

The Bethe-Salper equation is the equation for the two-particle propagator for which has the bound state condition that no particles are propagated to infinity. This localization conditon is required for a bound state to exist. To be more specific, the two-particle quantum field theory propagator can be written as

$$G(34;12) = \langle 0|T\{\psi_a^H(3)\psi_b^H(4)\bar{\psi}_a^H(1)\bar{\psi}_b^H(2)\}0\rangle \,, \tag{1}$$

where the superscript H indicate that the fields are in the Heisenberg picture. The Bethe-Salpter equation is obtained by writing Eq. 1 in the interaction picture where we obtain

$$\begin{aligned} G(34;12) &\equiv \langle 0|T\{\psi_a^H(3)\psi_b^H(4)\bar{\psi}_a^H(1)\bar{\psi}_b^H(2)\}|0\rangle \\ &= \langle 0|T\{\psi_a(3)\psi_b(4)\bar{\psi}_a(1)\bar{\psi}_b(2)S\}0\rangle \end{aligned} \tag{2}$$

with the S-matrix given by

$$\begin{aligned} S &= 1 + \sum_{n=1}^{\infty} \frac{(-i)^n}{n!} \int_{-\infty}^{\infty} dt_1 \cdots \int_{-\infty}^{\infty} dt_n T(H_I(t_1)\cdots H_I(t_n)) \\ &\equiv T\left(\exp\left[-i\int_{-\infty}^{\infty} H_I(t)dt\right]\right) \,, \end{aligned} \tag{3}$$

where the exponential form expressed in Eq 3 is a symbolic summary of the time ordered series with which it coincides when expanded in a power series in the interaction where the complete Hamiltonian can be written as H_0, containing the free particle Hamiltonian itself and H_I, the part containing interaction between the particles.

Bethe and Salpeter (as well as Schwinger) showed that Eq. 8 satisfies the following equation which is commonly referred to as the Bethe-Salpeter equation:

$$\begin{aligned} G(34;12) &= S'_a(x_3 - x_1)_F S'_b(x_4 - x_2)_F \\ &\quad + S'_a(x_3 - x_5)_F S'_b(x_4 - x_2)_F I(5,6;7,8) G(78;12) \,, \end{aligned} \tag{4}$$

where I is the irreducible part of H_I which connects particles 'a' and 'b' while the part which acts only on 'a' or 'b' iis called the reducible part and is the part which 'fully' dresses the propagators. Thus $S_a \to S'_a$ and $S_b \to S'_b$.

To obtain an equation for a bound state from Eq. 8 we need to insert a complete set of bound states $\sum_n = |n\rangle\langle n|$ into Eq. 8. For a bound state the inhomogenous term vanishes (otherwise the system would not be confined, thus not bound) and restricting ourselves to x_3^0 and $x_4^0 > x_1^0$ and x_2^0 and we define the two-particle wavefunction by $\phi_n(3,4) = \langle 0|T(\psi(3)\psi(4))|n\rangle$,

The Bethe-Salpeter equation can now be written as

$$\phi_n(x_3, x_4) = S'_a(x_3 - x_5)S'_b(x_4 - x_6)I(x_5x_6; x_7x_8)\phi_n(x_7, x_8)\,. \tag{5}$$

If we include the self-energy parts of each of the propagators as multiplicative factors of I, Eq. 5 can be written as

$$\phi_n(x_3, x_4) = S_a(x_3 - x_5)S_b(x_4 - x_6)\tilde{I}(56; 78)\phi_n(x_7, x_8)\,, \tag{6}$$

where the propagators appearing in Eq. 6 are the free propagtors and $\tilde{I}$ is the irreducible kernel multiplied by the self-energies of each of the particles. In C.M. coordinates,

$$S_a(x_3 - x_5) = \int d^4p_1 e^{ip_1\cdot(x_3-x_5)} S_a(p_1) \tag{7}$$

etc. where $p_1 = \eta_1 K + p$, p_2, $p_2 = \eta_2 K - p$, $\eta_1 + \eta_2 = 1$.

