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Poincaré on Mass-Energy Equivalence*

P. Moylan
Department of Physics, Pennsylvania State University, The Abington College, Abington, PA, USA

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Abstract. In his 1900 Festschrift article, dedicated to the 25th anniversary of Lorentz' dissertation, Henri Poincaré derived an equation expressing conservation of momentum in electromagnetic systems. I describe his proof and then proceed to an example illustrating it. It turns out that Newton's third law as a law solely between interacting charges does not always hold true. Rather we must also take into account the momentum stored in the electromagnetic fields in order to conserve momentum. This generalization of momentum conservation to include the momentum of the electromagnetic field led Poincaré in his paper to mass energy equivalence for the energy content of his "fluide fictif," making it probably the first assertion about $E = mc^2$ as we have come to understand it.

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1 Introduction

In his 1900 Festschrift article, *La théorie de Lorentz et le Principe de réaction* [1], Henri Poincaré addresses concerns about the status of Newton's third law in Lorentz' relativistically invariant electromagnetic theory of electrons, at least if one wishes to apply it solely to material objects. He derives an equation, namely Eq. (8), expressing conservation of momentum in electromagnetic systems, which equation explicitly contains a term expressing the contribution from the electromagnetic field. From this he concludes that electromagnetic fields carry momentum. This momentum stored in the electromagnetic field must be taken into account in order to uphold Newton's third law in modified form, which is an immediate corollary of Eq. (8).

After spelling out very clearly the limitations in electromagnetism of Newton's third law in its strict form as a law of action/reaction between pairs of material objects, Poincaré in his Festschrift article goes on to consider a particular example. It leads him to attribute to electromagnetic radiation a mass equal to E/c^2 where E is the total energy of the radiation. It is thus in this paper where for the first time a derivation of $E = mc^2$ may rightfully be considered to have been

*Dedicated to Vasil Valdemarov Tsanov (b. 1948, † 2017) given. This most famous formula of physics is described by Poincaré, not in equation form, but rather in words by considering "a light pulse emitted from a Hertzian oscillator and causing the emitter to suffer a recoil" for which he gives numerical calculations in which $E = mc^2$ is implicit[1] [2]. That Einstein essentially reproduces this very same derivation of Poincaré in his 1906 paper on $E = mc^2$ is an indisputable fact admitted by Einstein himself in that very same paper[2] [3].

Einstein's 1905 paper [4], which suffers from the fallacy of "circulus in probando" [5, 6] was probably an attempt to arrive at mass-energy equivalence in a way different from Poincaré's. Poincaré's Festschrift article was considered one of the most important and was one of the most widely studied papers by the physics community of that time, and it is impossible to imagine that Einstein had not familiarized himself with it by 1905, his *annus mirabilis*[3].

2 Poincaré's Revision of Newton's Third Law of Motion

Now to Poincaré's analysis of how the law of action/reaction is to be understood in Lorentz' theory. After a laudato on Lorentz' electron theory, Poincaré writes: "*Let us briefly review the calculation by which one shows that, in the theory of Lorentz, the principle of the equality of action and reaction is not correct, at least if one wishes to apply it solely to material objects.*" He then explains how he is led to this statement, which is what I now outline. This section is an *almost verbatim transcription* of Poincaré's Festschrift article, except for translation into English and "modernization" of his equations into present day form. (In this section, and only in this section, we use arrows over symbols to denote vectors. For the rest of the paper, vectors are designated by bold-faced letters.)

We start with the Lorentz force law

$$\vec{F} = \int \rho \left(\vec{E} + \vec{v} \times \vec{B} \right) dV \,, \tag{1}$$

where ρ is the volume charge density of all charged material objects and $\vec{v}$ represents their velocity field so that the current density is $\vec{j} = \rho\vec{v}$. Maxwell's equations in differential form are

$$\vec{\nabla} \cdot \vec{E} = \frac{\rho}{\epsilon_0} \,, \tag{2}$$

$$\vec{\nabla} \cdot \vec{B} = 0 \,, \tag{3}$$

[1] Neither does this most famous equation of physics appear explicitly in Einstein's first article on $E = mc^2$ from 1905 [4, 6].

