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On the Perfect Fluid Lorentzian Para-Sasakian Spacetimes

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Abstract. The object of the present paper is to study the properties of Ricci and Yamabe solitons on the perfect fluid LP-Sasakian spacetimes. Certain results related to the application of such spacetimes in the general relativity and cosmology are obtained.

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1 Introduction

In 1982, Hamilton [1] made the fundamental observation that the Ricci flow is an excellent tool for simplifying the structure of a manifold. It is a process which deforms the metric of a Riemannian manifold M by smoothing out the irregularities. It is given by

$$\frac{\partial g}{\partial t} = -2Ric, \tag{1.1}$$

where g is a Riemannian metric, Ric is the Ricci tensor and t is a time. Let $\phi_t : M \to M, t \in R$ be a family of diffeomorphisms, which is one parameter group of transformations, then it gives rise to a vector field called the infinitesimal generator and integral curves. The Ricci solitons move under the Ricci flow simply by diffeomorphisms of the initial metric, that is, they are stationary points of the Ricci flow in space of metrics of $\phi_t : M \to M$. Here the metric $g(t)$ is the pull back of the initial metric $g(0)$ of ϕ_t. A Ricci soliton on a Riemannian manifold (M, g) is a special solution to the Ricci flow and is a natural generalization of an Einstein metric which is defined as a triple (g, V, λ) with g a Riemannian metric, V a vector field and λ a real scalar such that

$$\mathfrak{L}_V g + 2S + 2\lambda g = 0, \tag{1.2}$$

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where S is a Ricci tensor and $\mathfrak{L}_V g$ is the Lie-derivative of the metric g along the vector field V on M and λ is a real number. A Ricci soliton (g, V, λ) on M is said to be shrinking, steady and expanding when λ is negative, zero and positive, respectively.

An Yamabe soliton is defined on a Riemannian (or pseudo-Riemannian) manifold (M, g) by a vector field V satisfying the equation

$$\frac{1}{2}(\mathfrak{L}_V g) = (\kappa - \lambda)g, \tag{1.3}$$

where κ is the scalar curvature and λ is a soliton constant [2]. An Yamabe soliton is said to be expanding, steady or shrinking, respectively, if $\lambda < 0, \ \ \lambda = 0$ or $\lambda > 0$. The study of the Yamabe flow appeared in the work of Hamilton [1] as a tool to construct Yamabe metric on the compact Riemannian manifolds. A time-dependent metric $g(\cdot\,, t)$ on a Riemannian (or pseudo-Riemannian) manifold M is said to evolve by the Yamabe flow if the metric g satisfies

$$\frac{\partial g(t)}{\partial t} = -\kappa g(t), \quad g(0) = g_0 \tag{1.4}$$

on M. An Yamabe soliton is a special soliton of the Yamabe flow that moves by one parameter family of diffeomorphisms ϕ_t generated by a fixed vector field V on M [3]. Ye [4] has found that a point-wise elliptic gradient estimate for the Yamabe flow on a locally conformally flat compact Riemannian manifold. Hui et al. [5] considered Kenmotsu manifolds and found some geometrical results of the Yamabe solitons. In case of the Ricci flow, the Yamabe soliton or the singularities of the Yamabe flow appeared naturally.

The significance of Yamabe flow lies in the fact that it is a natural geometric deformation to metric of constant scalar curvature. It is noted that the Yamabe flow corresponds to the fast diffusion case of porous medium equation (the plasma equation) in mathematical physics. Just as the Ricci soliton is a special solution of the Ricci flow, an Yamabe soliton is a special solution of the Yamabe flow that moves by one parameter family of diffeomorphisms ϕ_t generated by a fixed vector field V on M, and homothetic, $i.e.$, $g(\,;t) = \varsigma(t)\phi_*(t)g_0$.

Given an Yamabe soliton, if $V = Df$ holds for a smooth function f on M, the equation (1.3) becomes $Hess\, f = (r - \lambda)g$, where $Hess\, f$ denotes the Hessian of f and D denotes the gradient operator of g on M. In this case f is called the potential function of the Yamabe soliton and Df is said to be a gradient Yamabe soliton.

Definition 1.1. *A vector field X on an almost contact Riemannian manifold M is said to be an infinitesimal transformation [6] if there exists a smooth function υ on M such that*

$$(\mathfrak{L}_X \eta)(Y) = \upsilon \eta(Y) \tag{1.5}$$

for every smooth vector fields X and Y on M.

If $\upsilon = 0$, then X is called a strict infinitesimal transformation on M.

