Bulgarian Journal of Physics vol. **46** (2019) 303–312

Effects on the Equation of State through the Uniform Rotation of Neutron Stars

P.S. Koliogiannis, **Ch.C. Moustakidis**
Department of Theoretical Physics, Aristotle University of Thessaloniki, 54124 Thessaloniki, Greece

Received *23 October 2019*

Abstract. The observation and study of maximally-rotating neutron stars, close to mass-shedding limit, may be the key to understand the dense nuclear matter. While the theoretical predictions allow them to rotate extremely fast, the fastest observed one has a frequency of only 716 Hz. However, the theoretical study of maximally-rotating neutron stars bulk properties, along with future observations, will lead to useful insight at supra-nuclear densities and even more, to strong constraints on the nuclear equation of state. Furthermore, the time evolution of a neutron star may be an effective indicator of its final fate.

KEY WORDS: Neutron stars, Nuclear equation of state, Uniform rotation, Keplerian sequence

1 Introduction

Neutron stars, as an ideal laboratory for the strange physics of super-condensed matter, are the most important objectives for modern astrophysics [1–3]. Understanding the physics of such compact objects and the distortions of space-time around them can be of a crucial meaning, as the equation of state (EoS) at several times the normal nuclear matter density remains unknown.

Constraints on the EoS have come from the observations of a) the PSR J1614-2230 ($M = 1.908 \pm 0.016 M_{\odot}$) [4], b) the PSR J0348+0432 ($M = 2.01 \pm 0.04\ M_{\odot}$) [5], c) the PSR J0740+6620 ($M = 2.14^{+0.10}_{-0.09}\ M_{\odot}$) [6] and d) the PSR J2215+5135 ($M = 2.27^{+0.17}_{-0.15}\ M_{\odot}$) [7] by setting an upper limit to the maximum neutron star mass. This kind of observations is a powerful tool to constrain the EoS at high densities by assuming that the neutron star matter at its core is stiff enough to allow these masses. However, while the measurements of mass and radius of a non/slow-rotating neutron star would impose constraints on the EoS, the spin frequency of a maximally-rotating one, could be the key ingredient to understand the dense nuclear matter. At this moment, the fastest observed neutron star is J1748-244ad, with a spin frequency of 716 Hz [8], much lower than the theoretical predictions for hadronic matter.

Rotation, as it probes more properties of neutron stars, allow us to constrain the EoS at high densities [9–11]. To be more precise, accurate measurements of the spin frequency on neutron stars can lead to possible measurements of moment of inertia and Kerr parameter. Besides them, the significant meaning of the spin frequency lies with the determination of the Keplerian frequency, the frequency where the star would shed matter at its equator.

In this paper, we employ a large number of hadronic realistic EoSs based on phenomenological models, field theoretical and microscopic ones [12]. All of them predict the upper bound of the maximum neutron star mass, $M = 1.908 \pm 0.016\ M_{\odot}$ [4], while also reproducing accurately the bulk properties of symmetric nuclear matter. Except these realistic equations, we have also construct two EoSs, APR-1 and APR-2 [13], predicted by the Momentum-Dependent Interaction model (MDI). Among the advantages of this model, it reproduces the results of microscopic calculations of symmetric nuclear matter and neutron star matter at zero temperature, with the advantage of its extension to finite temperature. In addition, for completeness we use an EoS with appearance of hyperons at high densities (FSU2H) [14] and one suitable to describe quark stars based on MIT bag model [2, 3] (QS57.6). We systematically study the role of the dimensionless moment of inertia and Kerr parameter on the bulk properties of neutron stars. These two quantities are directly connected with the spin frequency of a neutron star and furthermore, the second one, can be a guide to distinguish different compact objects.

Moreover, we explore the time evolutionary rest mass sequences of neutron stars in order to examine the case where they considered to be progenitors of black holes. In particular, we construct two equilibrium sequences, normal and supramassive ones [15–17]. In the framework of General Relativity, normal evolutionary sequences have a spherical non-rotating end point, while supramassive ones, which have masses higher than the maximum mass of the non-rotating one, they don't have a non-rotating end point, and as a consequence they fade by collapsing to a black hole. The significance of the latter is that we can see how the neutron star will spend its lifetime and even more, we are in position to identify when a neutron star would collapse to a black hole by looking its spin up effect. Finally, we study the effects of the braking index on the EoS. We mainly focus on values near the Keplerian frequency (70% and more), where the braking index begins to be affected by the rest mass and significantly slows down the neutron star.