[2] Specifically, Einstein writes: "Although the simple formal considerations that have to be carried out to prove this statement are in the main already contained in a work by H. Poincaré, for the sake of clarity I shall not base myself upon that work."

[3] This is basically the conclusion of Alberto A. Martinez, an historian of science who has recently written a book about the subject [7].

$$\vec{\nabla} \times \vec{E} + \frac{\partial \vec{B}}{\partial t} = 0\,, \tag{4}$$

$$\vec{\nabla} \times \vec{B} - \mu_0 \epsilon_0 \frac{\partial \vec{E}}{\partial t} = \mu_0 \rho \vec{v}\,. \tag{5}$$

Using the first and last of Maxwell's equations in Eq. (1) we obtain

$$\vec{F} = \int \left\{ \epsilon_0 (\vec{\nabla} \cdot \vec{E})\vec{E} - \epsilon_0 \frac{\partial \vec{E}}{\partial t} \times \vec{B} + \frac{1}{\mu_0} (\vec{\nabla} \times \vec{B}) \times \vec{B} \right\}.$$

Now

$$\begin{aligned}(\vec{\nabla} \times \vec{B}) \times \vec{B} = -\vec{B} \times (\vec{\nabla} \times \vec{B}) &= \sum_{i=1}^{3} B_i \vec{\nabla} B_i - (\vec{B} \cdot \vec{\nabla})\vec{B} \\ &= \sum_{i=1}^{3} B_i \vec{\nabla} B_i - \frac{1}{2} \vec{\nabla} |\vec{B}|^2\,.\end{aligned}$$

Thus

$$\begin{aligned}\vec{F} &= \int \left\{ \epsilon_0 (\vec{\nabla} \cdot \vec{E})\vec{E} - \epsilon_0 \frac{\partial \vec{E}}{\partial t} \times \vec{B} + \frac{1}{\mu_0} \sum_{i=1}^{3} B_i \vec{\nabla} B_i - \frac{1}{2\mu_0} \vec{\nabla} |\vec{B}|^2 \right\} dV \\ &= \underbrace{\epsilon_0 \int dV \vec{B} \times \frac{\partial \vec{E}}{\partial t}}_{\vec{F}_1} + \underbrace{\frac{1}{\mu_0} \int dV \sum_{i=1}^{3} B_i \vec{\nabla} B_i}_{\vec{F}_2} \\ &\quad \underbrace{- \frac{1}{2\mu_0} \int dV \vec{\nabla} |\vec{B}|^2}_{\vec{F}_3} + \underbrace{\epsilon_0 \int dV (\vec{\nabla} \cdot \vec{E}) \vec{E}}_{\vec{F}_4}\,.\end{aligned}$$

Integration by parts on $\vec{F}_2$ gives

$$\vec{F}_2 = \frac{1}{\mu_0} \int d\sigma (\hat{n} \cdot \vec{B})\vec{B} - \frac{1}{\mu_0} \int dV \vec{B} (\vec{\nabla} \cdot \vec{B})$$

and the generalized Stokes' theorem applied to $\vec{F}_3$ gives

$$\vec{F}_3 = \frac{1}{2\mu_0} \int d\sigma \hat{n}\ (|\vec{B}|^2).$$

Since $\vec{\nabla} \cdot \vec{B} = 0$ we get

$$\vec{F}_2 - \vec{F}_3 = \frac{1}{2\mu_0} \int d\sigma \left\{ 2(\hat{n} \cdot \vec{B})\vec{B} - \hat{n}\ (|\vec{B}|^2 \right\}.$$

$$\vec{F}_4 = \epsilon_0 \int dV (\vec{\nabla} \cdot \vec{E}) \vec{E} = \epsilon_0 \int dV \nabla_k (E_k \vec{E}) - \epsilon_0 \int dV E_k \nabla_k \vec{E}$$
$$= \underbrace{\epsilon_0 \int d\sigma \vec{E} \cdot (\hat{n} \cdot \vec{E})}_{\vec{F}_4'} - \underbrace{\epsilon_0 \int dV \vec{E} \cdot \vec{\nabla} \vec{E}}_{\vec{F}_4''} .$$

