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Coherently Driven Nondegenerate Three-Level Laser with Noiseless Vacuum Reservoir

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Abstract. In this paper, the analysis of the quantum properties of cavity light produced by a coherently driven nondegenerate three-level laser in a closed cavity and coupled to a two-mode vacuum reservoir, is presented. Normal ordering of the noise operators associated with the vacuum reservoir is considered. Applying the solutions of the equations of evolution for the expectation values of the atomic operators and the quantum Langavin equations for the cavity mode operators, the quadrature squeezing, entanglement amplification, and the normalized second-order correlation function of the cavity radiation are obtained. The three-level laser generates squeezed light under certain conditions, with maximum intracavity squeezing being 43% below the vacuum-state level. Moreover, it is found that the photon numbers of a two-mode light beams are correlated.

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1 Introduction

Entanglement is one of the fundamental tools for quantum information processing and communication protocols. The generation and manipulation of entanglement has attracted a great deal of interest with wide applications in quantum teleportation, quantum dense coding, quantum computation, quantum error correction, and quantum cryptography [1–5]. Recently, much attention is given on the generation of continuous-variable entanglement to manipulate the discrete counterparts, quantum bits, to perform quantum information processing. In general, the degree of entanglement decreases when it interacts with the environment. But, the efficiency of quantum information processing highly depends on the degree of entanglement. Therefore, it is necessary to generate strongly entangled states which can survive from external noise.

In general, due to the result of the strong correlation between the cavity modes, a two-mode squeezed state violates certain classical inequalities and then can be used in preparing Einstein-Podolsky-Rosen (EPR)-type entanglement [6]. The

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steady state entanglement in a nondegenerate three-level laser has been studied when the atomic coherence is induced by initially preparing atoms in coherent superposition of the top an bottom levels [7–13] and when the top and bottom levels of the three-level atoms injected into a cavity are coupled by coherent light [14–21]. Moreover, Fesseha has studied the quantum properties of the light emitted by the three-level atoms available in a closed cavity and pumped to the top level at a constant rate by means of electron bombardment [22]. Furthermore, he studied the quantum properties of the light generated by a two-level laser in which the two-level atoms available in a closed cavity are pumped to the upper level by means of electron bombardment [23, 24].

In this paper, we study the squeezing and entanglement properties of the light generated by a coherently driven nondegenerate three-level laser with a closed cavity and coupled to a two-mode vacuum reservoir via a single-port mirror. In order to carry out our calculation, we put the noise operators associated with the vacuum reservoir in normal order. Thus, first we obtain the the quantum Langevin equations for the cavity mode operators. Then, employing the large-time approximation scheme, we obtain equations of evolution of the expectation values of atomic operators. Moreover, we determine the solutions of the equations of evolution of the expectation values of the atomic operators and the quantum Langevin equations for cavity mode operators. And applying the resulting solutions, we obtain the mean photon number, the quadrature squeezing, and the entanglement. Furthermore, applying the same solutions, we obtain the normalized second-order correlation function for the two-mode light.

2 The Model and the Quantum Langevin Equations

Here N nondegenerate three-level atoms in cascade configuration are available in a closed cavity. The top, intermediate, and bottom levels of the three-level atom are denoted by $|a\rangle_k$, $|b\rangle_k$, and $|c\rangle_k$, respectively (see Figure 1). When the atom makes a transition from level $|a\rangle_k$ to $|b\rangle_k$ and from levels $|b\rangle_k$ to $|c\rangle_k$ two photons with different frequencies are emitted. It is assumed that the cavity mode a is at resonance with transition $|a\rangle_k \rightarrow |b\rangle_k$ and the cavity mode b is at resonance with the transition $|b\rangle_k \rightarrow |c\rangle_k$, with top and bottom levels of the three-level atom coupled by coherent light. The interaction of a nondegenerate three-level atom with the coherent light and with the light modes a and b can be described by the Hamiltonian

$$\hat{H} = ig\big[\hat{\sigma}_a^{\dagger k}\hat{a} - \hat{a}^\dagger\hat{\sigma}_a^k + \hat{\sigma}_b^{\dagger k}\hat{b} - \hat{b}^\dagger\hat{\sigma}_b^k\big] + \frac{i\Omega}{2}\big[\hat{\sigma}_c^{\dagger k} - \hat{\sigma}_c^k\big], \tag{1}$$

where g is the coupling constant between the atom and cavity mode a or b, and $\hat{a}$ and $\hat{b}$ are the annihilation operators for light modes a and b. Here $\Omega = 2\varepsilon\lambda$, in which ε considered to be real and constant, is the amplitude of the driving coherent light and λ is the coupling constant between the driving coherent light

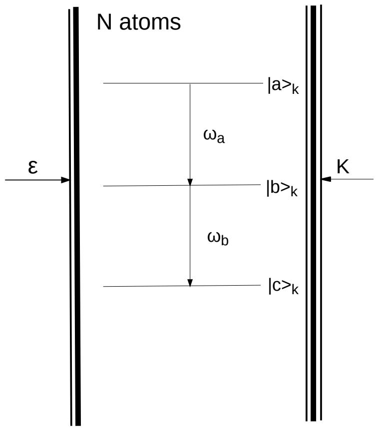

Figure 1. Scheme of N three-level atom inside a closed cavity and coupled a vacuum reservoir.

and the three-level atom. We also define that $\hat{\sigma}_a^k = |b\rangle_{k\ k}\langle a|$, $\hat{\sigma}_b^k = |c\rangle_{k\ k}\langle b|$, and$\hat{\sigma}_c^k = |c\rangle_{k\ k}\langle a|$ are lowering atomic operators.

