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Shell Model Analysis of Multiple $SU(3)$ Algebras in Nuclei

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Abstract. Rotational $SU(3)$ algebraic symmetry continues to generate new results in the shell model (SM). Interestingly, it is possible to have multiple $SU(3)$ algebras for nucleons occupying an oscillator shell η. Several different aspects of the multiple $SU(3)$ algebras are investigated using shell model and also deformed shell model based on Hartree-Fock single particle states with nucleons in sdg orbits giving four $SU(3)$ algebras. Numerical results showed that one of the $SU(3)$ algebras generates prolate shape, one oblate shape and the other two also generate prolate shape with one of them giving small quadrupole moments for low-lying levels. Conclusions from the shell model studies are also substantiated using sdg interacting boson model.

KEY WORDS: shell model, multiple SU(3) algebras, deformed shell model, quadrupole operator, sdg interacting boson model

1 Introduction

Elliott has recognized in 1958 that shell model (SM) admits $SU(3) \supset SO(3)$ algebra and this will generate rotational spectra in nuclei starting with the interacting particle picture [1,2]. By mid 60's it was recognized that the $SU(3)$ symmetry is useful for $1p$ and $2s1d$ shell nuclei but due to the strong spin-orbit force it will be a badly broken symmetry for $2p1f$ shell nuclei and beyond. Later it was recognized that for heavy deformed nuclei pseudo-$SU(3)$ based on pseudo spin and pseudo Nilsson orbits will be a useful symmetry [3,4]. Very recently, a proxy-$SU(3)$ scheme within SM was proposed with definite prediction for prolate dominance over oblate shape in heavy deformed nuclei [5,6]. In addition, in the multishell situation again $SU(3)$ appears within the $Sp(6,R)$ model [7] and this has given rise to the no-core-sympletic shell model [8]. Going beyond SM, a major basis for the interacting boson model (IBM) of atomic nuclei is that with s and d bosons the spectrum generating algebra (SGA) is $U(6)$ and it has $SU(3)$ as a subalgebra generating rotational spectra [9]. Similarly, proton-neutron IBM or IBM-2 [9], IBM-3 and IBM-4 [9, 10], sdgIBM [11] and $sdpf$IBM [12] all contain $SU(3)$. Similarly, for odd-A nuclei we have $SU^{BF}(3) \otimes SU^{F}(2)$ symmetry in the interacting boson-fermion model (IBFM) with Nilsson correspondence [13] and this extends to $SU(3)$ in IBFFM for odd-odd nuclei [14] and $SU(3)$ in IBF^{2}M for two quasi-particle excitations [15]. With $SU(3)$ appearing within both SM and IBM in a variety of ways, it is natural to look for new perspectives for $SU(3)$ symmetry in nuclei.

One curious aspect of $SU(3)$ in nuclei is that in a given oscillator shell η, there will be multiple $SU(3)$ algebras. Very early it is recognized that in SM with s and d orbits there will be two $SU(3)$ algebras [16] but its consequences are not explored in any detail. Similarly, in sdIBM there are two $SU(3)$ algebras [9] and they are applied in phase transition studies [17]. Finally, it was also recognized that there will be four $SU(3)$ algebras in sdgIBM [18]. Except for the sdIBM, properties of multiple $SU(3)$ algebras are not investigated in any detail in the past. In the present paper, following the recent investigation of multiple pairing algebras in SM and IBM [19], several different aspects of multiple $SU(3)$'s in SM are investigated. Now, we will give a preview.

In Section 2, multiple $SU(3)$ algebras in SM are identified and presented are the results of first information about their differences obtained using correlation coefficients between different $Q \cdot Q$ operators. In Section 3, spectra and electric quadrupole ($E2$) properties of these algebras are studied using three shell model examples. Additional insight into multiple $SU(3)$ algebras in SM is obtained using sdgIBM as presented in Section 4. Finally, Section 5 gives conclusions.