Now

$$\vec{E} \times (\vec{\nabla} \times \vec{E}) = E_i \vec{\nabla} E_i - (\vec{E} \cdot \vec{\nabla}) \vec{E}$$

and

$$E_i \vec{\nabla} E_i = \frac{1}{2} \vec{\nabla} |\vec{E}|^2 ,$$

so

$$\vec{F}_4'' = \underbrace{\frac{\epsilon_0}{2} \int \vec{\nabla} |\vec{E}|^2}_{\vec{Y}} + \underbrace{\epsilon_0 \int dV \left(\vec{E} \times \frac{\partial \vec{B}}{\partial t} \right)}_{\vec{Z}} ,$$

where in obtaining the last term of this equation we used the third of Maxwell's equations.

$$\vec{F}_1 - \vec{Z} = \epsilon_0 \int dV \vec{B} \times \frac{\partial \vec{E}}{\partial t} - \epsilon_0 \int dV \left(\vec{E} \times \frac{\partial \vec{B}}{\partial t} \right)$$
$$= \epsilon_0 \frac{d}{dt} \int dV \vec{B} \times \vec{B} = -\frac{1}{c^2} \frac{d}{dt} \int dV \left(\frac{1}{\mu_0} \vec{E} \times \vec{B} \right) = -\frac{1}{c^2} \frac{d}{dt} \int dV \vec{S} ,$$

where

$$\vec{S} = \frac{1}{\mu_0} \vec{E} \times \vec{B} .$$

is the Poynting vector. Finally Poincaré computes

$$\vec{F}_4' - \vec{Y} = \epsilon_o \int d\sigma \vec{E} (\hat{n} \cdot \vec{E}) - \frac{1}{2} \epsilon_0 \int dV \vec{\nabla} |\vec{E}|^2 =$$
$$\frac{\epsilon_0}{2} \int d\sigma \left\{ 2 \vec{E} (\hat{n} \cdot \vec{E}) - \hat{n} |\vec{E}|^2 \right\} .$$

Combining everything he finally obtains for the Lorentz force

$$\vec{F} = \epsilon_0 \frac{d}{dt} \int dV \vec{B} \times \vec{E} + (\vec{F}_2 - \vec{F}_3) + (\vec{F}_4' - \vec{Y}) .$$

If one extends the integration over all space the surface terms $\vec{F}_2 - \vec{F}_3$ and $\vec{F}_4' - \vec{Y}$ vanish and we are left with

$$\vec{F} = \epsilon_0 \frac{d}{dt} \int dV \vec{B} \times \vec{E} = -\frac{1}{c^2} \frac{d}{dt} \int dV \vec{S} . \tag{6}$$

If we define the electromagnetic momentum density $\vec{G}$ and electromagnetic momentum $\vec{P}_{EM}$ by

$$\vec{G} = \frac{1}{c^2}\vec{S}$$

and

$$\vec{P}_{EM} = \int dV \vec{G},$$

respectively, we can rewrite Eq. (6) as

$$\vec{F} = -\frac{d\vec{P}_{EM}}{dt} \,. \tag{7}$$

Now the Lorentz force represents the "rate of change of the momentum of all ponderable matter", i.e.,

$$\vec{F} := \frac{d\vec{P}_{\text{Matter}}}{dt}$$

so that we obtain out of Eq. (7)

$$\frac{d\vec{P}_{\text{Matter}}}{dt} + \frac{d\vec{P}_{EM}}{dt} = 0 \,. \tag{8}$$

This expresses the law of conservation of momentum in electromagnetism:

the total momentum of an isolated electromagnetic system, consisting not only of the momentum of the system of charges but also of the momentum of their electromagnetic fields, is constant.