We assume that the laser cavity is coupled to a vacuum reservoir via a single-port mirror. In addition, we carry out our calculation by putting the noise operators associated with the vacuum reservoir in normal order. Thus, the noise operators will not have any effect on the dynamics of the cavity mode operators [22–24]. Therefore, with the help of the expression (1), one can drop the noise operators and write the quantum Langevin equations for the operators $\hat{a}$ and $\hat{b}$ as

$$\frac{d\hat{a}}{dt} = -\frac{\kappa}{2}\hat{a} - i[\hat{a}, \hat{H}], \tag{2}$$

$$\frac{d\hat{b}}{dt} = -\frac{\kappa}{2}\hat{b} - i[\hat{b}, \hat{H}], \tag{3}$$

where κ is the cavity damping constant. With the aid of Eq. (1), we easily find

$$\frac{d\hat{a}}{dt} = -\frac{\kappa}{2}\hat{a} - g\hat{\sigma}_a^k, \tag{4}$$

$$\frac{d\hat{b}}{dt} = -\frac{\kappa}{2}\hat{b} - g\hat{\sigma}_b^k. \tag{5}$$

3 Equations of Evolution of Atomic Oprators

Employing the relation
$$\frac{d}{dt}\langle\hat{A}\rangle = -i\langle[\hat{A},\hat{H}]\rangle. \tag{6}$$
along with the result (1), one can readily establish that
$$\frac{d}{dt}\langle\hat{\sigma}_a^k\rangle = g(\langle\hat{\eta}_b^k\hat{a}\rangle - \langle\hat{\eta}_a^k\hat{a}\rangle + \langle\hat{b}^\dagger\hat{\sigma}_c^k\rangle) + \frac{\Omega}{2}\langle\hat{\sigma}_b^{\dagger k}\rangle, \tag{7}$$
$$\frac{d}{dt}\langle\hat{\sigma}_b^k\rangle = g(\langle\hat{\eta}_c^k\hat{b}\rangle - \langle\hat{a}^\dagger\hat{\sigma}_c^k\rangle - \langle\hat{\eta}_b^k\hat{b}\rangle) - \frac{\Omega}{2}\langle\hat{\sigma}_a^{\dagger k}\rangle, \tag{8}$$
$$\frac{d}{dt}\langle\hat{\sigma}_c^k\rangle = g(\langle\hat{\sigma}_b^k\hat{a}\rangle - \langle\hat{\sigma}_a^k\hat{b}\rangle) + \frac{\Omega}{2}\big[\langle\hat{\eta}_c^k\rangle - \langle\hat{\eta}_a^k\rangle\big], \tag{9}$$
$$\frac{d}{dt}\langle\hat{\eta}_a^k\rangle = g(\langle\hat{\sigma}_a^{\dagger k}\hat{a}\rangle + \langle\hat{a}^\dagger\hat{\sigma}_a^k\rangle) + \frac{\Omega}{2}\big[\langle\hat{\sigma}_c^k\rangle + \langle\hat{\sigma}_c^{\dagger k}\rangle\big], \tag{10}$$
$$\frac{d}{dt}\langle\hat{\eta}_b^k\rangle = g(\langle\hat{\sigma}_b^\dagger\hat{b}\rangle - \langle\hat{\sigma}_a^{\dagger k}\hat{a}\rangle - \langle\hat{a}^\dagger\hat{\sigma}_a^k\rangle + \langle\hat{b}^\dagger\hat{\sigma}_b^k\rangle), \tag{11}$$
$$\frac{d}{dt}\langle\hat{\eta}_c^k\rangle = -g(\langle\hat{\sigma}_b^\dagger\hat{a}\rangle + \langle\hat{b}^\dagger\hat{\sigma}_b^k\rangle) - \frac{\Omega}{2}\big[\langle\hat{\sigma}_c^k\rangle + \langle\hat{\sigma}_c^{\dagger k}\rangle\big], \tag{12}$$
where $\hat{\eta}_a^k = |a\rangle_k\, {}_k\langle a|$, $\hat{\eta}_b^k = |b\rangle_k\, {}_k\langle b|$, and $\hat{\eta}_c^k = |c\rangle_k\, {}_k\langle c|$.
It can be noted that expressions (7)–(12) are nonlinear and coupled differential equations. Therefore, it is not possible to obtain exact solutions. Then, employing the large-time approximation scheme on equations (4) and (5), one obtains
$$\hat{a} = -\frac{2g}{\kappa}\hat{\sigma}_a^k, \tag{13}$$
$$\hat{b} = -\frac{2g}{\kappa}\hat{\sigma}_b^k. \tag{14}$$

Now introducing equations (13) and (14) into (7)-(12) and sum over the N three-level atoms,it is possible to see that
$$\frac{d}{dt}\langle\hat{m}_a\rangle = -\gamma_c\langle\hat{m}_a\rangle + \frac{\Omega}{2}\langle\hat{m}_b^\dagger\rangle, \tag{15}$$
$$\frac{d}{dt}\langle\hat{m}_b\rangle = -\frac{\gamma_c}{2}\langle\hat{m}_b\rangle - \frac{\Omega}{2}\langle\hat{m}_a^\dagger\rangle, \tag{16}$$
$$\frac{d}{dt}\langle\hat{m}_c\rangle = -\frac{\gamma_c}{2}\langle\hat{m}_c\rangle + \frac{\Omega}{2}\big[\langle\hat{N}_c\rangle - \langle\hat{N}_a\rangle\big], \tag{17}$$
$$\frac{d}{dt}\langle\hat{N}_a\rangle = -\gamma_c\langle\hat{N}_a\rangle + \frac{\Omega}{2}\big[\langle\hat{m}_c\rangle + \langle\hat{m}_c^\dagger\rangle\big], \tag{18}$$
$$\frac{d}{dt}\langle\hat{N}_b\rangle = -\gamma_c\langle\hat{N}_b\rangle + \gamma_c\langle\hat{N}_a\rangle, \tag{19}$$
$$\frac{d}{dt}\langle\hat{N}_c\rangle = -\gamma_c\langle\hat{N}_b\rangle - \frac{\Omega}{2}\big[\langle\hat{m}_c\rangle + \langle\hat{m}_c^\dagger\rangle\big], \tag{20}$$

in which $\gamma_c = \frac{4g^2}{\kappa}$ is the stimulated emission decay constant,

$$\hat{m}_a = \sum_{k=1}^{N} \hat{\sigma}_a^k, \quad \hat{m}_b = \sum_{k=1}^{N} \hat{\sigma}_b^k, \quad \hat{m}_c = \sum_{k=1}^{N} \hat{\sigma}_c^k,$$

$$\hat{N}_a = \sum_{k=1}^{N} \hat{\eta}_a^k, \quad \hat{N}_b = \sum_{k=1}^{N} \hat{\eta}_b^k, \quad \hat{N}_c = \sum_{k=1}^{N} \hat{\eta}_c^k,$$

with the operators $\hat{N}_a$, $\hat{N}_b$, and $\hat{N}_c$ representing the number of atoms in the top, intermediate, and bottom levels, respectively. In addition, employing the completeness relation $\hat{I} = \hat{\eta}_a^k + \hat{\eta}_b^k + \hat{\eta}_c^k$, we get

$$\langle \hat{N}_a \rangle + \langle \hat{N}_b \rangle + \langle \hat{N}_c \rangle = N. \tag{21}$$