2 Phase Choice and Multiple $SU(3)$ Algebras in Shell Model

Let us consider the situation where valence nucleons in a nucleus occupying an oscillator shell with major shell number η. With the spin-isospin degrees of freedom for the nucleons, the SGA is $U(4\mathcal{N})$ and decomposing the space into orbital and spin-isospin (ST) parts, we have $U(4\mathcal{N}) \supset U(\mathcal{N}) \otimes SU(4)$. Here, $\mathcal{N} = (\eta+1)(\eta+2)/2$ and $SU(4)$ is the Wigner's spin-isospin $SU(4)$ algebra [10, 16, 20]. Also, for a given η, the single particle (sp) orbital angular momentum ℓ takes values $\ell = \eta, \eta-2, \ldots, 0$ or 1. As Elliott has established, we have $U(\mathcal{N}) \supset SU(3) \supset SO(3)$ where $SO(3)$ generates orbital angular momentum. The eight generators of $SU(3)$ are the orbital angular momentum operators L^1_q and quadrupole moment operators Q^2_q. In LST coupling and using fermion creation ($a^\dagger$) and annihilation (a) operators, the quadrupole operator is

$$Q^2_q = 2 \sum_{\ell_f, \ell_i} \frac{\langle \eta, \ell_f \mid\mid Q^2 \mid\mid \eta, \ell_i \rangle}{\sqrt{5}} \left(a^\dagger_{\ell_f \frac{1}{2} \frac{1}{2}} \tilde{a}_{\ell_i \frac{1}{2} \frac{1}{2}} \right)^{2,0,0} . \tag{1}$$

Note that $\tilde{a}_{\ell -m, \frac{1}{2}-m_s, \frac{1}{2}-m_t} = (-1)^{\ell - m + \frac{1}{2} - m_s + \frac{1}{2} - m_t} a_{\ell m, \frac{1}{2} m_s, \frac{1}{2} m_t}$ where m_s and m_t are the S_z and T_z quantum numbers for a single nucleon. For a single oscillator shell η, the quadrupole operator, with oscillator length parameter

$b = 1$, is equivalent to $Q^2_q = \sqrt{\frac{16\pi}{5}}\, r^2 Y^2_q(\theta,\phi)$. With this, the reduced matrix elements of Q^2 decompose into the radial part and angular part with the angular part given by [21], $-2\sqrt{\dfrac{\ell(\ell+1)(2\ell+1)}{(2\ell+3)(2\ell-1)}}$ for the $\ell \to \ell$ and $\sqrt{\dfrac{6(\ell+1)(\ell+2)}{(2\ell+3)}}$ for the $\ell \to \ell+2$ and $\ell+2 \to \ell$ matrix elements. More importantly, the corresponding radial matrix elements are $(2\eta+3)/2$ and $\alpha_{\ell,\ell+2}\sqrt{(\eta-\ell)(\eta+\ell+3)}$ with $\alpha_{\ell,\ell+2} = \alpha_{\ell+2,\ell} = \pm 1$. The phase factor $\alpha_{\ell,\ell+2}$ arises as there is freedom in choosing the phases of the radial wave functions of a 3D oscillator. In SM studies, the standard convention is to use $\alpha_{\ell,\ell+2} = -1$ for all ℓ [20–22]. However, Elliott in his $SU(3)$ introductory paper [1] and in sd as well as sdg IBM and IBFM the choice made is $\alpha_{\ell,\ell+2} = +1$ for all ℓ [9, 11, 13]. Thus, in general we have in SM,