The article is organized as follows. In Section 2 we briefly review the EoSs and the MDI model. In Section 3, the rotating configuration is presented and in addition, we describe two properties of the EoS, the dimensionless moment of inertia and Kerr parameter. A detailed study for the constant rest mass sequences is also provided. Finally, Section 4 contains the summarize and discussion of the present study.

2 Models of the Nuclear Equations Of State

In the present study we employed 23 hadronic realistic EoSs which are in consistent with the current observed limits of neutron star mass [12]. The models of these EoSs are phenomenological, field theoretical and microscopic. However, two of them, APR-1 and APR-2, have been constructed by the MDI model and the data provided by Akmal *et al.* [13] as part of this study. The main reason we used the MDI model is because it can extend to finite temperature, an important parameter for a future study of proto-neutron stars, core-collapse supernova and neutron stars mergers.

3 Uniform Rotation

The equilibrium equations for a rotating neutron star, in the framework of General Relativity, can be described a) by the stationary axisymmetric space-time metric [18]

$$ds^2 = -e^{2\nu}dt^2 + e^{2\psi}\left(d\phi - \omega dt\right)^2 + e^{2\mu}\left(dr^2 + r^2 d\theta^2\right) , \qquad (1)$$

where the metric functions ν, ψ, ω and μ depend only on the coordinates r and θ, and b) the matter inside the neutron star. If we neglect sources of non-isotropic stresses, as well as viscous ones and heat transport, then the matter inside the neutron star can be fully described by the stress-energy tensor and modeled as a perfect fluid [18],

$$T^{\alpha\beta} = \left(\varepsilon + P\right) u^{\alpha} u^{\beta} + P g^{\alpha\beta} , \qquad (2)$$

where u^{α} is the fluid's 4-velocity. The energy density and pressure is denoted as ε and P.

3.1 Moment of inertia

In rotating configuration neutron stars probe more aspects to be studied. Among them, there is the moment of inertia which is one of the most important quantities for neutron stars and in general, in pulsar analysis [19]. Moment of inertia is defined as

$$I = \frac{J}{\Omega} , \qquad (3)$$

where J is the angular momentum and Ω is the angular velocity. This property of neutron stars quantifies how fast an object can spin with a given angular momentum.

We studied the dimensionless moment of inertia dependence on the corresponding compactness both for the whole region of the Keplerian sequence of neutron stars and for the maximum mass configuration at this sequence. In Figure 1 we present the above results and a window (colored area) where these quantities can lie. In fact, this window is a way to constrain them in a narrow region and can be described by the empirical formula

$$I/MR^2 = p_1 + p_2\beta + p_3\beta^2 + p_4\beta^3 + p_5\beta^4\,, \tag{4}$$

where the coefficients for the two edges can be found in Table 1.

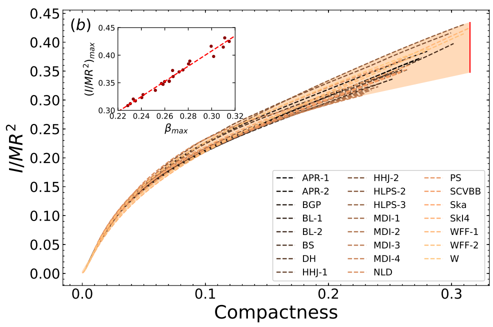

Figure 1. Dimensionless moment of inertia dependence on the corresponding compactness parameter of a maximally-rotating neutron star for the 23 EoSs. Red line corresponds to the best linear fit. The data at the maximum mass configuration are also presented with red circles.

For the maximum mass configuration at the Keplerian sequence (Figure 1 inside) the relation is given by the equation

$$\left(I/MR^2\right)_{\rm max} = -0.006 + 1.379\beta_{\rm max}\,. \tag{5}$$

Eq. (5) provide us with a universal relation between these quantities at the maximum mass configuration for the Keplerian sequence.