Furthermore, using the definition and setting for any k

$$\hat{\sigma}_a^k = |b\rangle\langle a|, \tag{22}$$

we have

$$\hat{m}_a = N|b\rangle\langle a|. \tag{23}$$

One can find following the same procedure described in the above that $\hat{m}_b = N|c\rangle\langle b|$, $\hat{m}_c = N|c\rangle\langle a|$, $\hat{N}_a = N|a\rangle\langle a|$, $\hat{N}_b = N|b\rangle\langle b|$, $\hat{N}_c = N|c\rangle\langle c|$. Moreover, using the definition $\hat{m} = \hat{m}_a + \hat{m}_b$ and taking into account this result, we observe that $\hat{m}^\dagger\hat{m} = N(\hat{N}_a + \hat{N}_b)$, $\hat{m}\hat{m}^\dagger = N(\hat{N}_b + \hat{N}_c)$, and $\hat{m}^2 = N\hat{m}_c$. For N three-level atoms, equations (4) and (5) rewritten as

$$\frac{d\hat{a}}{dt} = -\frac{\kappa}{2}\hat{a} + \lambda^{'}\hat{m}_a, \tag{24}$$

$$\frac{d\hat{b}}{dt} = -\frac{\kappa}{2}\hat{b} + \lambda^{''}\hat{m}_b, \tag{25}$$

in which $\lambda^{'}$ and $\lambda^{''}$ are constants whose values remain to be determined. Furthermore, using equations (13) and (14) and sum over all atoms, the commutation relations of the cavity mode operators are

$$[\hat{a}, \hat{a}^\dagger] = \frac{\gamma_c}{\kappa}\big[\hat{N}_b - \hat{N}_a\big], \tag{26}$$

$$[\hat{b}, \hat{b}^\dagger] = \frac{\gamma_c}{\kappa}\big[\hat{N}_c - \hat{N}_b\big], \tag{27}$$

where $[\hat{a}, \hat{a}^\dagger] = \sum_{k=1}^{N}[\hat{a}, \hat{a}^\dagger]_k$, and $[\hat{b}, \hat{b}^\dagger] = \sum_{k=1}^{N}[\hat{b}, \hat{b}^\dagger]_k$ stand for the commutators $\hat{a}$ and $\hat{a}^\dagger$, and $\hat{b}$ and $\hat{b}^\dagger$. On the other hand, employing the steady-state solutions of (24) and (25), one can easily verify that

$$[\hat{a}, \hat{a}^\dagger] = N\bigg(\frac{2\lambda^{'}}{\kappa}\bigg)^2\bigg(\hat{N}_b - \hat{N}_a\bigg), \tag{28}$$

$$[\hat{b}, \hat{b}^\dagger] = N\bigg(\frac{2\lambda^{''}}{\kappa}\bigg)^2\bigg(\hat{N}_c - \hat{N}_b\bigg). \tag{29}$$

Thus inspection of equations (26) and (27) with (28) and (29) shows that

$$\lambda' = \lambda'' = \pm\frac{g}{\sqrt{N}}. \tag{30}$$

Hence in view of this result, equations (24) and (25) can be rewritten as

$$\frac{d\hat{a}}{dt} = -\frac{\kappa}{2}\hat{a} + \frac{g}{\sqrt{N}}\hat{m}_a, \tag{31}$$

$$\frac{d\hat{b}}{dt} = -\frac{\kappa}{2}\hat{b} + \frac{g}{\sqrt{N}}\hat{m}_b. \tag{32}$$

Now adding Eqs. (26) and (27), and the sum of (31) and (32), we get

$$[\hat{c}, \hat{c}^\dagger] = \frac{\gamma_c}{\kappa}[\hat{N}_c - \hat{N}_a], \tag{33}$$

$$\frac{d\hat{c}}{dt} = -\frac{\kappa}{2}\hat{c} + \frac{g}{\sqrt{N}}\hat{m}, \tag{34}$$

in which $\hat{c} = \hat{a} + \hat{b}$.

Furthermore, employing large-time approximation scheme on equations (13) and (14), one can obtain $\langle\hat{m}_a(t)\rangle = \langle\hat{m}_b(t)\rangle = 0$. Hence, the steady-state solutions of the expectation values of the cavity mode operators described in equations (31), (32), and (34) is found to be $\langle\hat{a}(t)\rangle_{ss} = \langle\hat{b}(t)\rangle_{ss} = \langle\hat{a}(t)\rangle_{ss} = 0$. Therefore, in view of the linear equations described by expressions (31), (32) and (33) with the corresponding solutions, we claim that $\hat{a}(t)$, $\hat{b}(t)$, and $\hat{c}(t)$ are Gaussian variables with zero means.