$$
\begin{aligned}
L^1_q &= 2\sum_\ell \sqrt{\frac{\ell(\ell+1)(2\ell+1)}{3}} \left(a^\dagger_{\ell\frac{1}{2}\frac{1}{2}}\tilde{a}_{\ell\frac{1}{2}\frac{1}{2}}\right)^{1,0,0}_q , \\
Q^2_q(\boldsymbol{\alpha}) &= -2(2\eta+3)\sum_\ell \sqrt{\frac{\ell(\ell+1)(2\ell+1)}{5(2\ell+3)(2\ell-1)}} \left(a^\dagger_{\ell\frac{1}{2}\frac{1}{2}}\tilde{a}_{\ell\frac{1}{2}\frac{1}{2}}\right)^{2,0,0}_q \\
&\quad + \sum_{\ell<\eta} 2\alpha_{\ell,\ell+2}\sqrt{\frac{6(\ell+1)(\ell+2)(\eta-\ell)(\eta+\ell+3)}{5(2\ell+3)}} \\
&\quad \times \left[\left(a^\dagger_{\ell\frac{1}{2}\frac{1}{2}}\tilde{a}_{\ell+2,\frac{1}{2}\frac{1}{2}}\right)^{2,0,0}_q + \left(a^\dagger_{\ell+2,\frac{1}{2}\frac{1}{2}}\tilde{a}_{\ell\frac{1}{2}\frac{1}{2}}\right)^{2,0,0}_q\right] ; \\
\boldsymbol{\alpha} &= (\alpha_{0,2}, \alpha_{2,4}, \ldots, \alpha_{\eta-2,\eta}) \;\; \text{for} \;\; \eta \;\; \text{even} , \\
\boldsymbol{\alpha} &= (\alpha_{1,3}, \alpha_{3,5}, \ldots, \alpha_{\eta-2,\eta}) \;\; \text{for} \;\; \eta \;\; \text{odd} , \\
\boldsymbol{\alpha} &= (\pm 1, \pm 1, \ldots) .
\end{aligned}
\tag{2}
$$

Now, the most important result that can be proved by using the tedious but straight forward angular momentum algebra is that the eight operators $(L^1_q, Q^2_{q'}(\boldsymbol{\alpha}))$ commute among themselves generating $SU(3)$ algebras independent of $\boldsymbol{\alpha}$'s. Thus, we have multiple $SU(3)$ algebras $SU^{\boldsymbol{\alpha}}(3)$ in SM spaces generated by the operators in Eq. (2). Clearly for a given η, there will be $2^{\left[\frac{\eta}{2}\right]}$ number of $SU(3)$ algebras; $\left[\frac{\eta}{2}\right]$ is the integer part of $\eta/2$. Then, we have two $SU(3)$ algebras in sd ($\eta = 2$) and pf ($\eta = 3$) shells, four $SU(3)$ algebras in sdg ($\eta = 4$) and pfh ($\eta = 5$) shells, eight $SU(3)$ algebras in $sdgi$ ($\eta = 6$) and $pfhj$ ($\eta = 7$) shells and so on. Thus, the first non-trivial situation that is not discussed in literature before is sdg or $\eta = 4$ shell with four $SU(3)$ algebras $SU^{(-,-)}(3)$, $SU^{(+,-)}(3)$, $SU^{(-,+)}(3)$ and $SU^{(+,+)}(3)$. Here, $\boldsymbol{\alpha} = (\alpha_{sd}, \alpha_{dg})$ and $(-,-)$ means $(\alpha_{sd}, \alpha_{dg}) = (-1,-1)$ and similarly for other choices of $(\alpha_{sd}, \alpha_{dg})$. In the reminder of this paper, we will use the example of $\eta = 4$ shell in investigating some aspects of multiple $SU(3)$ algebras in SM. Also, for convenience we will use $\hat{Q} = \frac{1}{2}Q$ where appropriate. An important property of the $\hat{Q}^2(\boldsymbol{\alpha}) \cdot \hat{Q}^2(\boldsymbol{\alpha})$

operator is that it is related to the quadratic Casimir invariant (C_2) of $SU^{\boldsymbol{\alpha}}(3)$ in a simple manner, $-\hat{Q}^2(\boldsymbol{\alpha}) \cdot \hat{Q}^2(\boldsymbol{\alpha}) = -C_2(SU^{\boldsymbol{\alpha}}(3)) + \frac{3}{4}\, L \cdot L$; the eigenvalue of $C_2(SU^{\boldsymbol{\alpha}}(3))$ over a $SU^{\boldsymbol{\alpha}}(3)$ irreducible representation (irrep) $(\lambda\mu)$ is $\lambda^2 + \mu^2 + \lambda\mu + 3(\lambda + \mu)$.