3.2 Kerr parameter

Einstein's field equations provided us with the Kerr space-time which is uniquely specified by the gravitational mass and the angular momentum. The existence of a rotating black hole is determined through the Kerr bound ($J \leq GM^2/c$). If the Kerr bound holds, then we have a rotating black hole, otherwise, a naked singularity is formed. While its validation is an open problem, the cosmic-censorship conjecture asserts that a generic gravitational collapse cannot form a naked singularity. This hypothesis is the main reason that astrophysical black holes should satisfy the Kerr bound [19, 20].

As the gravitational collapse of a massive rotating neutron star creates a black hole with almost the same gravitational mass and angular momentum, due to mass-energy and angular momentum conservation, we studied the Kerr parameter dependence on the gravitational mass both for the whole region of the Keplerian sequence of neutron stars and for the maximum mass configuration at this sequence. The relation which describes the Kerr parameter is defined as [19, 20]

$$\mathcal{K} = \frac{cJ}{GM^2} \, . \tag{6}$$

From Figure 2, it is clear that the maximum value for neutron stars, considering the stiffest EoS, is around 0.75, much lower than the 0.998 [21] which applies for black holes. We present also a window (colored area) where these quantities can lie. The window can be described by the empirical formula

$$\mathcal{K} = p_6 + p_7 \coth\left[p_8 \left(\frac{M_{\rm max}}{M_\odot}\right)\right] \, , \tag{7}$$

where the coefficients for the two edges can be found in Table 1. By constraining these quantities, we could impose strong constraints on the dense nuclear matter.

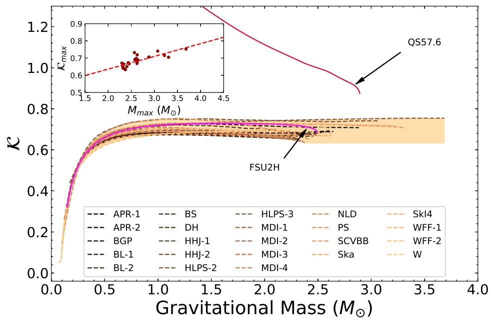

Figure 2. Kerr parameter dependence on the gravitational mass of a maximally-rotating neutron star. (Inside) The Kerr parameter values as a function of the corresponding gravitational mass at the maximum mass configuration. The red line corresponds to the best linear trend. The data at maximum mass configuration for the 23 hadronic EoSs are presented with red circles. FSU2H and QS57.6 EoSs are also indicated with the two solid lines.

For the maximum mass configuration at the Keplerian sequence (Figure 2 inside) the relation is given by the equation

$$\mathcal{K}_{\rm max} = 0.488 + 0.074 \left(\frac{M_{\rm max}}{M_\odot}\right) \, . \tag{8}$$

Eq. (8) provide us with a universal relation between these quantities at the maximum mass configuration for the Keplerian sequence. Constraining the Kerr parameter at neutron stars has a twofold meaning. First, a maximum value at Kerr parameter can lead to possible limits on compactness of neutron stars and second, can be a criteria to determine the final fate of a rotating compact object.

3.3 Constant rest mass sequences

The time evolutionary sequences of a neutron star, known as constant rest mass sequences, are lines that extend from the Keplerian sequence to the non-rotating end point or at the axisymmetric instability limit [15–17]. Sequences that are below the rest mass value that corresponds to the maximum mass configuration at the non-rotating model, they have a non-rotating member, and as a consequence, are stable and terminate at the non-rotating model sequence. Above this value, none of the sequences have a non-rotating member. Instead, they are unstable and terminate at the axisymmetric instability limit. The latter ones, are called supramassive sequences as their mass exceeds the maximum mass of the non-rotating configuration. In particular, if a neutron star spin-up by accretion and becomes supramassive, then it would subsequently spin-down along the constant rest mass sequence until it reaches the axisymmetric instability limit and collapse to a black hole. There is a case, where some relativistic stars could be born as supramassive ones, or even more, become one as a result of a binary merger [18].