In addition, the steady-state solutions of the atomic operators are found to be

$$\langle\hat{N}_a\rangle_{ss} = \left[\frac{\Omega^2}{\gamma_c^2 + 3\Omega^2}\right]N, \tag{35}$$

$$\langle\hat{N}_b\rangle_{ss} = \left[\frac{\Omega^2}{\gamma_c^2 + 3\Omega^2}\right]N, \tag{36}$$

$$\langle\hat{N}_c\rangle_{ss} = \left[\frac{\gamma_c^2 + \Omega^2}{\gamma_c^2 + 3\Omega^2}\right]N, \tag{37}$$

$$\langle\hat{m}_c\rangle_{ss} = \left[\frac{\Omega\gamma_c}{\gamma_c^2 + 3\Omega^2}\right]N. \tag{38}$$

Up on setting $\eta = \frac{\Omega}{\gamma_c}$, we can rewrite Eqs. (35)-(38) as

$$\langle\hat{N}_a\rangle_{ss} = \left[\frac{\eta^2}{1 + 3\eta^2}\right]N, \tag{39}$$

$$\langle\hat{N}_b\rangle_{ss} = \left[\frac{\eta^2}{1 + 3\eta^2}\right]N, \tag{40}$$

$$\langle \hat{N}_c \rangle_{ss} = \left[\frac{1+\eta^2}{1+3\eta^2} \right] N, \tag{41}$$

$$\langle \hat{m}_c \rangle_{ss} = \left[\frac{\eta}{1+3\eta^2} \right] N. \tag{42}$$

Initially (when $\Omega = 0$), all the atoms are on the lower level ($\langle \hat{N}_c \rangle_{ss} = N$) while the number of atoms on the top and intermediate levels are zero.

4 Photon Statistics

We next seek to calculate the mean and variance of the photon number at steady state. We will also consider some special cases of interest.

4.1 Mean photon number

To learn about the brightness of the generated light, it is necessary to study the mean number of photon pairs describing the two-mode cavity radiation that can be defined as

$$\bar{n} = \langle \hat{c}^\dagger \hat{c} \rangle, \tag{43}$$

where $\hat{c}$ is the annihilation operator of the two-mode cavity. Employing the steady-state solution of equation (34), it can be found to be

$$\bar{n} = \frac{\gamma_c}{k} \left[\langle \hat{N}_a \rangle_{ss} + \langle \hat{N}_b \rangle_{ss} \right]. \tag{44}$$

With the aid of equations (39) and (40), one can readily show that

$$\bar{n} = \left(\frac{2\gamma_c}{k} N \right) \left[\frac{\eta^2}{1+3\eta^2} \right]. \tag{45}$$

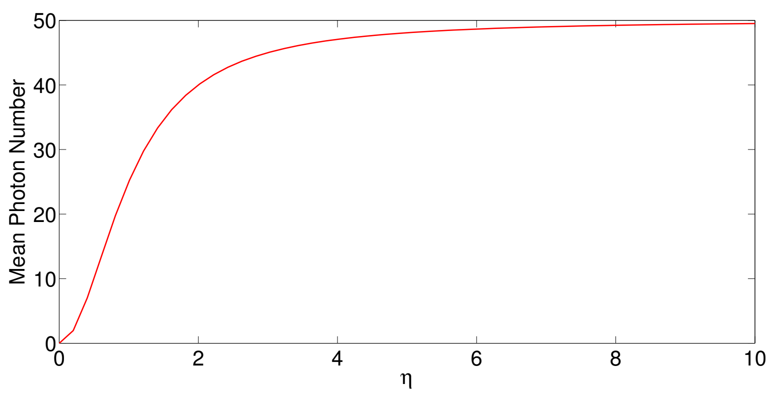

Figure 2. (Color online) Plots of $\bar{n}$ vs. η for $\gamma_c = 0.4$, $\kappa = 0.8$, and $N = 50$.

It is not difficult to see for $\Omega \gg \gamma_c$ that

$$\bar{n} = \frac{2\gamma_c}{3\kappa} N. \tag{46}$$

We see from Figure 2 that the mean photon number of the two-mode light increases with η.

4.2 Two-mode photon-number variance

Here we proceed to study the steady-state photon number variance of the two-mode light beam, produced by the coherently driven nondegenerate three-level laser in a closed cavity and coupled to a two-mode vacuum reservoir. The photon number variance for the two-mode cavity light is expressible as

$$(\Delta n)^2 = \langle \hat{c}^\dagger \hat{c} \hat{c}^\dagger \hat{c} \rangle - \langle \hat{c}^\dagger \hat{c} \rangle^2. \tag{47}$$

Since $\hat{c}$ is Gaussian variable with zero mean, the variance of the photon number can be written as

$$(\Delta n)^2 = \langle \hat{c}^\dagger \hat{c} \rangle \langle \hat{c} \hat{c}^\dagger \rangle + \langle \hat{c}^{\dagger 2} \rangle \langle \hat{c}^2 \rangle. \tag{48}$$

With the aid of the steady-stae solution of Eq. (34), one can easily establish that

$$\langle \hat{c} \hat{c}^\dagger \rangle = \frac{\gamma_c}{\kappa} \left[\langle \hat{N}_b \rangle + \langle \hat{N}_c \rangle \right] \tag{49}$$

and

$$\langle \hat{c}^2 \rangle = \frac{\gamma_c}{\kappa} \langle \hat{m}_c \rangle. \tag{50}$$

Since $\langle \hat{m}_c \rangle$ is real, then $\langle \hat{c}^2 \rangle = \langle \hat{c}^{\dagger 2} \rangle$. Therefore, with the aid of Eqs. (31), (49) and (50), Eq. (48) turns out to be

$$(\Delta n)^2 = \left(\frac{\gamma_c}{\kappa}\right)^2 \left[\left(\langle \hat{N}_a \rangle + \langle \hat{N}_b \rangle \right) \left(\langle \hat{N}_b \rangle + \langle \hat{N}_c \rangle \right) + \langle \hat{m}_c \rangle^2 \right]. \tag{51}$$

Furthermore, upon substituting of Eqs. (39)-(42) into Eq. (51), we see that

$$(\Delta n)^2 = \left(\frac{\gamma_c}{\kappa} N \right)^2 \left[\frac{3\eta^2 + 4\eta^4}{1 + 6\eta^2 + 9\eta^4} \right]. \tag{52}$$

This is the steady-state photon number variance of the two-mode light beam, produced by the coherently driven nondegenerate three-level laser with a closed cavity and coupled to a two-mode vacuum reservoir (see Figure 3).