Poincaré's statement about the law of action/reaction should now be clear, since from Eq. (8) it can happen that in cases where there is a nonzero electromagnetic momentum the total momentum due only to the material objects may not be constant and hence lead to a violation of Newton's third law. We now turn to such an example.

3 Example: A Moving Line Charge and an Electron

It is convenient to use cylindrical coordinates (ρ, ϑ, z) for the point described by the endpoint of the displacement vector $\mathbf{r}$ from the origin. The cartesian coordinates of the point are (x, y, z). At some time t_0 the electron is along the x axis at a distance d from the origin and the line charge is along the z axis of the coordinate system as shown in Figure 1. At the given time t_0 the electron has velocity $\mathbf{v}_q$ and the line charge moves with speed u along the positive z direction. It carries a uniform positive linear charge density, λ_0.

In cylindrical coordinates the charge's position and its velocity are given, respectively, by

$$\mathbf{r}_q = d\hat{i} = d(\cos\vartheta\hat{\mathbf{r}} - \sin\vartheta\hat{\vartheta})$$

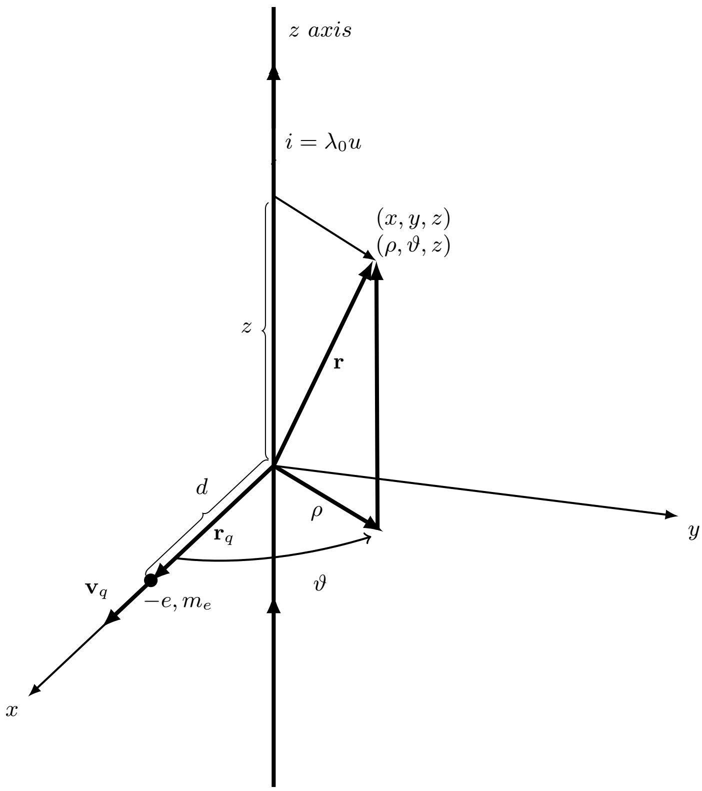

Figure 1. Moving line charge and an electron. The moving line charge is located along the z axis and the effective current of the moving line charge is $i = \lambda_0 u$.

and

$$\mathbf{v}_q = \frac{d\mathbf{r}_q}{dt} = v(\hat{\mathbf{r}} \cos\vartheta - \hat{\vartheta} \sin\vartheta) \, ,$$

where $v = v_q = |\mathbf{v}_q|$ is the magnitude of the electron's velocity. In the following we work nonrelativistically (or, equivalently, we work relativistically but only keeping at each step the lowest order in v/c), so that, at a generic point

$$\mathbf{r} = \rho\hat{\mathbf{r}} + z\hat{k} \, ,$$

the electric field of the charge and the magnetic field produced by the charge are, respectively [8]