3 Fermon-Fermion Case-QED

$$\begin{aligned}
G(34;12) &\equiv \langle 0|T\{\psi_a^H(3)\psi_b^H(4)\bar{\psi}_a^H(1)\bar{\psi}_b^H(2)\}0\rangle \\
&= \langle 0|T\{\psi_a(3)\psi_b(4)\bar{\psi}_a(1)\bar{\psi}_b(2)S\}0\rangle \\
&= \langle 0|T\{\psi_a(3)\psi_b(4)\bar{\psi}_a(1)\bar{\psi}_b(2)\}0\rangle \\
&\quad + \sum_{n=1}^{\infty}\langle 0|T\{\psi_a(3)\psi_b(4)\bar{\psi}_a(1)\bar{\psi}_b(2) \\
&\quad \times \frac{(-i)^n}{n!}\int_{-\infty}^{\infty} dt_1 \cdots \int_{-\infty}^{\infty} dt_n T(H_I(t_1)\cdots H_I(t_n))\}0\rangle\,,
\end{aligned} \tag{8}$$

where H is given by

$$H = H_0 + H_I \tag{9}$$

with $H_0 = i\gamma^a\partial + m_a + i\gamma^b\partial + m_b$

$$\begin{aligned}
S_a(x_3 - x_5)S_b(x_4 - x_5) &= \int e^{(X-X')\cdot K} e^{-i(p_1-p_2)} \\
&\quad \times \frac{1}{[\gamma^a \cdot \eta_1 K + p - m_a][\gamma^b \cdot \eta_2 K - p - m_b]}\,.
\end{aligned} \tag{10}$$

Separating rhs by partial fractions gives [5]

$$\frac{1}{[\gamma^a \cdot \eta_1 K + p - m_a][\gamma^b \cdot \eta_2 K - p - m_b]} = \Big(\frac{1}{[\gamma^a \cdot \eta_1 K + p - m_a]} + \frac{1}{[\gamma^b \cdot \eta_2 K - p - m_b]}\Big) \times \frac{1}{[K - H^a(\mathbf{p}) - H^b(\text{-}\mathbf{p})]}\,, \quad (11)$$

where K is the energy eigenvalue. Equation 6 can be written as

$$[K - H^a(\mathbf{p}) - H^b(\text{-}\mathbf{p}) - \tilde{I}_0(x)]\phi_n(x) = 0\,. \quad (12)$$

$\tilde{I}$ is the irreducible kernel which is combined with first term on the right side of Eq. 11.

This was the initial form the Bethe-Salpeter equation and its perturbation theory was developed. It relied on the fact that the reciprocal of each of the free particles propagators, is linear in momentum.

4 Generalization Needed to Accommodate Vector Mesons as Bound State Constituents

It is the Eq. 12 that connects quantum field treatment of bound states with both the non-relativistic treatment as well as the external field approximation using the Dirac equation. This above treatment relied on the inverse of the fermion propagators being linear in momentum. We want our formalism to describe bound states having constituents of scalar particles as well as charged vector mesons. In the standard form the inverse of these propagators are quadratic and not linear in the momentum. To have such particles described as constituents in the Bethe-Salpeter equation, in need to find an alternative form to express the propagators in which the momentum appears linear in the inverse propagator. In this way, we will have generalized the Bethe-Salpeter equation that the constituent particles could, in fact, be any of the particles occurring in the standard model. It is also conceivable that this formalism describe yet unobserved bound states whose constituents are predicted by supersymmetry. Up to this time, applications of the Bethe-Salpeter equation had been confined to interacting fermions via QED or quarks interacting with gluons. It needn't be so.

4.1 Bethe-Salpeter equation with spinless particle constituents [6] [7]

4.1.1 Two-component representation of the Klein-Gordon propagator

The Klein-Gordon propagator can be defined as

$$\Delta_F(x - x') = -i\langle 0|T[\phi(x)\phi(x')]|0\rangle\,, \quad (13)$$

where ϕ is the solution of the Klein-Gordon equation [i.e. $(\partial_x^2 + m^2)\phi(x) = 0$]. In momentum space Eq. 13 can be written

$$
\begin{aligned}
\Delta_F(x - x') &= \int \frac{d^4p}{(2\pi)^4} e^{-ip\cdot(x-x')} \frac{1}{p^2 - m^2 + i\epsilon} \\
&= \int \frac{d^3p}{(2\pi)^3} \int_{-\infty}^{\infty} \frac{dp_0}{2\pi} e^{-ip\cdot(x-x')} \\
&\quad \times \frac{1}{2E_p} \left(\frac{1}{(p_0 - E_p - i\epsilon)} - \frac{1}{(p_0 + E_p - i\epsilon)} \right) , \qquad (14)
\end{aligned}
$$

where in Eq. 14 we have made a partial fraction decomposition of the propagator ($E_p = \sqrt{\mathbf{p}^2 + m^2}$). This propagator takes a free scalar particle state at x' and tranforms it to x, i.e.