Moreover, we note that for $\eta \gg 1$, Eq. (52) reduces to

$$(\Delta n)^2 = \left[\frac{2\gamma_c}{3\kappa} N \right]^2 \tag{53}$$

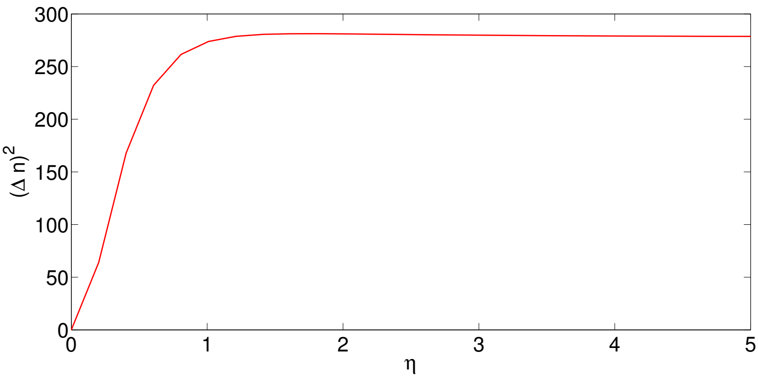

Figure 3. (Color online) Plots of $(\Delta n)^2$ vs. η for $\gamma_c = 0.4$, $\kappa = 0.8$, and $N = 50$.

and in view of Eq. (46), we have

$$(\Delta n)^2 = \bar{n}^2, \tag{54}$$

which represents the normally-ordered variance of the photon number for chaotic light.

5 Quadrature Fluctuations

In this section, we seek to obtain the quadrature variance and squeezing of the two-mode light in a closed cavity produced by a coherently driven nondegenerate three-level laser.

5.1 Quadrature variance

The squeezing properties of the two-mode cavity light are described by two quadrature operators

$$\hat{c}_+ = \hat{c}^\dagger + \hat{c}, \tag{55}$$

$$\hat{c}_- = i(\hat{c}^\dagger - \hat{c}), \tag{56}$$

where $\hat{c} = \hat{a} + \hat{b}$.

Making use of the well-known definition of the variance of an operator, the variances of the quadrature operators (55) and (56) are found to have the form

$$(\Delta\hat{c}_\pm)^2 = \langle\hat{c}\hat{c}^\dagger\rangle + \langle\hat{c}^\dagger\hat{c}\rangle \pm \langle\hat{c}^2\rangle \pm \langle\hat{c}^{\dagger 2}\rangle \mp \langle\hat{c}\rangle^2 \mp \langle\hat{c}^\dagger\rangle^2 - 2\langle\hat{c}\rangle\langle\hat{c}^\dagger\rangle. \tag{57}$$

With the aid of the steady-state solution of equation (34), one can easily establish that

$$(\Delta\hat{c}_\pm)^2 = \frac{\gamma_c}{k}\big[N + \langle\hat{N}_b\rangle_{ss} \pm 2\langle\hat{m}_c\rangle_{ss}\big], \tag{58}$$

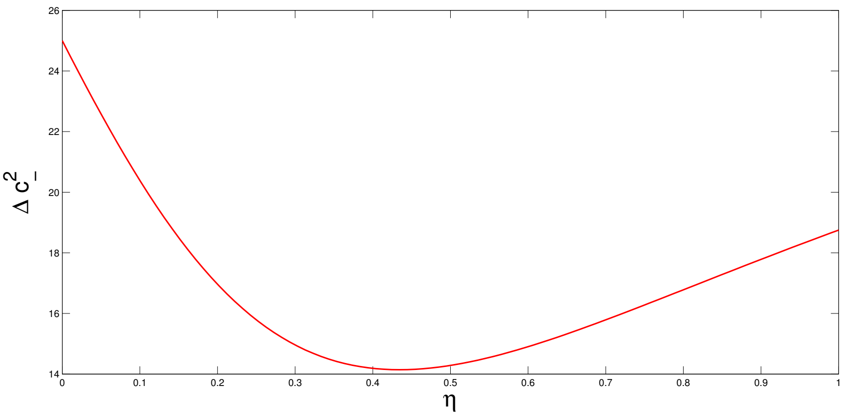

Figure 4. (Color online) Plot of $(\Delta c_-)^2$ vs. η for $\gamma_c = 0.4$, $k = 0.8$, and $N = 50$.

from which follows

$$(\Delta \hat{c}_\pm)^2 = \frac{\gamma_c}{k} N \left[\frac{4\eta^2 \pm 2\eta + 1}{1 + 3\eta^2} \right] \tag{59}$$

and for $\Omega \gg \gamma_c$

$$(\Delta \hat{c}_\pm)^2 = \frac{4\gamma_c}{3k} N = 2\bar{n}, \tag{60}$$

where $\bar{n}$ is given by equation (46). It can be seen that expression (60) represents the normally ordered quadrature variance for chaotic light. Moreover, for the case in which the deriving coherent light is absent, one can see that

$$(\Delta \hat{c}_+)^2_v = (\Delta \hat{c}_-)^2_v = \frac{\gamma_c}{k} N, \tag{61}$$

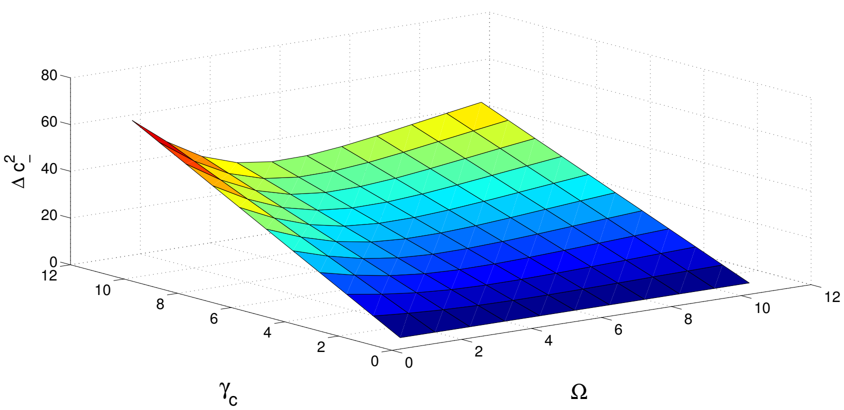

Figure 5. (Color online) Plot of $(\Delta c_-)^2$ vs Ω and γ_c for $k = 0.8$, $N = 50$.

which is the normally ordered quadrature variance of the two-mode cavity light in vacuum state. It is also observed that, the uncertainty in the plus and minus quadratures are equal and satisfy the minimum uncertainty relation.