$$\mathbf{E}_q = \frac{q}{4\pi\epsilon_0} \frac{\mathbf{r} - \mathbf{r}_q}{|\mathbf{r} - \mathbf{r}_q|^3} = -\frac{e}{4\pi\epsilon_0} \frac{(\rho - d\cos\vartheta)\hat{\mathbf{r}} + d\sin\vartheta\hat{\vartheta} + z\hat{k}}{(\rho^2 + d^2 + z^2 - 2\rho d\cos\vartheta)^{3/2}} \, , \tag{9}$$

$$\begin{aligned}
\mathbf{B}_q &= \frac{\mu_0}{4\pi}\frac{q\mathbf{v}_q \times (\mathbf{r}-\mathbf{r}_q)}{|\mathbf{r}-\mathbf{r}_q|^3} \\
&= -\frac{\mu_0}{4\pi}\frac{ev(\hat{\mathbf{r}}\cos\vartheta - \hat{\vartheta}\sin\vartheta) \times ((\rho - d\cos\vartheta)\hat{\mathbf{r}} + d\sin\vartheta\hat{\vartheta} + z\hat{k})}{(\rho^2+d^2+z^2-2\rho d\cos\vartheta)^{3/2}} \\
&= \frac{\mu_0}{4\pi}\frac{evz(\sin\vartheta\hat{\mathbf{r}} + \cos\vartheta\hat{\vartheta}) - ev\rho\sin\vartheta\hat{k}}{(\rho^2+d^2+z^2-2\rho d\cos\vartheta)^{3/2}}\,.
\end{aligned} \tag{10}$$

Similarly the electric and magnetic fields at the point $\mathbf{r}$ created by the moving line charge are

$$\mathbf{E}_\ell = \frac{2k\lambda_0}{\rho}\hat{\mathbf{r}} \tag{11}$$

and

$$\mathbf{B}_\ell = \frac{\mu_0\lambda_0 u}{2\pi\rho}\hat{\vartheta} \tag{12}$$

respectively.

The magnetic force on the electron due to the line charge is clearly nonzero; however, by symmetry, the magnetic field $\mathbf{B}_q$ created by the electron exerts no net force on the moving line charge. *Thus Newton's third law for the action/reaction pair consisting of the electron and the line charge is violated, since there is a magnetic force exerted on the point charge by the magnetic field* $\mathbf{B}_\ell$ *of the line charge.*

We need to show instead that Eq. (8) is true. We distinguish forces on the electron from forces on the line charge by putting primes on the forces on the line charge. Specifically, the total electromagnetic force on the electron due to the line charge is

$$\mathbf{F} = \mathbf{F}_E + \mathbf{F}_B = q\mathbf{E}_\ell + q\mathbf{v}_q \times \mathbf{B}_\ell = -\frac{2ke\lambda_0}{d}\hat{\imath} - \frac{\mu_0 e\lambda_0 vu}{2\pi d}\hat{k} \tag{13}$$

and for the total electromagnetic force on the line charge due to the electron

$$\mathbf{F}' = \mathbf{F}'_E = \int_{-\infty}^{\infty} \lambda_0\,\mathbf{E}_q dz = \frac{2ke\lambda_0}{d}\hat{\imath}\,. \tag{14}$$

Written out in terms of the individual forces of our problem Eq. (8) reads

$$\mathbf{F} + \mathbf{F}' + \frac{d\mathbf{P}_{EM}}{dt} = 0\,. \tag{15}$$

Subsituting our results for $\mathbf{F}$ and $\mathbf{F}'$ given in Eqns. (13) and (14) into this equation gives

$$-\frac{\mu_0 e\lambda_0 vu}{2\pi d}\hat{k} + \frac{d\mathbf{P}_{EM}}{dt} = 0\,, \tag{16}$$

which is what we need to show.