$$
\phi(x) = \int d^4x' \Delta_F(x - x')\phi(x') \, . \qquad (15)
$$

It is easy to verify that if $\phi(x')$ is a positive energy eigenstate, then the second term in Eq. 14 acting on ϕ vanishes and likewise, if ϕ is a negative energy state, then the first term of Eq. 14 acting on it vanishes. Thus we can describe the action of Eq. 15 as acting on a two-component wave function

$$
\Phi = \begin{pmatrix} \phi^+ \\ \phi^- \end{pmatrix} , \qquad (16)
$$

where the upper component represents a positive energy eigenstate and the lower component, a negative eigenstate. We can now write Eq. 14 as

$$
\begin{aligned}
\Delta_F(x - x') = \int \frac{d^4p}{(2\pi)^4} & e^{-ip\cdot(x-x')} \\
& \times \left(\frac{\Lambda^+}{2E_p(p_0 - E_p + i\epsilon)} - \frac{\Lambda^-}{2E_p(p_0 + E_p + i\epsilon)} \right) , \qquad (17)
\end{aligned}
$$

where

$$
\Lambda^\pm = \frac{1}{2}(1 + \pm\beta_s), \qquad \beta_s = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} . \qquad (18)
$$

In this two component representation, the quantum field representation of Φ, for a charged, spinless field is given by

$$
\Phi(x) = \int \frac{d^3k}{\sqrt{2E_k(2\pi)^3}} \{ u a_+(k) e^{-ik\cdot x} + v a_-^\dagger(k) e^{ik\cdot x} \} \qquad (19)
$$

with $k_0 = E_k$,

$$
u = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad \text{and} \quad v = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \qquad (20)
$$

and $[a_{\pm}(k), a^{\dagger}_{\pm}(k')] = \delta(\mathbf{k} - \mathbf{k'})$; $[a_{\pm}(k), a_{\pm}(k')] = 0$, etc. with these definitions, one finds in a straightforward manner that $-i\langle 0|T[\Phi(x)\Phi(x')]|0\rangle$ leads directly to Eq. 17. Furthermore, we can write the equation defining $\Delta_F(x - x')$ in this representation by

$$2E_p(p_0(\Lambda^+ - \Lambda^-) - E_p)\Delta_F(x - x') = \delta(x - x')\,. \tag{21}$$

Alternatively, we coould write Eq. 17 in a more compact form as

$$\Delta_F(x - x') = \int \frac{d^4p}{(2\pi)^4} e^{-ip\cdot(x-x')} \frac{1}{2p_0} \left(\frac{1}{p_0 - \beta_s E_p + i\epsilon} \right). \tag{22}$$

In what follows this form will be very convenient to use. It should, however, be noted that despite the appearance of a pole at $p_0 = 0$ in Eq. 22 none actually occurs and the expression is finite as it must be at $p_0 = 0$ as it must be. This can be seen by writing the integrand of Eq. 22 in a form similar to Eq. 17. The form of Eq. 22 is somewhat reminiscent of the Dirac propagator and correspondingly leads to a simpler equation for $\Delta_F(x - x')$ than Eq. 21. That is,

$$2p_0(p_0 - \beta_s E_p)\Delta_F(x - x') = \delta(x - x')\,. \tag{23}$$

4.1.2 Application: Bethe-Salpeter equation for a boson & fermion

Here we give an example of how the previous section can be used to write the Bethe-Salpeter equation for the interaction of a scalar particle & a fermion. Eq. 6 for the case of a boson and a fermion bound system can be written as

$$\phi_K(x) = G_K(x, x')I_K(x', x'')\phi_K(x'')\,, \tag{24}$$

where $G_K(x, x')$ for the case of a fermion and a spinless particle is given by

$$G_K(x, x'') = \int \frac{d^4p}{(2\pi)^4} \frac{e^{-ip\cdot(x-x')}}{(2\eta_a K - p_0)} \frac{1}{[K - H^a(p) - \beta_s E_b]} \times \left(\frac{1}{\eta_a K + p_0 - H^{(a)} + i\epsilon} + \frac{1}{\eta_b K - p_0 - E_a + i\epsilon} \right). \tag{25}$$

Using Equations 24 and 25, we can write

$$[K - H^a(p) - \beta_s]\phi_K(x) = \tilde{\Lambda}_k(x, x'')\gamma_0 I_K(x', x'')\phi_K(x'') \tag{26}$$

$$\tilde{\Lambda}_K(x, x'') = \int \frac{d^4p}{(2\pi)^4} \frac{e^{-ip\cdot(x-x')}}{2(\eta_a K - p_0)} \times \left(\frac{1}{\eta_a K + p_0 - H^a(p) + i\epsilon} + \frac{1}{\eta_b K - p_0 - E_b + i\epsilon} \right). \tag{27}$$

Here we have designated a as the fermion and b as the scalar particle. Eq. 26 is the Bethe-Salpeter equation containing a scalar particle and a fermion. The generalization to two scalar particles is trivial.