5.2 The quadrature squeezing

The quadrature squeezing of the two-mode cavity light relative to the quadrature variance of the two-mode vacuum light can be defined as

$$S = \frac{(\Delta\hat{c}_{\pm})^2_v - (\Delta\hat{c}_{-})^2}{(\Delta\hat{c}_{\pm})^2_v}, \tag{62}$$

where $(\Delta\hat{c}_{\pm})^2_v$ is the quadrature variance in vacuum state given by equation (61). Taking into account equations (59) and (61), (62) yields

$$S = \frac{2\eta - \eta^2}{1 + 3\eta^2}. \tag{63}$$

We observe that in Eq. (63), unlike the mean photon number, the quadrature squeezing does not depend on the number of atoms. This implies that the quadrature squeezing of the cavity light is independent of the number of photons.

The plot in Figure 6 shows that the maximum squeezing of the cavity light is 43% degree of squeezing and occurs when the three-level laser is operating at $\eta = 0.4$. Hence one can observe that a coherently driven light produced by a nondegenerate three-level laser can exhibit equal degree of squeezing when, for example, compared to the light generated by a three-level laser in which the three-level atoms available in a closed cavity are pumped to the top level by means of electron bombardment [20, 24].

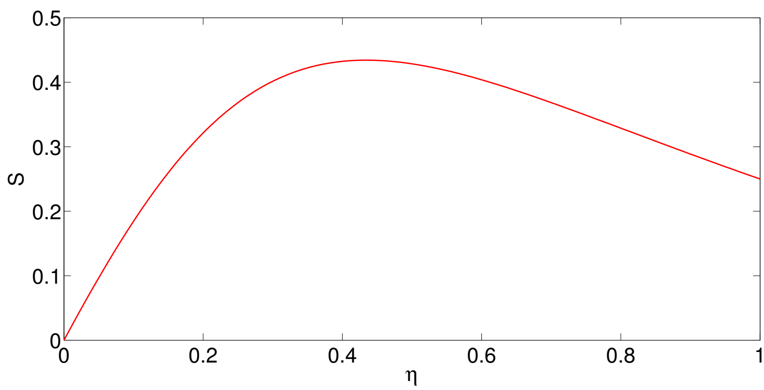

Figure 6. (Color online) Plots of the quadrature squeezing vs. η for $\gamma_c = 0.4$.

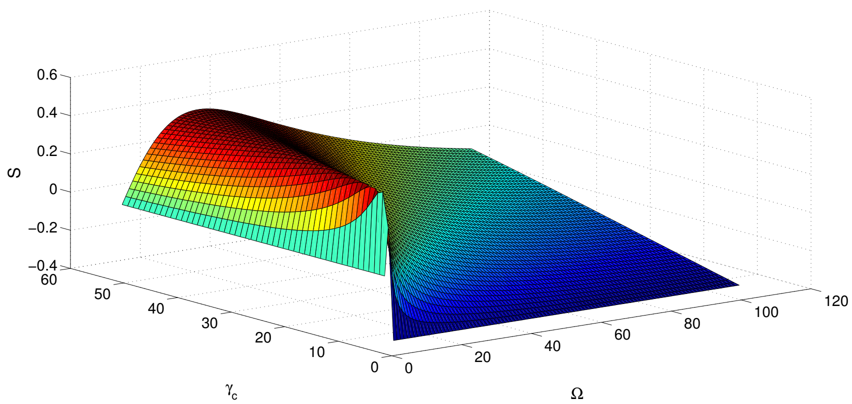

Figure 7. (Color online) Plot of the quadrature squeezing vs. Ω and γ_c.

6 Entanglement Properties of the Two-Mode Light

The quantum entanglement between the two cavity modes a and b proposed by Duan-Giedke-Cirac-Zoller (DGCZ) [25], which is a sufficient condition for entangled quantum states. According to DGCZ, a quantum state of a system is said to be entangled if the sum of the variances of the EPR-like quadrature operators, $\hat{u}$ and $\hat{v}$, satisfy the inequality

$$(\Delta\hat{u})^2 + (\Delta\hat{v})^2 < 2N, \tag{64}$$

where

$$\hat{u} = \hat{x}_a - \hat{x}_b, \qquad \hat{v} = \hat{p}_a + \hat{p}_b, \tag{65}$$

with

$$\hat{x}_a = (\hat{a}^\dagger + \hat{a})/\sqrt{2}, \qquad \hat{x}_b = (\hat{b}^\dagger + \hat{b})/\sqrt{2},$$
$$\hat{p}_a = i(\hat{a}^\dagger - \hat{a})/\sqrt{2}, \qquad \hat{p}_b = i(\hat{b}^\dagger - \hat{b})/\sqrt{2}$$

being the quadrature operators for modes $\hat{a}$ and $\hat{b}$. Taking into account (65), equation (64) yields

$$\Delta\hat{u}^2 + \Delta\hat{v}^2 = 2\frac{\gamma_c}{\kappa}[N + \langle\hat{N}_b\rangle_{ss} - \langle\hat{m}_c\rangle_{ss}]. \tag{66}$$

Thus in view of equation (65), togather with (58), the sum of the variances of $\hat{u}$ and $\hat{v}$ can be expressed as

$$\Delta\hat{u}^2 + \Delta\hat{v}^2 = 2\Delta c_-^2, \tag{67}$$

where Δc_-^2 is given by (58). One can readily see from this result that the degree of entanglement is directly proportional to the degree of squeezing of the two-mode light (see Figure 8).