Definition 1.2. *A vector field V on an n-dimensional semi-Riemannian manifold (M, g) is said to be a conformal vector field if*

$$\mathfrak{L}_V g = 2\psi\, g$$

holds for the smooth function ψ [7]. It is observed that the conformal vector field V on (M, g) satisfies the relations

$$(\mathfrak{L}_V S)(X, Y) = -(n-2)g(\nabla_X D\psi, Y) + (\triangle\psi)g(X, Y) \tag{1.6}$$

and

$$\mathfrak{L}_V \kappa = -2\psi r + 2(n-1)\triangle\psi \tag{1.7}$$

for all vector fields X and Y on M, where D and $\triangle$ denote the gradient operator and the Laplacian operator, respectively.

2 Preliminaries

Let M^n be an n-dimensional differentiable manifold endowed with a $(1,1)$-tensor field ϕ, a contravariant vector field ξ, a covariant vector field η and a Lorentzian metric g of type $(0,2)$ such that for each point $p \in M^n$ the tensor $g_p : T_pM \times T_pM \to \Re$ is an inner product of signature $(-, +, +, ..., +)$, where T_pM denotes the tangent space of M^n at p and $\Re$ is the real number space which satisfies

$$\phi^2(X) = X + \eta(X)\xi, \quad \eta(\xi) = -1, \tag{2.1}$$

$$g(X, \xi) = \eta(X), \quad g(\phi X, \phi Y) = g(X, Y) + \eta(X)\eta(Y) \tag{2.2}$$

for all vector fields X, Y on M^n, then the structure (ϕ, ξ, η, g) is known as the Lorentzian almost paracontact structure and the manifold M^n equipped with the structure (ϕ, ξ, η, g) is called the Lorentzian almost paracontact manifold [8]. In the Lorentzian almost paracontact manifold M^n, the following relations hold [8].

$$\phi\xi = 0, \quad \eta(\phi X) = 0, \quad \Omega(X, Y) = \Omega(Y, X), \tag{2.3}$$

where $\Omega(X, Y) = g(X, \varphi Y)$. If the Lorentzian almost paracontact manifold M^n satisfies

$$\begin{aligned}(\nabla_Z \Omega)(X, Y) = \alpha\big[&\{g(X, Z) + \eta(X)\eta(Z)\}\eta(Y) \\ &+ \{g(Y, Z) + \eta(Y)\eta(Z)\}\eta(X)\big]\end{aligned} \tag{2.4}$$

and

$$\Omega(X, Y) = \frac{1}{\alpha}(\nabla_X \eta)(Y) \tag{2.5}$$

for all vector fields X, Y Z on M^n, where ∇ denotes the operator of covariant differentiation with respect to the Lorentzian metric g and α is a non-zero scalar function, then M^n is called an LP-Sasakian manifold with the coefficient α [9]. An LP-Sasakian manifold with the coefficient $\alpha = 1$ is an LP-Sasakian manifold [8]. If a vector field V satisfies the equation

$$\nabla_X V = \alpha X + A(X)V, \tag{2.6}$$

where α is a non-zero scalar function and A is a non-zero 1-form, then V is called a torse-forming vector field [10].

Let ξ is a unit torse-forming vector field in the Lorentzian manifold M^n. Then we have

$$\nabla_X \xi = \alpha X + A(X)\xi. \tag{2.7}$$

Since $g(\xi, \xi) = -1$, which implies that $g(\nabla_X \xi, \xi) = 0$. From (2.7), we get

$$A(X) = \alpha\eta(X). \tag{2.8}$$

Also,

$$(\nabla_X \eta)(Y) = \nabla_X \eta(Y) - \eta(\nabla_X Y) = g(Y, \nabla_X \xi). \tag{2.9}$$

In view of (2.7)-(2.9), we yield

$$(\nabla_X \eta)(Y) = \alpha[g(X, Y) + \eta(X)\eta(Y)]. \tag{2.10}$$

Especially, if η satisfies

$$(\nabla_X \eta)(Y) = \varepsilon \{g(X, Y) + \eta(X)\eta(Y)\}, \quad \varepsilon^2 = 1. \tag{2.11}$$

Then M^n is called a Lorentzian special para-Sasakian manifold (briefly, LSP-Sasakian manifold) [9]. In particular, if α satisfies (2.10) and the equation

$$\nabla_X \alpha = d\alpha(X) = \sigma\eta(X), \tag{2.12}$$

where σ is a smooth function and η is the 1-form, then ξ is called a concircular vector field.