4.2 Bethe-Salpeter equation in which one of the constituents is a vector meson

Our example will be taken from the electro-weak theory. With the gauge fixing term [8]

$$\mathcal{L}_{gf} = -\frac{1}{2\xi}(\partial_\mu A^\mu - \xi M \phi_2)^2 \,, \tag{28}$$

where ξ is a parameter determing the gauge. Depending on the value of ξ we get a class of gauges called t' Hooft gauges. Adding Eq.28 to the electro-weak Lagrangian we have

$$\begin{aligned}\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{M^2}{2}A^\mu A_\mu - \frac{1}{2\xi}\partial_\mu A^\mu \partial_\nu A^\nu + \frac{1}{2}(\partial_\mu \phi_1)^2 \\ - \frac{1}{2}\lambda a^2 \phi_1^2 + \frac{1}{2}(\partial_\mu \phi_2)^2 - \frac{\xi}{2}M^2\phi_2^2 + \text{coupling terms.}\end{aligned} \tag{29}$$

The quadratic term in A^μ is

$$\frac{1}{2}A^\mu[g_{\mu\nu}(\Box + M^2) - \partial_\mu\partial_\nu(1 - \frac{1}{\xi})]A^\nu \,. \tag{30}$$

The vector propagator follows:

$$D_{\mu\nu} = \frac{1}{k^2 - M^2}\left[-g_{\mu\nu} + (1-\xi)\frac{k_\mu k_{nu}}{k^2 - \xi M^2}\right] . \tag{31}$$

If we choose the gauge $\xi \to \infty$, we obtain

$$D_{\mu\nu} = \frac{1}{k^2 - M^2}\left(-g_{\mu\nu} + \frac{k^\mu k^\nu}{M^2}\right) . \tag{32}$$

From Eq. 22, we know that

$$\frac{1}{k^2 - M^2 + i\epsilon} \to \frac{1}{2k_0}\left(\frac{1}{k_0 - \beta_s E_k + i\epsilon}\right) . \tag{33}$$

Hence we can write the vector meson propagator by [9]

$$D_{\mu\nu} = \frac{1}{2k_0}\left(\frac{1}{k_0 - \beta_s E_k}\right)\left(-g_{\mu\nu} + \frac{k^\mu k^\nu}{M^2}\right) , \tag{34}$$

where $E_k = \sqrt{k^2 + M^2}$, and where M is the mass of the vector meson.

4.2.1 Bethe-Salpeter equation for a vector-meson and a fermion

As an example , we write the Bethe-Salpeter equation for a fermion and a vector meson particle

$$\begin{aligned}&[K - \alpha \cdot k - \beta m_a - \beta_s E_b(k)]\, \phi_K^\mu(x) \\ &\qquad = \Lambda_K(x, x')\gamma_b^0 \left(-g^{\mu\nu} + \frac{k^\mu k^\nu}{m_b^2} \right) I_K(x', x'')\phi_{K.\nu}(x'') \,. \end{aligned} \tag{35}$$

In a similar manner, we could have written the Bethe-Salpeter equation for a bound state of two vector mesons or a vector meson and a scalar particle.

Spontaneous symmetry breaking induces further complications as will be discussed elsewhere. In principle, except for this caveat, we have formulated an equation that could be used for QCD as well as the electroweak.

References

[1] E. Salpeter and H. Bethe (1951) *Phys. Rev.* **84** 1232.
[2] J. Schwinger (1951) *Proc. Natl. Acad. Sci. U.S.A.* **37** 452; **37** 455.
[3] C. Sommerfeld (1957) *Phys. Rev.* **107** 328; (1958) *Ann. Phys. (New York)* **5** 26.
[4] T. Fulton, D.A. Owen and W.W. Repko (1970) *Phys. Rev. Lett.* **24** 1035.
[5] D.A. Owen (1990) *Phys. Rev. D* **42** 3534.
[6] D.A. Owen (1994) *Found. Phys.* **24** 273.
[7] M. Halpert and D.A. Owen (1994) *J. Phys. G: Nucl. Part. Phys.* **20** 51.
[8] L. Ryder (1996) *Quantum Field Theory* 2nd Edition (Cambridge University Press).
[9] D.A. Owen (1997) *Found. Phys.* **27** 1.