The linear momentum stored in the electric and magnetic fields is

$$\mathbf{P} = \frac{1}{c^2}\int \mathbf{S} dV \,, \tag{17}$$

where

$$\mathbf{S} = \frac{1}{\mu_0}\mathbf{E}\times\mathbf{B} \tag{18}$$

is the Poynting vector and the integration extends over all space. $\mathbf{E}$ and $\mathbf{B}$ are the total electric and magnetic fields due to the electron and line charge

$$\mathbf{E} = \mathbf{E}_q + \mathbf{E}_\ell \qquad \mathbf{B} = \mathbf{B}_q + \mathbf{B}_\ell \,,$$

so

$$\mathbf{S} = \frac{1}{\mu_0}(\mathbf{E}_q + \mathbf{E}_\ell)\times(\mathbf{B}_q + \mathbf{B}_\ell) \\ = \frac{1}{\mu_0}\left(\mathbf{E}_q\times\mathbf{B}_q + \mathbf{E}_q\times\mathbf{B}_\ell + \mathbf{E}_\ell\times\mathbf{B}_q + \mathbf{E}_\ell\times\mathbf{B}_\ell\right).$$

We expect that only the interaction terms, i.e. the two middle terms in the lower line of this equation, contribute to the force balance, and so we neglect the self-interaction terms $\mathbf{E}_q\times\mathbf{B}_q$ and $\mathbf{E}_\ell\times\mathbf{B}_\ell$ in this equation. Thus the linear momentum in the fields responsible for the force balance is

$$\mathbf{P} = \frac{1}{c^2}\int(\mathbf{E}_q\times\mathbf{B}_\ell + \mathbf{E}_\ell\times\mathbf{B}_q)dV \,. \tag{19}$$

We claim that the second term gives vanishing contribution to the integral. To see this we compute

$$\mathbf{E}_\ell\times\mathbf{B}_q = \Big(\frac{2k\lambda_0}{\rho}\hat{\mathbf{r}}\Big)\times\Big(-\frac{\mu_0}{4\pi}\frac{evz(\sin\vartheta\hat{\mathbf{r}} + \cos\vartheta\hat{\vartheta}) - ev\rho\sin\vartheta\hat{k}}{(\rho^2+d^2+z^2-2\rho d\cos\vartheta)^{3/2}}\Big) \\ = \frac{2k\mu_0 evz\lambda_0\cos\vartheta\hat{k}}{4\pi\rho(\rho^2+d^2+z^2-2\rho d\cos\vartheta)^{3/2}} \\ + \frac{2k\mu_0 ev\rho\lambda_0\sin\vartheta\hat{\vartheta}}{4\pi\rho(\rho^2+d^2+z^2-2\rho d\cos\vartheta)^{3/2}}\,. \tag{20}$$

To establish the claim we observe that the first term in this expression gives zero when integrated over z from $z=-\infty$ to $z=+\infty$ and the second term gives zero when integrated over ϑ from 0 to 2π. We can perform the integrations for the first term in Eq. (19) either by hand or by using Gradsteyn and Ryshik [9] and we obtain

$$\mathbf{P}_{EM} = -\frac{\mu_0 e\lambda_0 u}{2\pi}\ln\frac{R}{d}\hat{k}\,.$$

Thus,

$$\frac{d\mathbf{P}_{EM}}{dt} = \frac{\mu_0 e \lambda_0 v u}{2\pi d}\hat{k} \ .$$

Substitution of this result into Eq. (16) leads to the desired result.

Conclusions and Futher Developments

It is also possible to study conservation of angular momentum in electromagnetism along lines similar to the analysis just presented for conservation of linear momentum. In this case we would need to take into account the contribution of the angular momentum of the electromagnetic field to the total angular momentum of the system. An example illustrating this can be found in the Chapter 17 of the Feynman Lectures on Physics [10]. It involves a circular rotating disk and a circular solenoid carrying a current sitting on top of the disk and placed in such a way that the solenoid is concentric with the disk. Feynman doesn't solve the problem, rather he leaves it as an exercise to the reader, stating only: "When you figure it out, you will have discovered an important principle of electromagnetism." This important principle is, just as it was in our case for linear momentum, that the angular momentum of the electromagnetic field must be taken into account in order to uphold the conservation of angular momentum [11].