In an LP-Sasakian manifold $M^n(\phi, \xi, \eta, g)$ with a coefficient α, the following relations

$$\eta(R(X, Y)Z) = (\alpha^2 - \sigma)[g(Y, Z)\eta(X) - g(X, Z)\eta(Y)], \tag{2.13}$$

$$S(X, \xi) = (n - 1)(\alpha^2 - \sigma)\eta(X), \tag{2.14}$$

$$R(X, Y)\xi = (\alpha^2 - \sigma)[\eta(Y)X - \eta(X)Y], \tag{2.15}$$

$$R(\xi, Y)X = (\alpha^2 - \sigma)[g(X, Y)\xi - \eta(X)Y], \tag{2.16}$$

$$(\nabla_X \phi)(Y) = \alpha[g(X, Y)\xi + 2\eta(X)\eta(Y)\xi + \eta(Y)X], \tag{2.17}$$

$$S(\phi X, \phi Y) = S(X, Y) + (n - 1)(\alpha^2 - \sigma)g(X, Y), \tag{2.18}$$

hold for all vector fields X, Y, Z on M^n, where R and S denote, respectively, the curvature tensor and the Ricci tensor of the manifold (M^n, g) [11].

In 1983, O'Neill [12] discussed the application of semi-Riemannian geometry in the theory of relativity. The curvature structure of the spacetime is studied by Kaigorodov [13]. These ideas of the general theory of spacetime are extended by Raychaudhary et al. [14]. Chaki and Roy [15] studied the spacetimes with the covariant constant energy momentum tensor.

3 Ricci solitons on $(M^n, \phi, \xi, \eta, g)$

We call the notion of Ricci soliton from [1]. Thus from (1.2) we have

$$(\mathfrak{L}_V g)(X, Y) + 2S(X, Y) + 2\lambda g(X, Y) = 0, \tag{3.1}$$

where $\mathfrak{L}_V$ is the Lie-derivative operator along the vector field V and λ is a real constant. We have two natural situations regarding the vector field V : $V \in Span\{\xi\}$ and $V \perp \xi$. In our study we consider only the case $V = \xi$. With reference to (2.7) and (2.8), equation (3.1) reduces to

$$S(X, Y) = -(\alpha + \lambda)g(X, Y) - \alpha\,\eta(X)\eta(Y), \tag{3.2}$$

$$S(X, \xi) = S(\xi, X) = -\lambda\eta(X), \quad S(\xi, \xi) = \lambda, \tag{3.3}$$

$$QX = -(\alpha + \lambda)X - \alpha\eta(X)\xi, \tag{3.4}$$

$$\kappa = -\lambda n - (n-1)\alpha, \quad Q\xi = -\lambda\xi, \tag{3.5}$$

$$\lambda = -(n-1)(\alpha^2 - \sigma), \tag{3.6}$$

where κ is the scalar curvature of M^n and $\alpha^2 - \sigma \neq 0$. To achieve our goal, we prove the following results.

Theorem 3.1. *Suppose (M^n, g), $n > 1$, is an n-dimensional LP-Sasakian manifold with coefficient α and it admits a symmetric tensor field h of type $(0, 2)$ and a skew-symmetric tensor ϕ. If h is parallel with respect to ∇ on M^n, then the structure (ϕ, ξ, η, g) possesses a Ricci soliton (g, ξ, λ).*

Proof. Let us consider that

$$h(X, Y) = (\mathfrak{L}_\xi g)(X, Y) + 2S(X, Y). \tag{3.7}$$

In view of (2.7), (2.8) and (3.2), equation (3.7) reduces to

$$h(X, Y) = -2\lambda\,g(X, Y). \tag{3.8}$$

Replacing $X = Y = \xi$ in (3.8), we yield

$$h(\xi, \xi) = 2\lambda = -2(n-1)(\alpha^2 - \sigma) \neq 0. \tag{3.9}$$

In [16], Singh et al. proved that "On an LP-Sasakian manifold M, a second order symmetric parallel tensor is a constant multiple of associated metric tensor" (see Theorem 2.1, p-384, [16]). This fact together with the above results and the equation (3.1) give the statement of the Theorem 3.1. □

Corollary 3.2. *An n-dimensional LP-Sasakian manifold $M^n(\phi, \xi, \eta, g)$, $n > 1$, endowed with coefficient α and a Ricci soliton (g, ξ, λ) is an η-Einstein manifold and the Ricci soliton (g, ξ, λ) is shrinking and expanding according as $\alpha^2 > \sigma$ and $\alpha^2 < \sigma$, respectively.*

In particular, for $\alpha = 1$, the equation (3.9) gives $\lambda = -(n-1) < 0$, provided $n > 1$. Thus we can state:

Corollary 3.3. *A Ricci soliton (g, ξ, λ) on an n-dimensional LP-Sasakian manifold, $n > 1$, is always shrinking.*