Before turning to full shell model calculations, in order to gain some insight into the differences between different $SU^{\boldsymbol{\alpha}}(3)$ algebras, correlation coefficient ζ in m nucleon spaces between different $Q(\boldsymbol{\alpha}) \cdot Q(\boldsymbol{\alpha})$ operators are examined. Given any two operators $\mathcal{O}_1$ and $\mathcal{O}_2$, ζ the cosine of the angle between the two operators is $\zeta = \langle \widetilde{\mathcal{O}_1}^\dagger \widetilde{\mathcal{O}_2} \rangle^m / \big[\langle \widetilde{\mathcal{O}}_1^\dagger \widetilde{\mathcal{O}}_1 \rangle^m \langle \widetilde{\mathcal{O}}_2^\dagger \widetilde{\mathcal{O}}_2 \rangle^m \big]^{1/2}$ [23]. Here $\langle - - \rangle^m$ is m-particle space average and $\widetilde{\mathcal{O}}$ is the traceless (in m particle space) part of $\mathcal{O}$; see [23]. We have calculated for various particle numbers $\zeta[(\alpha_{sd}, \alpha_{dg}), (\alpha'_{sd}, \alpha'_{dg})]$ between the operators $Q^2(\alpha_{sd}, \alpha_{dg}) \cdot Q^2(\alpha_{sd}, \alpha_{dg})$ and $Q^2(\alpha'_{sd}, \alpha'_{dg}) \cdot Q^2(\alpha'_{sd}, \alpha'_{dg})$ in sdg ($\eta = 4$) shell for all possible combinations of α's and (α')'s. We found that $\zeta[(-,-),(+,-)] \sim 0.35 - 0.39$, $\zeta[(-,-),(-,+)] \sim 0.08 - 0.14$, $\zeta[(-,-),(+,+)] \sim 0 - 0.07$ and $\zeta[(+,+),(-,+)] \sim 0.35 - 0.39$. These show that $Q^2(-,-) \cdot Q^2(-,-)$ is fairly well correlated with $Q^2(+,-) \cdot Q^2(+,-)$ and similarly, $Q(+,+) \cdot Q(+,+)$ and $Q(-,+) \cdot Q(-,+)$ pairs, Thus, $SU^{(-,-)}(3)$ and $SU^{(+,-)}(3)$ are expected to give similar results but quite different from $SU^{(+,+)}(3)$ and $SU^{(-,+)}(3)$. This is seen in the results of detailed calculations presented in the next section.

3 Results for Spectra, Quadrupole Moments and $E2$ Transition Strengths from Shell Model and Deformed Shell Model

With the sdg example, we have four $Q \cdot Q$ Hamiltonians,

$$
\begin{aligned}
H_Q^{(-,-)} &= -\hat{Q}^2(-,-) \cdot \hat{Q}^2(-,-), \quad & H_Q^{(+,-)} &= -\hat{Q}^2(+,-) \cdot \hat{Q}^2(+,-), \\
H_Q^{(-,+)} &= -\hat{Q}^2(-,+) \cdot \hat{Q}^2(-,+), \quad & H_Q^{(+,+)} &= -\hat{Q}^2(+,+) \cdot \hat{Q}^2(+,+).
\end{aligned} \tag{3}
$$

In this section we will present the results generated by these four H's that include energies of the yrast levels, quadrupole moments $Q_2(J)$ of these levels and the $B(E2)$'s along the yrast line for J up to 10. Used for this purpose are the Antoine shell model code [24] and also deformed shell model (DSM). Starting with the same model space, sp energies and two-body interaction as in SM, in DSM first the Hartree-Fock (HF) sp spectrum and there by the lowest-energy prolate or oblate intrinsic state is obtained. Generating excited intrinsic states by particle-hole excitations over the lowest intrinsic state, band mixing calculations with angular momentum projection are carried out. Ref. [10] gives full details and many applications of DSM; latest application of DSM is to dark matter studies [25]. Note that DSM is particularly important for bringing out shape information in a transparent manner and also it is useful for larger particle numbers where SM calculations are impractical.