Both of these examples have much to say about the momentum of radiation and its ensuing consequence, namely the principle of the inertia of energy at least for electromagnetic fields. It seems clear that Poincaré as early as 1900 fully understood this, in particular, the implications of Eq. (8) for mass-energy equivalence, namely $E = mc^2$.[5]

[5]The history of $E = mc^2$ is quite a bit more involved than that discussed here. After the discovery by Maxwell and others of electromagnetic energy, radiation pressure, the Poynting vector and Poynting's theorem [12] on energy transport in electromagnetic systems, it became clear that the electromagnetic energy of an electrified body must contribute to its inertia. As early as 1881 J.J. Thomson argued that the backreaction of the magnetic field of a charged sphere would impede its motion and result in an apparent mass increase of the sphere [13]. Shortly thereafter, Heaviside proved that the mass increase of a moving sphere with uniform surface charge distribution was $m = (4/3)E_0/c^2$ where E_0 is the electromagnetic energy of a stationary sphere [14]. Subsequently, attempts were made to describe the electron solely in terms of its electromagnetic field, well-known electromagnetic models being those of Abraham and Lorentz [15, 16]. Out of many of these models came the puzzling factor of 4/3 seen already in Heaviside's calculations. Poincaré, on the other hand, in his Festschrift article attributes momentum to electromagnetic radiation and was led to correctly conclude $m = E/c^2$ with E being the energy of the radiation. Poincaré discovered a means of accounting for the discrepancy in his 1906 "Rendiconti paper" [17] by introducing compensating forces of non-electromagnetic origin which today have come to be known as Poincaré stresses. A somewhat complete explanation of the $4/3$ factor for the electromagnetic mass in terms of relativistic covariance was given by the young Enrico Fermi in 1922 [18, 19].

Apparently, the first serious attempt to extend mass-energy equivalence beyond the confines of electromagnetic models was made by Poincaré in his Festschrift article. Thus Ives ascribes to Poincaré the first derivation of energy-mass equivalence as it applies to radiation [20]. Further development along this more general line of thought were papers by Hasenöhrl and Planck on the inertia of a cavity containing radiation [21, 22]. Additionally, there appeared Einstein's early con-

In addition to describing a generalization of the principle of action/reaction for electromagnetic systems and the ensuing implications of mass-energy equivalence for radiation, Poincaré's Festchrift article contains at least one other extremely important discovery. It is distant clock synchronization using light signals [29], in other words, that which has come to be called by most *Einstein clock synchronization*. Reignier's lucid account of this matter, i.e. Ref. [29], makes it quite clear that it is Poincaré and not Einstein who should be given credit for first coming up with clock synchronization in special relativity.

In fact, much of special relativity in its entirety can be traced back to the writings of Poincaré. For an interesting perspective on this from one of the great theoretical physicists of our time, i.e. on the priority dispute between Poincaré and Einstein regarding the subject of special relativity, I mention a recent article by Academician Ivan Todorov [28]. He paints a very positive picture regarding Poincaré's role in the creation of special relativity. Interestingly, at the end of his paper, rather than committing himself to a conclusion one way or the other regarding the priority dispute, he simply quotes a passage from Freeman Dyson, another great physicist of our time, like-minded in his estimation of Poincaré's contributions to the subject.

As a final note we point out that Poincaré's Festschrift paper compare Lorentz's theory to an electromagnetic theory proposed by Hertz [30]. Hertz's electromagnetic theory, which Poincaré in his Festschrift paper discards in favor of Lorentz's, satisfies Newton's third law in its usual sense as a law between material masses only [1]. Undoubtedly it is the theory of Hertz that is described in his book, *Untersuchungen Über Die Ausbreitung Der Elektrischen Kraft* [30,31] which appeared in 1892, two years before Hertz' death at 36 that Poincaré is referring to. There Hertz presented in a chapter dealing with the electrodynamics of moving bodies a Galilean invariant theory of electromagnetism. In that chapter one finds an original set of four equations comprising what today we would call a “deformation of Maxwell's theory" in the sense that Maxwell's equations (i.e. Maxwell's theory) are recovered as a limiting case of Hertz' more gen-