4 Perfect fluid LP-Sasakian spacetimes

In this section, we consider the Lorentzian para-Sasakian spacetime, that is, a 4-dimensional LP-Sasakian manifold with a constant coefficient α. Since α is a constant, then from (2.12) we get $\sigma = 0$ and thus the equation (2.14) reduces to

$$S(X, \xi) = 3\alpha^2 \eta(X). \tag{4.1}$$

This shows the Ricci tensor of the Lorentzian para-Sasakian spacetime possesses the eigen value $3\alpha^2$. If l^2 denotes the square of length of the Ricci tensor, then

$$l^2 = \sum_{i=1}^{n} S(Qe_i, e_i), \tag{4.2}$$

where Q is the symmetric endomorphism of the tangent space at a point corresponding to the Ricci tensor S and $\{e_i\}_{i=1}^{n}$ is an orthonormal basis of the tangent space at each point of the manifold. If we put $X = Y = e_i$, $1 \le i \le n$, in (3.2) we get

$$\kappa = 4a_1 + b_1, \tag{4.3}$$

where $a_1 = -(\alpha + \lambda)$ and $b_1 = \alpha$. Again from (3.2) and (4.1), we obtain

$$S(\xi, \xi) = -b_1 - a_1 = \lambda = -3\alpha^2. \tag{4.4}$$

In view of (3.2), (4.2) and (4.4), we yield

$$l^2 = 3a_1^2 + (b_1 + a_1)^2. \tag{4.5}$$

This implies that l^2 is a constant and consequently $\mathfrak{L}_X l^2 = 0$. It is well known that a compact Riemannian manifold of dimension greater than 2 with constant scalar curvature admits an infinitesimal non-isometric conformal transformation X such that $\mathfrak{L}_X l^2 = 0$, then it is an Einstein manifold [7].

By considering this fact together with the above results, we state the following theorem.

Theorem 4.1. *If a Lorentzian para-Sasakian spacetime together with the Ricci soliton (g, ξ, λ) admits an infinitesimal non-isometric conformal transformation, then the manifold is Einstein and the soliton is always shrinking.*

In light of the equation (4.3) and the Theorem 4.1, we state the following corollaries.

Corollary 4.2. *An η-Einstein Lorentzian para-Sasakian spacetime along with an infinitesimal non-isometric conformal transformation does not admits a proper Ricci soliton (g, ξ, λ).*

Corollary 4.3. *A Lorentzian para-Sasakian spacetime equipped with the Ricci soliton (g, ξ, λ) possesses a constant scalar curvature.*

Let us suppose that the Lorentzian para-Sasakian spacetime is a perfect fluid spacetime. Then on it, the Einstein's field equation with the cosmological term μ is given by

$$S(X, Y) - \frac{\kappa}{2} g(X, Y) + \mu g(X, Y) = \tau T(X, Y) \tag{4.6}$$

for all vector fields X, Y, where τ is the gravitational constant and T is the energy momentum tensor of type $(0, 2)$. The energy momentum tensor T is said to describe a perfect fluid [12] if

$$T(X, Y) = (\rho + p)A(X)A(Y) + pg(X, Y),$$

where ρ is the energy density function, p is the isotropic pressure function of the fluid and A is a non-zero 1-form such that $g(X, V) = A(X)$ for all X; V being the flow vector field of the fluid. The energy density ρ and the pressure p can be describe in the sense that if ρ and p vanishes identically, then the matter of the fluid is not pure and dust, respectively.

In the LP-Sasakian spacetime, we consider the characteristic vector field ξ as the flow vector field of the fluid, then the energy momentum tensor takes the form

$$T(X, Y) = (\rho + p)\eta(X)\eta(Y) + pg(X, Y). \tag{4.7}$$

With the help of the equations (3.2), (4.3), (4.4) and (4.6), we get

$$T(X, Y) = \frac{1}{\tau}\{(\mu + \frac{1}{2}\alpha - 3\alpha^2)g(X, Y) - \alpha\eta(X)\eta(Y)\}. \tag{4.8}$$

Thus we are going to state the following theorem.

Theorem 4.4. *If a perfect fluid LP-Sasakian spacetime satisfies the Einstein field equation with a cosmological term μ, then the energy momentum tensor of the space is given by (4.8).*

From the equations (4.4), (4.7) and (4.8), we find

$$\lambda = -\{\tau\,\rho + \mu + \frac{3}{2}\alpha\}.$$

Thus the Ricci soliton (g, ξ, λ) on a perfect fluid LP-Sasakian spacetime to be shrinking, steady or expanding accordingly $\tau\,\rho + \mu + \frac{3}{2}\alpha\ \ >, \ < \ \ or \ \ = 0$, respectively.