In Figure 3, we present the normal and supramassive region of a neutron star and the corresponding constant rest mass sequences.

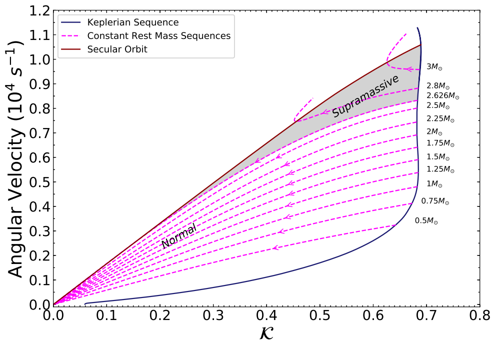

Figure 3. Normal and supramassive evolutionary sequences of constant rest mass as the dependence of the angular velocity on the Kerr parameter for the APR-1 EoS. The Keplerian sequence is presented with the blue curve while the constant rest mass sequences are presented with the fuchsia dashed lines. The quasi-radial stability limit is presented with the purple solid line.

From Figure 3, we can see that if we have a neutron star with rest mass in the white region, it would evolve toward stable configuration at the non-rotating end point, but if we have a star in the colored area, it would subsequently spin-up and evolve toward catastrophic collapse to a black hole [22–24].

In all cases, neutron stars which evolve along normal evolutionary sequences, never spin-up as they lose angular momentum. In contradiction to them, neutron stars on supramassive ones, because their unstable portion is always at higher angular velocity than the stable one, at the same value of angular momentum, must spin-up with angular momentum loss in the neighborhood of the stability limit. If the neutron star is massive enough, then the evolutionary sequence (supramassive) exhibit an extended region where spin-up is allowed. This effect may provide us an observable precursor to gravitational collapse to a black hole [19, 20].

As a follow up to Figure 3, we have constructed the last stable rest mass sequence (LSRMS) for the 23 realistic EoSs, as shown in Figure 4. This sequence is the one that divides the normal from supramassive evolutionary sequences. In Figure 4, we present a window (colored area) where the last stable rest mass sequence can lie and because this sequence is the one that corresponds to the maximum mass configuration at the non-rotating model, this is also the region where the EoS can lie, constraining with this way, simultaneously, the spin parameter and the angular velocity.

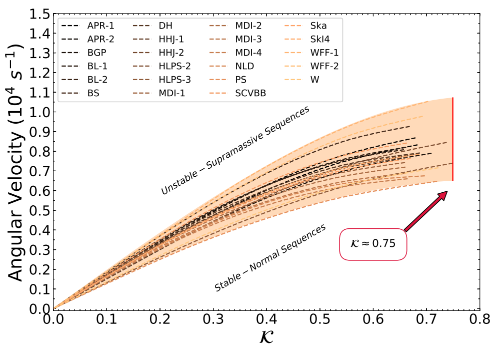

Figure 4. Last stable rest mass sequences for the 23 EoSs as the dependence of the angular velocity on the Kerr parameter. Supramassive and normal area are shown to guide the eye. The maximum value of the Kerr parameter is also noted.

The window can be described by the empirical formula

$$\Omega = \left(p_9\mathcal{K} + p_{10}\mathcal{K}^2 + p_{11}\mathcal{K}^3\right) 10^4 \quad (\mathrm{s}^{-1}) , \tag{9}$$

where the coefficients for the two edges are shown in Table 1. It is clear from Figure 4 and Eq.(9), that if we have a measurement of angular velocity, or spin parameter, we could extract the interval where the other parameter can lie.

As a consequence, by constraining simultaneously these two quantities, we could significantly narrow the existing area of EoS.

Table 1. Coefficients of Eq.(4), Eq.(7) and Eq.(9) for the window presented in Figure 1, Figure 2 and Figure 4, respectively.