With $SU(3)$ symmetry of the H_Q Hamiltonians, the shell model space for a m nucleon system decomposes into $SU(3)$ irreps. If we have identical nucleons (protons or neutrons), the ground band belongs to the leading $SU(3)$ irrep (λ_H, μ_H) with spin $S = 0$ and $J = L$ for even m (similarly with $S = 1/2$ for odd m). It is easy to write a formula for obtaining (λ_H, μ_H) as given in [26]. Note that for m nucleons with isospin T, we need to consider the lowest spin-isospin $SU(4)$ irrep allowed for this system and this will then give (λ_H, μ_H) [26]. The eigenstates of H_Q are $|m; (\lambda_H\mu_H)KL; S : JT\rangle$ and the $(\lambda_H\mu_H) \to L$ reduction is well known giving the K label. It is easy to see that the energies of the yrast levels in a even m system (assuming spin $S = 0$ and then $J = L$) are simply $E(J = L) = -(\lambda_H^2 + \mu_H^2 + \lambda_H\mu_H + 3(\lambda_H + \mu_H)) + \frac{3}{4}J(J+1)$. In the present paper we will only consider even m systems with $(\lambda_H\mu_H) = (\lambda 0)$ and then λ is even. Therefore, they generate the ground $K = 0$ band with $J = 0$, 2, 4, ..., λ. The ground state energy $E_{gs} = (\lambda^2 + 3\lambda)$ and the energies of the J levels with respect to E_{gs} are just $3J(J+1)/4$. In addition, if we choose the $E2$ transition operator to be the Q of one of the H_Q, then formulas for $Q_2(J)$ and $B(E2)$ will be simple for the $(\lambda, 0)$ irrep of the corresponding $SU(3)$ algebra. Just as it is considered in SM and DSM codes, we will take the $E2$ operator T^{E2} for identical nucleon systems to be

$$T^{E2} = Q_q^2(-,-)\; e_{\text{eff}}\, b^2 \tag{4}$$

where e_{eff} is effective charge. Using the results in [2,13], we have for $H_Q^{(-,-)}$ in Eq. (3) with T^{E2} in Eq. (4),

$$\begin{aligned} &Q((\lambda 0)J) = -\frac{J}{2J+3}(2\lambda+3)\, e_{\text{eff}} b^2 \;, \\ &B(E2; (\lambda 0)J \to J-2) = \\ &\quad = \frac{5}{16\pi}\left\{\frac{6J(J-1)(\lambda-J+2)(\lambda+J+1)}{(2J-1)(2J+1)}\right\}(e_{\text{eff}})^2 b^4 \;. \end{aligned} \tag{5}$$

However, for systems with valence protons and neutrons the $E2$ transition operator is taken to be

$$T^{E2} = \left[e_{\text{eff}}^p\, Q_q^2(-,-;p) + e_{\text{eff}}^n\, Q_q^2(-,-;n)\right]\, b^2 \tag{6}$$

where e_{eff}^p and e_{eff}^n are proton and neutron effective charges. Again, in the situation the eigenstates obtained for $H_Q^{(-,-)}$ [they belong to $SU^{(-,-)}(3)$] for the $K = 0$ band are of the form $|(\lambda_p, 0)(\lambda_n, 0)(\lambda_p + \lambda_n, 0)K = 0, L, S = 0, J = L\rangle$ and using the T^{E2} in Eq. (6), formulas for $Q(J)$ and $B(E2)$'s are given by Eq. (5) but with e_{eff} replaced by X where

$$X = \frac{e_{\text{eff}}^p\left(\lambda_p^2 + 3\lambda_p + \lambda_p\lambda_n\right) + e_{\text{eff}}^n\left(\lambda_n^2 + 3\lambda_n + \lambda_n\lambda_p\right)}{(\lambda^2 + 3\lambda)}\;, \quad \lambda = \lambda_p + \lambda_n\,. \tag{7}$$