Corollary 4.5. *Suppose that the perfect fluid Lorentzian para-Sasakian spacetimes equipped with the Ricci soliton (g, ξ, λ) satisfy the Einstein field equations with the cosmological term μ, then the Ricci soliton (g, ξ, λ) is shrinking, steady or expanding if $\tau\,\rho + \mu + \frac{3}{2}\alpha\ \ >, \ < \ \ or \ \ = 0$, respectively.*

Theorem 4.6. *If a perfect fluid LP-Sasakian spacetime satisfies Einstein field equation along with a cosmological term, then it is quasi Einstein. Also, the perfect fluid LP-Sasakian spacetime to be dust if and only if the Lie-derivative of the energy momentum tensor with respect to ξ vanish.*

Proof. From (4.7) and (4.8), we yield

$$\left\{\frac{2\mu + \alpha - 6\alpha^2 - 2p\tau}{2\tau}\right\} g(X, Y) = (\rho + p + \frac{\alpha}{\tau})\eta(X)\eta(Y), \tag{4.9}$$

which gives

$$\mu = \frac{1}{4}\{(3p - \rho)\tau + 3\alpha(4\alpha - 1)\}. \tag{4.10}$$

If we put $X = Y = \xi$ in (4.9), we obtain

$$\mu = -\tau\rho + 3\alpha^2 - \frac{3}{2}\alpha. \tag{4.11}$$

Combining the equations (4.10) and (4.11), we get

$$(\rho + p) = -\frac{\alpha}{\tau}, \quad \tau \neq 0. \tag{4.12}$$

Using (4.7) and (4.12) in (4.6), we conclude that

$$S = \left(\frac{\kappa}{2} - \mu + p\tau\right) g - \alpha\,\eta \otimes \eta.$$

Taking $X = Y = e_i,\ 1 \leq i \leq 4$, in the above equation we have

$$\kappa = 4\mu - \alpha + p\tau. \tag{4.13}$$

In consequence of the tast two results, we find

$$S = \left(\mu - \frac{\alpha}{2} + \frac{3}{2}p\tau\right) g - \alpha\,\eta \otimes \eta, \tag{4.14}$$

which shows that the perfect fluid LP-Sasakian spacetime under consideration is quasi Einstein (see [17–19] and the references there in). Also from the equations (4.7) and (4.12), we find

$$T = p\,\tau g - \alpha\,\eta \otimes \eta, \tag{4.15}$$

provided α and τ are non-zero. The Lie derivative of (4.15) with respect to ξ gives

$$\begin{aligned}(\mathfrak{L}_\xi T)(X,Y) = p\,\tau(\mathfrak{L}_\xi g)(X,Y) - \alpha\{(\mathfrak{L}_\xi g)(X,\xi)g(Y,\xi)\\ +g(X,\mathfrak{L}_\xi\xi)g(Y,\xi) + g(X,\xi)(\mathfrak{L}_\xi g)(Y,\xi) + g(X,\xi)g(Y,\mathfrak{L}_\xi\xi)\}\end{aligned}$$

for all vector fields X and Y. It is obvious that $(\mathfrak{L}_\xi g)(X,Y) = 2\,\alpha\{g(X,Y) + \eta(X)\eta(Y)\}$ and $\mathfrak{L}_\xi\xi = 0$ and hence the above equation assumes the form

$$(\mathfrak{L}_\xi T)(X,Y) = p\,\tau(\mathfrak{L}_\xi g)(X,Y). \tag{4.16}$$

In general, $g(\phi X, \phi Y) \neq 0$ on an LP-Sasakian manifold and therefore ξ is not a Killing vector field on the perfect fluid LP-Sasakian spacetime, that is, $\mathfrak{L}_\xi g \neq 0$. Thus from the equation (4.16), we can see that $(\mathfrak{L}_\xi T)(X,Y) = 0$ if and only if $p = 0$, provided $\tau \neq 0$. Hence the statement of the Theorem 4.6 is proved. □

Corollary 4.7. *Let the perfect fluid LP-Sasakian spacetimes satisfy the Einstein field equation with a cosmological term. If the Lie-derivative of the energy momentum tensor with respect to ξ vanishes, then the acceleration vector of the fluid and the expansion scalar vanish.*

Proof. The energy density equation and the force equation for the perfect fluid are given by

$$\xi\rho = -(p+\rho)div\xi$$

and

$$(\rho + p)\nabla_\xi\xi = -grad\,p - (\xi p)\xi,$$

respectively [12]. Equations (4.12) and (4.16) with the last two equations give $div\,\xi = 0$ and $\nabla_\xi\xi = 0$. It is well known that $div\,\xi$ and $\nabla_\xi\xi$ represent the expansion scalar and the acceleration vector, respectively, of the perfect fluid. Hence the statement of the Corollary 4.7 is proved. □