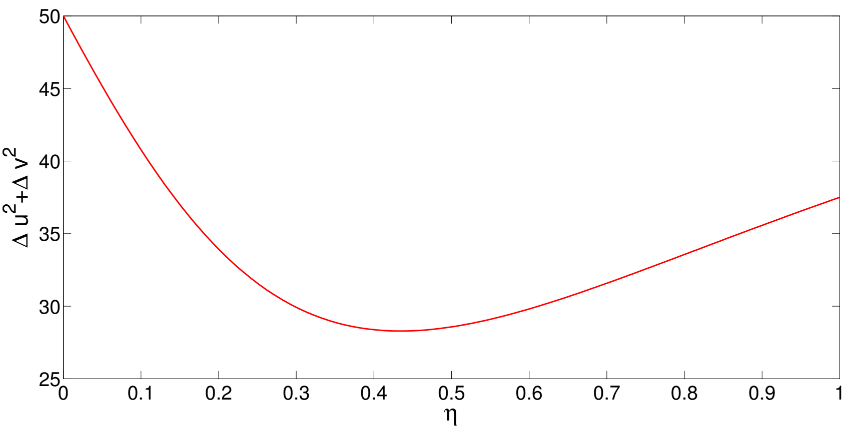

Figure 8. (Color online) Plot of the $\Delta\hat{u}^2 + \Delta\hat{v}^2$ vs. η for $\gamma_c = 0.4$, $\kappa = 0.8$, and $N = 50$.

One can immediately notice that, this particular entanglement measure is directly related the two-mode squeezing. This direct relationship shows that whenever there is a two-mode squeezing in the system there will be entanglement in the system as well. It is noted that the entanglement disappears when the squeezing vanishes. This is due to the fact that the entanglement is directly related to the squeezing as given by (58). It also follows that like the mean photon number and quadrature variance the degree of entanglement depends on the number of atom. With the help of the criterion (64) that a significant entanglement between the states of the light generated in the cavity. This is due to the strong correlation between the radiation emitted when the atoms decay from the upper energy level to the lower via the intermediate level.

In the following, the sum of the variances of a pair of EPR-type operators $\Delta\hat{u}^2 + \Delta\hat{v}^2$ is plotted against the amplitude of the driving coherent light so that the available entanglement is clearly evident for various values of η between 0 and 1.

7 Normalized Second-Order Correlation Functions

The second-order correlation function of the separate mode as well as for the superposition of the two modes of the cavity radiation at equal time can also be investigated using [26, 27]

$$g^{(2)}_{(a,b)}(0) = \frac{\langle\hat{a}^\dagger\hat{a}\hat{b}^\dagger\hat{b}\rangle}{\langle\hat{a}^\dagger\hat{a}\rangle\langle\hat{b}^\dagger\hat{b}\rangle}. \tag{68}$$

Since $\hat{a}$ and $\hat{b}$ are Gaussian variables with vanishing means, then the normalized second-order correlation function for the two-mode light takes, at steady-state, the form

$$g^{(2)}_{(a,b)}(0) = 1 + \frac{\langle \hat{b}\hat{a}\rangle\langle \hat{a}^{\dagger}\hat{b}^{\dagger}\rangle}{\langle \hat{a}^{\dagger}\hat{a}\rangle\langle \hat{b}^{\dagger}\hat{b}\rangle}. \tag{69}$$

It then follows that

$$g^{(2)}_{(a,b)}(0) = 1 + \frac{\langle \hat{m}_c\rangle^2_{ss}}{\langle \hat{N}_a\rangle_{ss}\langle \hat{N}_b\rangle_{ss}}, \tag{70}$$

so that in view of (39), (40), and (42), one obtains

$$g^{(2)}_{(a,b)}(0) = 1 + \eta^2. \tag{71}$$

It can be that from this result the second-order correlation function of the two-mode light does not depend on the number of atoms.

Now it is essential to calculate the second-order correlation function for the individual mode to have an insight for the previous result. To this end, the second order correlation function for mode a is given by

$$g^{(2)}_{(a,a)}(0) = \frac{\langle : \hat{n}_a\hat{n}_a :\rangle}{\langle \hat{n}_a\rangle^2}, \tag{72}$$

where :: represent normal ordering and $\hat{n}_a = \hat{a}^{\dagger}\hat{a}$ is the photon number operator for mode a. Since $\hat{a}$ is a Gaussian variable with vanishing mean, one can easily verify that

$$g^{(2)}_{(a,a)}(0) = 2. \tag{73}$$

Similarly, the second-order correlation function for mode b is found to be

$$g^{(2)}_{(b,b)}(0) = 2. \tag{74}$$

From the expressions 73 and (74), we note that the second-order correlation function for light in a chaotic state. So, the cavity modes a and b are separately in a chaotic or thermal state.

Figure 9 show that the second-order correlation function $g^{(2)}_{(a,b)}(0)$ increases when η increases. It can be observed from Figure 10 that the second-order correlation function increases when Ω increases and when γ_c decreases.

Furthermore, in order to quantify the correlation between the two modes, we introduce the linear correlation coefficient in terms of a covariance as [28]

$$J_{(\hat{n}_a,\hat{n}_b)} = \frac{cov(\hat{n}_a, \hat{n}_b)}{\sqrt{\Delta\hat{n}_a^2}\sqrt{\Delta\hat{n}_b^2}}, \tag{75}$$

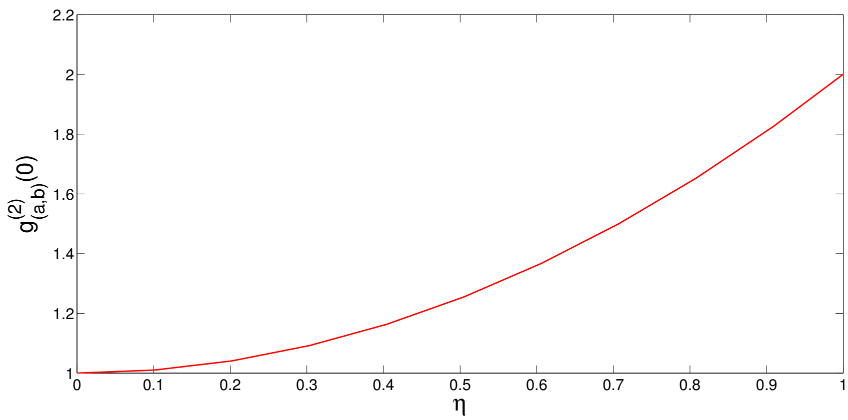

Figure 9. (Color online) Plot of the $g^{(2)}_{(a,b)}(0)$ of the two-mode cavity light vs η.