It is important to stress that in the event we use the eigenstates of other $H_Q^{\boldsymbol{\alpha}}$, the ground band generated by them will belong to the $(\lambda 0)$ irrep of the corresponding $SU^{\boldsymbol{\alpha}}(3)$. However, then the Q's in T^{E2} in Eqs. (4) and (6) are no longer generators of these $SU^{\boldsymbol{\alpha}}(3)$'s and hence the formulas in Eqs. (5) and (7) will not apply. Here, we will present the results of full SM calculations and also those from DSM for the four H_Q's for yrast spectra, $Q(J)$'s and $B(E2)$'s.

3.1 SM and DSM results for multiple $SU(3)$ algebras: $(sdg)^{6_p}$ example

In our first example, we have analyzed a system of 6 protons in $\eta = 4$ shell, i.e. $(sdg)^{6_p}$ system by carrying out SM calculations using the four H_Q Hamiltonians in the full SM space (matrix dimension in the m-scheme is $\sim 10^5$). For this system, the leading $SU(3)$ irrep is $(18, 0)$ with $S = 0$. Then, $E_{gs} = -378$ and SM calculations for all four H_Q's are in agreement with this $SU(3)$ result. Also, the excitation energies of the ground band members ($J = 0, , 2, 4, 6, \ldots$) are seen to follow for all the four H_Q's the $3J(J+1)/4$ law as given by $SU(3)$. Thus, it is verified by explicit SM calculations that all the four H_Q's give $SU(3)$ symmetry. Though the energy spectra are same, the wavefunctions of the yrast J states are different. This is established by calculating $Q(J)$ and $B(E2)$'s for the ground band members using T^{E2} given by Eq. (4). In all the calculations, $e_{\text{eff}} = 1e$ and $b^2 = A^{1/3}\,fm^2$ with $A = 86$ are used. Results for the four H_Q's are given in Table 1. It is easy to see that the results for $H_Q^{(-,-)}$ are in complete agreement with the $SU(3)$ formulas given by Eq. (5). This is expected as T^{E2} in Eq. (4) is a generator of $SU^{(-,-)}(3)$ generated by $H_Q^{(-,-)}$. However, the results from the other three H_Q's are quite different and do not follow the $SU(3)$ results in Eq. (5) as the T^{E2} chosen is not a generator of the $SU(3)$'s generated by the three H_Q's. It is seen from Table 1 that the results for $Q(J)$ and $B(E2)$'s from $H_Q^{(+,-)}$ are closer to those from $H_Q^{(-,-)}$ and this is consistent with the correlation coefficients discussed in Section 2. The $B(E2)$'s from $H_Q^{(-,+)}$ are much

Table 1. Shell model results for quadrupole moments $Q(J)$ and $B(E2; J \to J-2)$ values for the ground $K = 0^+$ band members for a system of 6 protons in $\eta = 4$ shell. Results are given for the four Hamiltonians in Eq. (3). In the table $(-,-)$ means we are using the wavefunctions obtained using $H_Q^{(-,-)}$ and similarly others.