Theorem 4.8. *Suppose a perfect fluid LP-Sasakian spacetime satisfies the Einstein field equation with a cosmological term. If the Lie-derivative of the energy momentum tensor with respect to ξ vanishes, then*

$$(\nabla_X S)(Y,Z) = \tau(\nabla_X T)(Y,Z).$$

Proof. From (2.1), (4.1), (4.12) and (4.16), the equation (4.14) turns into the form

$$S(X,Y) = (\mu - \frac{\alpha}{2})g - \alpha\,\eta(X)\eta(Y).$$

Change Y by ξ in the above equation, we get $\mu = 3\alpha^2 - \frac{\alpha}{2}$. Hence the above equation becomes

$$S(X,Y) = \alpha(3\alpha - 1)g - \alpha\,\eta(X)\eta(Y). \tag{4.17}$$

The covariant derivative of the above equation gives

$$(\nabla_X S)(Y,Z) = -\alpha^2\{\eta(Z)g(X,Y) + \eta(Y)g(X,Z) + 2\eta(X)\eta(Y)\eta(Z)\}.$$

Also taking the covariant derivative of (4.15), we obtain

$$(\nabla_X T)(Y,Z) = \rho\,\alpha\{\eta(Z)g(X,Y) + \eta(Y)g(X,Z) + 2\eta(X)\eta(Y)\eta(Z)\}.$$

The last two equations together with the equations (4.12) and (4.16) give

$$(\nabla_X T)(Y,Z) = \tau\,(\nabla_X S)(Y,Z). \tag{4.18}$$

□

A $(0,2)$-type tensor $\mathfrak{A}$ on a semi-Riemannian manifold is said to cyclic parallel and Codazzi tensors if

$$(\nabla_X\mathfrak{A})(Y,Z) + (\nabla_Y\mathfrak{A})(Z,X) + (\nabla_Z\mathfrak{A})(X,Y) = 0$$

and

$$(\nabla_X\mathfrak{A})(Y,Z) = (\nabla_Y\mathfrak{A})(X,Z)$$

for all vector fields X, Y and Z, respectively. By considering these definitions and Theorem 4.8, we can state the following corollaries.

Corollary 4.9. *Let a perfect fluid LP-Sasakian spacetime satisfies the Einstein field equation with a cosmological term. If the Lie-derivative of the energy momentum tensor with respect to ξ vanishes, then the Ricci tensor is of Codazzi type if and only if the energy momentum tensor is also Codazzi type.*

Corollary 4.10. *Assume that the perfect fluid LP-Sasakian spacetime satisfies the Einstein field equation with a cosmological term and the Lie-derivative of the energy momentum tensor with respect to ξ vanishes. Then the necessary and sufficient condition for the Ricci tensor to be cyclic parallel is that the energy momentum tensor is cyclic parallel.*

5 Yamabe solitons on the LP-Sasakian spacetimes

This section deals with the properties of Yamabe solitons on the LP-Sasakian spacetimes. Now, we prove the existence of the Yamabe soliton (g, V, λ), for $V = \xi$, on an LP-Sasakian spacetime in following theorem.

Theorem 5.1. *There does not exist an Yamabe soliton (g, ξ, λ) on an LP-Sasakian spacetime with the constant coefficient α.*

Proof. If possible, we suppose that the LP-Sasakian spacetimes admit the Yamabe solition (g, ξ, λ), then the equation (1.3) for $V = \xi$ gives

$$\frac{1}{2}(\mathfrak{L}_\xi g)(X, Y) = (\kappa - \lambda)g(X, Y) \tag{5.1}$$

implies that

$$g(\nabla_X \xi, Y) + g(X, \nabla_Y \xi) = 2(\kappa - \lambda)g(X, Y). \tag{5.2}$$

Keeping in mind (2.7) and (2.8), equation (5.2) reduces to

$$\alpha\{g(X, Y) + \eta(X)\eta(Y)\} = (\kappa - \lambda)g(X, Y). \tag{5.3}$$

Putting $X = \xi$ in (5.3), we get $\lambda = \kappa$, which shows that $\mathfrak{L}_\xi g = 0$, that is, the vector field ξ is a Killing vector field, which is inadmissible because $(\mathfrak{L}_\xi g)(X, Y) = \alpha\{g(X, Y) + \eta(X)\eta(Y)\} \neq 0$ (in general). Hence the Theorem 5.1 is proved. ☐

Theorem 5.2. *Every infinitesimal contact transformation on a Lorentzian para-Sasakian spacetime equipped with the Yamabe soliton (g, V, λ) is an infinitesimal strict contact transformation.*

Proof. Let (M, g) be a Lorentzian para-Sasakian spacetime, that is, a 4-dimensional LP-Sasakian manifold with constant coefficient α. It is obvious from the equations (4.13) and (4.14) that the scalar curvature of the spacetime is constant. Thus the equations (1.6) and (1.7) gives

$$(\mathfrak{L}_V S)(X, Y) = -(n-2)g(\nabla_X D\kappa, Y) + (\triangle\kappa)g(X, Y) \tag{5.4}$$

and

$$\psi = \kappa - \lambda.$$

From (5.4), we find that

$$(\mathfrak{L}_V S)(X, Y) = 0. \tag{5.5}$$

Adopting $Y = \xi$ in (5.5), we yield

$$(\mathfrak{L}_V S)(X, \xi) = 0. \tag{5.6}$$

On the other hand

$$(\mathfrak{L}_V S)(X,\xi) = \mathfrak{L}_V(S(X,\xi)) - S(\mathfrak{L}_V X,\xi) - S(X,\mathfrak{L}_V\xi).$$