where $\Delta\hat{n}_a^2$ and $\Delta\hat{n}_b^2$ are the variances of the photon number for modes a and b, respectively. So, the covariance of the photon numbers is defined by

$$cov(\hat{n}_a, \hat{n}_b) = \langle\hat{n}_a\hat{n}_b\rangle - \langle\hat{n}_a\rangle\langle\hat{n}_b\rangle. \tag{76}$$

One can easily verify, using the fact that $\hat{a}$ and $\hat{b}$ are Gaussian variables, in the steady state that

$$cov(\hat{n}_a, \hat{n}_b) = \langle\hat{b}\hat{a}\rangle_{ss}\langle\hat{a}^\dagger\hat{b}^\dagger\rangle_{ss}. \tag{77}$$

Since the cavity modes are separately in a chaotic state the variances of the photon numbers obey the relation for a chaotic state, $\Delta\hat{n}_a^2 = \langle\hat{n}_a\rangle + \langle\hat{n}_a\rangle^2$ and $\Delta\hat{n}_b^2 = \langle\hat{n}_b\rangle + \langle\hat{n}_b\rangle^2$. On account of this fact and (77), the correlation function

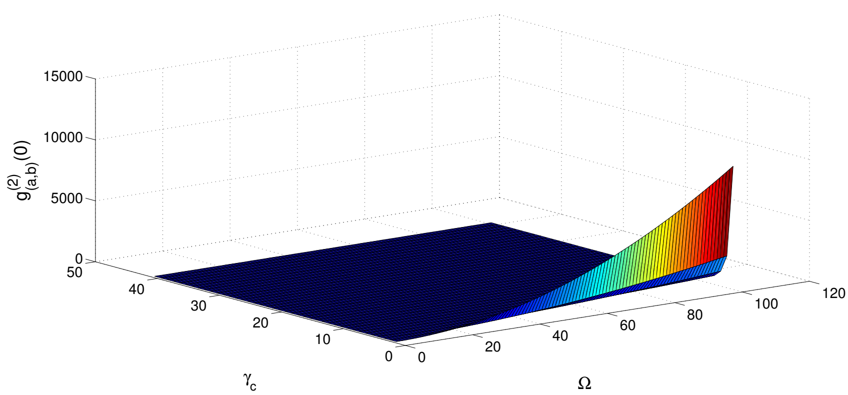

Figure 10. (Color 0nline) Plot of the $g^{(2)}_{(a,b)}(0)$ of the two-mode cavity light vs Ω and γ_c.

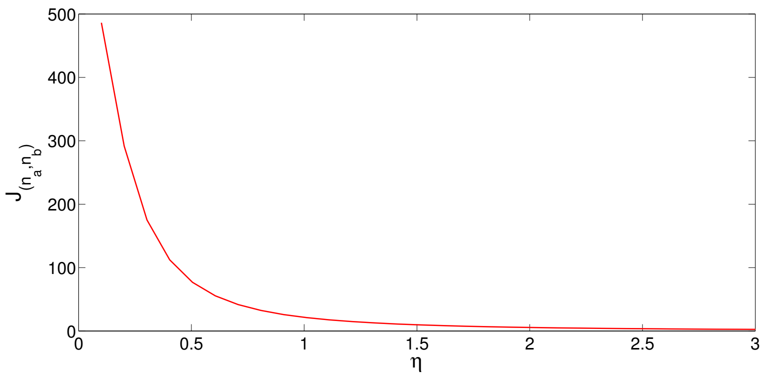

Figure 11. (Color online) Plot of the $J_{(\hat{n}_a,\hat{n}_b)}$ vs. η for $\gamma_c = 0.4$, $\kappa = 0.8$, and $N = 50$.

can be rewritten as

$$J_{(\hat{n}_a,\hat{n}_b)} = \frac{\langle \hat{b}\hat{a}\rangle_{ss}\langle \hat{a}^{\dagger}\hat{b}^{\dagger}\rangle_{ss}}{\sqrt{\langle \hat{n}_a\rangle_{ss} + \langle \hat{n}_a\rangle_{ss}^2}\sqrt{\langle \hat{n}_b\rangle_{ss} + \langle \hat{n}_b\rangle_{ss}^2}}. \tag{78}$$

In Figure 11, the linear correlation coefficient versus η is plotted. It is also found from this figure that for η very close to 0 the the intermode correlation would be significantly large, since the mean photon numbers of the light in modes a and b are very close to zero when initially almost all atoms are populated in the lower level. Moreover, similar to the second-order correlation function, the plots of Figure 11 show that the linear correlation coefficient vanishes when $\Omega < 0.05$.

8 Conclusion

In conclusion, the squeezing and entanglement properties of a non-degenerate three-level laser driven by coherent light and coupled to a two-mode vacuum reservoir via a single-port mirror whose closed cavity contains N non-degenerate three-level atoms, are thoroughly analyzed. It is carried out the analysis by putting the noise operators associated with the vacuum reservoir in normal order and by considering the interaction of the three-level atoms with the vacuum reservoir outside the cavity. The interaction Hamiltonian and the quantum Langevin equations for the cavity light are obtained. Applying these equations, the equations of evolution of the cavity mode and the atomic operators are solved. Making use of the steady-state solutions of atomic and cavity mode operators, the quadrature variance, the quadrature squeezing, and the entanglement for the two-mode cavity light, at steady state, are determined. In addition, the normalized second-order correlation function is obtained for the individual mode as well as for the superposition of the two modes. Finally, it is obtained that the linear correlation coefficient between the two modes.

The analysis showed that the intracavity quadrature squeezing is enhanced due to the driven coherent light. It is found that the squeezing and entanglement in the two-mode light is directly related. As a result, an increase in the degree of squeezing directly implies an increase in the degree of entanglement. This shows that whenever there is squeezing in the two-mode light, there exists entanglement in the system. In addition, it has shown that the photons in the laser cavity are highly correlated and the degree of photon number correlation as the pumping, Ω, increases.

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