J	$Q(J)\ efm^2$				J	$B(E2; J \to J-2)\ e^2fm^4$			
	$(-,-)$	$(+,-)$	$(-,+)$	$(+,+)$		$(-,-)$	$(+,-)$	$(-,+)$	$(+,+)$
2_1^+	-49.18	-33.90	-1.85	13.44	2_1^+	585.97	291.34	0.42	42.20
4_1^+	-62.59	-40.16	-4.71	17.72	4_1^+	815.31	388.79	1.38	58.58
6_1^+	-68.85	-39.78	-9.12	19.97	6_1^+	853.68	377.33	3.90	61.14
8_1^+	-72.48	-37.06	-14.96	20.46	8_1^+	827.24	325.68	9.24	58.93
10_1^+	-74.84	-34.57	-21.99	18.28	10_1^+	760.52	254.56	18.38	53.81

smaller in magnitude. Moreover, $H_Q^{(-,-)}$ generates prolate shape and $H_Q^{(+,+)}$ oblate as seen clearly from Table 1. Quadrupole moments show that $H_Q^{(+,-)}$ and $H_Q^{(-,+)}$ also generate prolate shapes but the deformation from $H_Q^{(-,+)}$ is quite small for the low-lying levels. To gain more insight, we have performed DSM calculations with results as follows.

Starting with the same model space, sp energies and two-body interaction as in SM, in DSM calculations are carried out using the lowest intrinsic state. As shown in Figure 1, the four H_Q's generate the same HF sp spectrum and the lowest intrinsic state is obtained by putting two protons each in the $1/2_1$, $1/2_2$ and $3/2_1$ states. The intrinsic quadrupole moments given in the figure caption show that $H_Q^{(-,-)}$ generates prolate shape and $H_Q^{(+,+)}$ generates oblate shape in agreement with SM. It is important to emphasize that the intrinsic quadrupole moments are calculated using $T^{E2} = Q_q^2(-,-)\, b^2$ as the quadrupole operator.

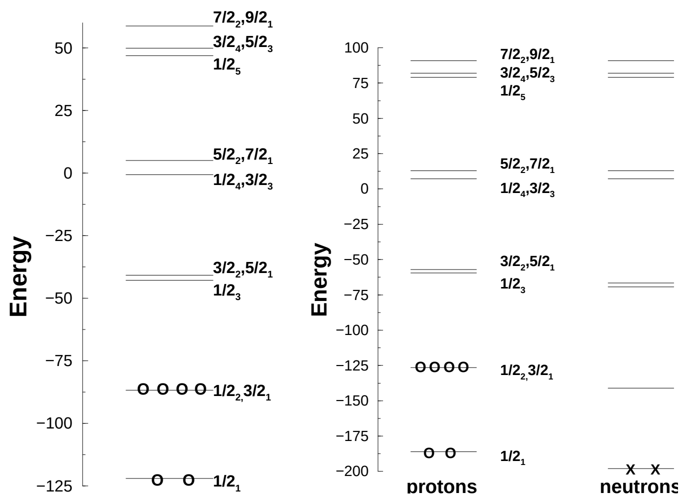

Figure 1. Hartree-Fock sp spectrum and the lowest intrinsic state for the $(sdg)^{6p}$ system (left side figure) and for the $(sdg)^{6p,2n}$ system (right side figure) generated by the four H_Q operators in Eq. (3). In the figure, the symbol $\times$ denotes neutrons and o denotes protons. Shown in the figure are the k values of the sp orbits and each orbit is doubly degenerate with $|k\rangle$ and $|-k\rangle$ states. The spectra are same for all the four Hamiltonians although the sp wavefunctions are different. Note that the energies in the figures are unit less and the unit MeV has to be put back after multiplying with an appropriate scale factor if the results are used for a real nucleus. The intrinsic quadrupole moments (in units of b^2), calculated using T^{E2} defined by Eq. (4), for $H_Q^{(-,-)}$, $H_Q^{(+,-)}$, $H_Q^{(-,+)}$ and $H_Q^{(+,+)}$ are 35.85, 24.4, 1.83, -9.63 respectively. Similarly, for the $(sdg)^{6p,2n}$ system, with T^{E2} in Eq. (6), the intrinsic quadrupole moments (in units of b^2) are 51.92, 26.44, 9.4, -16.08 respectively.

The ground state energy and the excitation energies for the yrast levels for all the four H_Q's are same as the exact $SU(3)$ values within 1% deviation. Similarly, the results for $Q(J)$'s and $B(E2)$'s are within 1-2% of the SM values. Thus, for larger particle systems where SM calculations are not possible, one can use with confidence DSM for further insights; see Section 3.3.