By use of (1.5), (4.1) and (5.6), the above equation reduces to

$$S(X,\mathfrak{L}_V\xi) = 3\alpha^2(\mathfrak{L}_V\eta)(X) = 3\alpha^2\upsilon\eta(X). \tag{5.7}$$

Replacing $X = \xi$ in (5.7), we have

$$S(\xi,\mathfrak{L}_V\xi) = -3\upsilon\alpha^2. \tag{5.8}$$

Keeping in mind (4.1) and (5.8), we get

$$\eta(\mathfrak{L}_V\xi) = -\upsilon. \tag{5.9}$$

Again (1.5) and (5.9) yield

$$(\mathfrak{L}_V\eta)(\xi) = \upsilon, \tag{5.10}$$

which implies that

$$\mathfrak{L}_V(\eta(\xi)) - \eta(\mathfrak{L}_V\xi) = \upsilon. \tag{5.11}$$

In view of (5.9) and (5.11), we get $\upsilon = 0$. Thus the Definition 1.1 together with the equation (5.11) prove the statement of the Theorem 5.2. □

Next, we suppose that the potential vector field V of the Yamabe soliton (g,V,λ) is point-wise collinear with ξ, that is, $V = \beta\xi$ for some smooth function β. Then we have

$$\begin{aligned}\nabla_X V &= \nabla_X(\beta\xi)\\ &= (X\beta)\xi + \alpha\beta\,\phi^2(X).\end{aligned} \tag{5.12}$$

From (2.1), (2.2), (2.7) and (5.12), we have

$$(\mathfrak{L}_V g)(X,Y) = (X\beta)\eta(Y) + (Y\beta)\eta(X) + 2\alpha\beta\{g(X,Y) + \eta(X)\eta(Y)\}. \tag{5.13}$$

Using (1.3) in (5.13), we get

$$(X\beta)\eta(Y) + (Y\beta)\eta(X) + 2\alpha\beta\{g(X,Y) + \eta(X)\eta(Y)\} = 2(\kappa-\lambda)g(X,Y). \tag{5.14}$$

The contraction of the equation (5.14) over X and Y follows that

$$\xi\beta = 4(\kappa-\lambda) - 3\alpha\beta. \tag{5.15}$$

Putting $Y = \xi$ in (5.14) and using (5.15), it implies

$$X\beta = \{2(\kappa-\lambda) - 3\alpha\beta\}\eta(X). \tag{5.16}$$

Again replacing $X = \xi$ in (5.16), we find

$$\xi\beta = -2(\kappa - \lambda) + 3\alpha\beta. \tag{5.17}$$

In view of (5.15) and (5.17), we get $\lambda = \kappa - \alpha\beta$. By using this fact in (5.16), we have

$$X\beta = -\alpha\beta\eta(X).$$

This implies that

$$g(D\beta, X) = -\alpha\beta g(X, \xi).$$

That is,

$$D\beta = -\alpha\,\beta\xi = -\alpha V.$$

Thus we can state the following results.

Theorem 5.3. *If a Lorentzian para-Sasakian spacetime admits the Yamabe soliton (g, V, λ), then the potential vector field V and the gradient of the function β are linearly dependent.*

Corollary 5.4. *If a Lorentzian para-Sasakian spacetime admits the Yamabe soliton (g, V, λ) and the potential vector field V is point-wise collinear with ξ, then the manifold possesses the space of constant curvature.*

6 An Example

Example 6.1. *Let us consider a 4-dimensional differentiable manifold $M = \{(x, y, z, t) \in \Re^4 : (x, y, z, t) \neq 0\}$, where (x, y, z, t) is the standard coordinate in the 4-dimensional real space $\Re^4$. Let (e_1, e_2, e_3, e_4) be a set of linearly independent vector fields at each point of M, and is defined by*

$$e_1 = e^{x-at}\frac{\partial}{\partial x}, \quad e_2 = e^{y-at}\frac{\partial}{\partial y}, \quad e_3 = e^{z-at}\frac{\partial}{\partial z}, \quad e_4 = \frac{\partial}{\partial t},$$

where a is a non-zero constant. Define the Lorentzian metric g on M as:

$$g_{ij} = g(e_i, e_j) = \begin{cases} 0 & \text{if} \quad i \neq j \\ -1 & \text{if} \quad i = j = 4 \\ 1 \text{ otherwise} & \end{cases}.$$