Edges	p_1	p_2	p_3	p_4	p_5	p_6	p_7	p_8	p_9	p_{10}	p_{11}
Upper	0.005	4.01	-24.79	86.66	-110.33	0.86	-0.12	1.54	1.94	0.117	-1.058
Lower	0.005	3.38	-17.45	49.68	-55.36	0.86	-0.21	2.67	1.35	-0.305	-0.449

3.4 Effects of the braking index on the equation of state

The angular velocity of a rotating neutron star is decreasing very slowly with the time as various mechanisms take effect (dipole radiation, charged particles ejection, gravitational waves radiation) [25,26]. In this study, assuming the simplest model, the angular velocity is given by the form

$$\dot{\Omega} \equiv \frac{d\Omega}{dt} = -\mathcal{J}\Omega^n \tag{10}$$

and the braking index is equal to $n = 3$ (based on the assumption of a constant dipolar magnetic field). In order to see how the rest mass affects the braking index, we present at Figure 5 five representative EoSs for different rest masses.

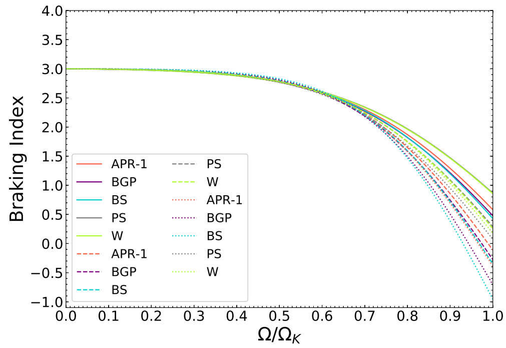

Figure 5. Braking index dependence on the angular velocity for five representative EoSs with constant rest masses. The lines correspond to the $M_{\rm max}^{\rm gr} = 1.45 M_\odot$, the dashed lines to the $M_{\rm max}^{\rm gr} = 2 M_\odot$ and the dotted lines to the $M_{\rm max}^{\rm gr} = 2.2 M_\odot$. The superscript "gr" corresponds to the gravitational mass and the subscript "max" to the maximum mass configuration.

From Figure 5, it is clear that the rest mass plays an important role on the braking index, i.e. by increasing the rest mass value, the braking index decreases more sharply. This effect will remain valid for all EoSs studied in this paper.

4 Discussion and Conclusions

Uniformly rotating neutron stars sequences for a large number of hadronic EoSs have been computed. For the numerical integration of the equilibrium equations, we used the public RNS code [27] by Stergioulas and Friedman [28]. In particular, we have calculated their gravitational mass, moment of inertia and Kerr parameter. Relations between these bulk properties and regions which constrain them have been established and shown in the corresponding figures. We have also constructed the normal and supramassive evolutionary sequences of constant rest mass for a specific EoS and determining the stability region of a neutron star.

From an astrophysical point of view, we have computed the maximum possible value of the Kerr parameter in neutron stars at 0.75, and concluded with this way that the gravitational collapse of a rotating neutron star, constrained to mass-energy and angular momentum conservation [29], cannot lead to a maximally rotating Kerr black hole ($\mathcal{K}_{B.H.} = 0.998$). In addition, the relations that connect the dimensionless moment of inertia with the compactness can lead to possible strong constraints in one of the open problems in astrophysics, the determination of radius. An important finding is also the construction of the LSRMS. This sequence, as it is the one that corresponds to the maximum mass configuration of the non-rotating model, can give us useful insight of the region where the EoS can lie and constrain both the angular velocity and Kerr parameter.

Furthermore, we have studied the effects of the braking index on the EoS. Besides the fact that it merely affects the EoS until 70% of the Keplerian angular velocity, after that value, braking index is strongly affected by the rest mass and may provide us with useful insight on the constitution of the dense nuclear matter.

We consider in this study also a limiting case, the FSU2H and QS56.7 EoSs. Figure 2 shows that the dependence of Kerr parameter on the gravitational mass for the quark star EoS is significantly different. Not only leads to higher values, but also has a different approach. This specific curve of quark stars could be a useful indicator to distinguish them from neutron stars. As far as concerning the hyperonic EoS, similar behavior with the hadronic ones is presented. However, a detailed study, which is in progress, is needed.

In near future, neutron star mergers and measurements of gravitational waves will be able to provide us with the Keplerian frequency. In fact, the remnant formed in the immediate aftermath of the GW170817 merger, although is believed to have been differentially rotating and not uniformly, it contains sufficient angular momentum to be near its mass-shedding limit [30]. The observational measurement of Keplerian frequency along with the theoretical predictions, would provide us with strong constraints on the dense nuclear matter.