tributions to the subject, the two papers which were already mentioned above, also attempting to establish mass-energy equivalence in general form. Also we should perhaps mention that Einstein throughout his life relentlessly devoted much time and effort to establishing $E = mc^2$ as a univeral law applicable to all forms of energy and matter. In this sense, it seems justified to associate the name of Einstein above all others with mass-energy equivalence. In fact, according to Ohanian, Einstein published in his lifetime no less than 7 papers on the subject, the last one appearing as late as 1946 [23]. Ohanian notes that, in some of his subsequent papers dealing with the problem, he repeated his original 1905 circular reasoning mistake, apparently never really understaning his error in logic. This raises the question as to whether or not a a proof of something in physics can be mathematically incorrect but, at the same time, correct in some sense from a physical standpoint, which presumably is the point of the argument made by Stachel and Torretti claiming, counter to Ives, Jammer and a whole host of others, that Einstein's 1905 paper is not flawed [24]. For more details on the history of $E = mc^2$ we refer the reader to the works of Sir E.T. Whittaker [25] and Max Jammer [5, 26]. For a recent article on the subject with a somewhat complete list of relevant references we refer to the article by Hecht [27].

eral theory, exactly in the same way as a Galilean invariant theory is a limiting case of a relativistic theory when $c \to \infty$ or as classical physics is of quantum mechanics when $\hbar \to 0$. Hertz' theory is thus an example of Minkowski's anti-deformation thesis roughly 20 years before Minkowski's description of it in his famous address to the 80th Assembly of German Natural Scientists at Cologne in 1908, where he introduced to the world space-time diagrams, and approximately 75 years before Segal [32] and Inönü and Wigner [33] introduced deformations and contractions of Lie groups in physics. Minkowski's anti-deformation thesis is simply that an existing physical theory such as Newtonian mechanics can be viewed as a limiting case of a more encompassing one such as special relativity.

Acknowledgements

This article was inspired by conversations I had with Vasil Tsanov, to whom I have dedicated this article. Poincaré's important and underestimated contributions to special relativity was an issue that Tsanov was very passionate about and, surprisingly to me, something about which he knew very much, even though he was not a physicist by training. He considered it an injustice that Poincaré has been not given enough due credit as far as the creation of special relativity is concerned. It was from my conversations with Tsanov where I first learned about the long ago discarded Galilean invariant theory of electromagnetism due to Heinrich Hertz, which, as mentioned above, is most certainly that which Poincaré, in his 1900 Festschrift article is referring to when he compares it to Lorentz' theory and judges in favor of Lorentz's.

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[2] ibid, in Poincaré's own words: "It is easy to evaluate that recoil quantitatively. If the device has a mass of 1 kg and if it emits three million joules in one direction with the velocity of light, the speed of the recoil is 1 cm/sec. In other terms, if the energy produced by a machine of 3,000 watts is emitted in a single direction, a force of one dyne is needed in order to hold the machine in place despite the recoil." Thus, in equation form $0 = -Mv_{\mathrm{recoil}} + mc$ where M is the mass of the recoiling object, $v_{\mathrm{recoil}} = .01$ m/s is its recoil speed, mc is the electromagnetic momentum and m is the mass of the fictitious fluid (the electromagnetic pulse). We obtain from this momentum balance equation that $m = Mv_{\mathrm{recoil}}/c$. Using Poincaré's value for the energy of the light pulse which is 3×10^6 J, we conclude that E and m must necessarily be related as follows: $E = mc^2$. Poincaré even goes further; he also utilizes the relation between power and force to get the recoil force on the device: the power of the machine producing the directional pulse of electromagnetic energy is 3,000 W and so the force is $F = 3000W/c = 10^{-5}$N which is 1 dyne.

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