Let η be the 1-form associated with the Lorentzian metric by

$$\eta(X) = g(X, e_4)$$

for any $X \in \Gamma(TM)$, where $\Gamma(TM)$ is the set of all smooth vector fields on M. If the $(1,1)$-tensor field ϕ is defined by

$$\phi(e_1) = e_1, \qquad \phi(e_2) = e_2, \qquad \phi(e_3) = e_3, \qquad \phi(e_4) = 0,$$

then by the linearity properties of ϕ and g, we can easily verify the following relations:

$$\eta(e_4) = -1, \quad \phi^2(X) = X + \eta(X)e_4, \quad g(\phi X, \phi Y) = g(X, Y) + \eta(X)\eta(Y),$$

for any $X, Y \in \Gamma(TM)$. Thus for $e_4 = \xi$, the structure (ϕ, ξ, η, g) leads to a Lorentzian paracontact structure and the manifold M endowed with the Lorentzian paracontact structure is known as the Lorentzian paracontact manifold of dimension 4.

The Lie derivative of vector fields X and Y is denoted by $[X, Y]$, and defined by $[X, Y] = X\,Y - Y\,X$. The non-vanishing components of the Lie bracket are calculated as:

$$[e_1, e_4] = ae_1, \quad [e_2, e_4] = ae_2, \quad [e_3, e_4] = ae_3.$$

If ∇ denotes the Levi-Civita connection with respect to the Lorentzian metric tensor g. Then for $e_4 = \xi$, the Koszul's formula gives

$$\begin{array}{llll}
\nabla_{e_1}e_1 = a\,e_4, & \nabla_{e_1}e_2 = 0, & \nabla_{e_1}e_3 = 0, & \nabla_{e_1}e_4 = ae_1, \\
\nabla_{e_2}e_1 = 0, & \nabla_{e_2}e_2 = a\,e_4, & \nabla_{e_2}e_3 = 0, & \nabla_{e_2}e_4 = ae_2, \\
\nabla_{e_3}e_1 = 0, & \nabla_{e_3}e_2 = 0, & \nabla_{e_3}e_3 = a\,e_4, & \nabla_{e_3}e_4 = ae_3, \\
\nabla_{e_4}e_1 = 0, & \nabla_{e_4}e_2 = 0, & \nabla_{e_4}e_3 = 0, & \nabla_{e_4}e_4 = 0.
\end{array}$$

If $X \in \chi(M)$, then it can be written as $X = a_1e_1 + a_2e_2 + a_3e_3 + a_4e_4$, where $a_i \in \Re$, $i = 1, 2, 3, 4$. From the above equations it can be easily verify that $\nabla_X e_4 = a\{X + \eta(X)e_4\}$ holds for each $X \in \chi(M)$. Thus the Lorentzian paracontact manifold is an LP-Sasakian manifold of dimension 4 with coefficient $\alpha = a \neq 0$. In light of the above equation, we found that the non-vanishing components of the curvature tensor are given by

$$\begin{array}{lll}
R(e_1, e_2)e_1 = -a^2e_2, & R(e_1, e_3)e_1 = -a^2e_3, & R(e_1, e_4)e_1 = -a^2e_4, \\
R(e_1, e_2)e_2 = a^2e_1, & R(e_2, e_3)e_2 = -a^2e_3, & R(e_2, e_4)e_2 = -a^2e_4, \\
R(e_1, e_3)e_3 = a^2e_1, & R(e_2, e_3)e_3 = a^2e_2, & R(e_3, e_4)e_3 = -a^2e_4, \\
R(e_1, e_4)e_4 = -a^2e_1, & R(e_2, e_4)e_4 = -a^2e_2, & R(e_3, e_4)e_4 = -a^2e_3.
\end{array}$$

The Ricci tensor S of M is defined as $S(X, Y) = \sum_{i=1}^{4} \varepsilon_i\, g(R(e_i, X)Y, e_i)$, where $\varepsilon_i = g(e_i, e_i)$. Thus we have

$$S(e_i, e_j) = \begin{bmatrix} 3a^2 & 0 & 0 & 0 \\ 0 & 3a^2 & 0 & 0 \\ 0 & 0 & 3a^2 & 0 \\ 0 & 0 & 0 & -3a^2 \end{bmatrix}, \quad i, j = 1, 2, 3, 4.$$

and the scalar curvature $\kappa = \sum_{i=1}^{4} S(e_i, e_j) = 6a^2$, which shows that the Lorentzian para Sasakian spacetime of dimension 4 possesses the constant scalar curvature and hence the equations (2.9)-(2.11), (2.13)-(2.18) and (4.3) are verified.

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