Acknowledgements

One of the authors (Ch.C.M) would like to thank the Bulgarian National Science Fund (BNSF) for its support under contract KP-06-N28/6. This work was partially supported by the COST action PHAROS (CA16214) and the DAAD Germany-Greece grant ID 57340132.

References

[1] S.L. Shapiro, S.A. Teukolsky (1983) "*Black Holes, White Dwarfs, and Neutron Stars*". John Wiley and Sons, New York
[2] N.K. Glendenning (2000) "*Compact Stars: Nuclear Physics, Particle Physics, and General Relativity*". Springer, Berlin
[3] P. Haensel, A.Y. Potekhin, D.G. Yakovlev (2007) "*Neutron Stars 1: Equation of State and Structure*". Springer-Verlag, New York
[4] Z. Arzoumanian *et al.* (2018) *Astrophys. J.* **235** 37
[5] J. Antoniadis, P.C. Freire, N. Wex, T.M. Tauris, R.S. Lynch, *et al.* (2013) *Science* **3**40 1233232
[6] H.T. Cromartie, E. Fonseca, *et al.* (2019) *Nature Astronomy*
[7] M. Linares, T. Shahbaz, J. Casares (2018) *Astrophys. J.* **859** 54
[8] J.W.T. Hessels, S.M. Ransom, I.H. Stairs, P.C.C. Freire, V.M. Kaspi, F. Camile (2006) *Science* **311** 1901
[9] J.L. Friedman, J.R. Ipser, L. Parker (1986) *Astrophys. J.* **304** 115
[10] N.K. Glendenning, (1992) *Phys. Rev. D* **46** 4161
[11] L.M. Lattimer, M. Prakash, D. Masak, A. Yahil (1990) *Astrophys. J.* **355** 241
[12] P.S. Koliogiannis, C.C. Moustakidis (2018) *Astroph. Space Sci.* **364** 52
[13] A. Akmal, V.R. Pandharipande, D.G. Ravenhall (1998) *Phys. Rev. C* **58** 3
[14] L. Tolos, M. Centelles, A. Ramos (2017) *Publ. Astron. Soc. Aus.* **34** 065
[15] G.B. Cook, S.L. Shapiro, S.A. Teukolsky (1994) *Astrophys. J.* **422** 227
[16] G.B. Cook, S.L. Shapiro, S.A. Teukolsky, (1994) *Astrophys. J. Let.* **423** L117
[17] G.B. Cook, S.L. Shapiro, S.A. Teukolsky (1994) *Astrophys. J.* **424** 823
[18] V. Paschalidis, N. Stergioulas (2017) *Liv. Rev. Rel.* **20** 7
[19] F. Cipolletta, C. Cherubini, S. Filippi, J.A. Rueda, R. Ruffini (2015) *Phys. Rev. D* **92** 023007
[20] K.W. Lo, L.M. Lin (2011) *Astrophys. J.* **728** 12
[21] K.S. Thorne (1974) *Astrophys. J.* **191** 507-519
[22] M. Shibata (2003) *Astrophys. J.* **595** 992-999
[23] P.D. Lasky, B. Haskell, V. Ravi (2014) *Phys. Rev. D* **89** 047302
[24] V. Ravi, P.D. Lasky (2014) *Mon. Not. R. Astron. Soc.* **441** 2433-2439
[25] O. Hamil, J.R. Stone, M. Urbanec, G. Urbancova (2015) *Phys. Rev. D* **91** 063007
[26] R.N. Manchester, G.B. Hobbs, A. Teoh, M. Hobbs (2005) *Astron. J.* **29** 1993-2006
[27] https://www.gravity.phys.uwm.edu/rns/
[28] N. Stergioulas, J.L. Friedman (1995) *Astrophys. J.* **444** 306
[29] L. Baiotti, I. Hawke, L. Rezzolla, E. Schnetter (2015) *Phys. Rev. Lett.* **94** 131101
[30] J.M. Lattimer (2019) *Universe* **5(7)** 159