3.2 SM results for multiple $SU(3)$ algebras: $(sdg)^{(6_p,2_n)T=2}$ example

In our second example, $(sdg)^{6_p,2_n}$ system is considered and carried out SM calculations using the four H_Q Hamiltonians in the full SM space (dimension in the m-scheme is $\sim 2 \times 10^7$). For this system, the leading $SU(3)$ irrep is $(26,0)$ with $S = 0$ and $T = 2$. It is seen that for all the four H_Q's the SM calculations give $E_{gs} = -754$ and the ground band members follow the $3J(J+1)/4$ law as given by $SU(3)$. Going beyond these, $Q(J)$ and $B(E2)$'s for the ground band members are calculated using T^{E2} in Eq. (6) with $e^p_{\text{eff}} = 1.5e$, $e^n_{\text{eff}} = 0.5e$, $b^2 = A^{1/3}\,fm^2$ and $A = 88$. Now, Eq. (5) combined with Eq. (7) will apply for the states from $H_Q^{(-,-)}$ with the irrep $(26,0)$ arising from the $(18,0)$ irrep of the protons and the $(8,0)$ irrep of the two neutrons. It is seen that the results for $H_Q^{(-,-)}$ are in complete agreement with the exact $SU(3)$ formulas. However, the results from the other three H_Q's are quite different. For example $Q(2_1^+)$ values (in efm^2 unit) are -83.34, -50.54, -8.84 and 23.96 for $H_Q^{(p,q)}$ with $(p,q) = (-,-)$, $(+,-)$, $(-,+)$ and $(+,+)$ respectively. Similarly, $B(E2; 2_1^+ \to 0_1^+)$ values (in e^2fm^4 unit) are 1687.7, 629.26, 17.14 and 40.57 respectively. Thus, again the results for $Q(J)$ and $B(E2)$'s from $H_Q^{(+,-)}$ are closer to those from $H_Q^{(-,-)}$ and the $B(E2)$'s from $H_Q^{(-,+)}$ are much smaller in magnitude. Moreover, $H_Q^{(-,-)}$ generates prolate shape and $H_Q^{(+,+)}$ oblate as in the previous example. Finally, let us mention that we have also carried out DSM calculations for this example (see Figure 1) and all the DSM results are found to be in close agreement with SM results.

3.3 DSM results for multiple $SU(3)$ algebras: $(sdg)^{(6_p,6_n)T=0}$ example

In our final example we have considered a system of 12 nucleons with $T = 0$ in $\eta = 4$ shell, i.e. $(sdg)^{(6_p,6_n)T=0}$ system. Here the dimension in the m-scheme is $\sim 10^{10}$ and therefore SM calculations are not possible with our computational facilities. In this example, DSM gives the results for the four H_Q's. Carrying out DSM calculations, it is found that the four H_Q's generate the same HF sp spectrum as shown in Figure 1 for six proton system except for a scale factor. Using the lowest intrinsic state, it is seen from the intrinsic quadrupole moments for the four H's that $H_Q^{(-,-)}$ generates prolate shape and $H_Q^{(+,+)}$ generates oblate shape in agreement with the previous SM examples. The ground state energy for the system is found to be -1402.4 for all four H_Q's against the $SU(3)$ value

Table 2. Deformed shell model results for quadrupole moments $Q(J)$ and $B(E2; J \rightarrow J-2)$ values for the ground $K = 0^+$ band members for a system of 6 protons and 6 neutrons (with $T = 0$) in $\eta = 4$ shell. Results are given for the four Hamiltonians in Eq. (3). Numbers in the brackets in the second column are exact $SU(3)$ results for $H_Q^{(-,-)}$.

J	$Q(J)\ efm^2$			
	$(-,-)$	$(+,-)$	$(-,+)$	$(+,+)$
2_1^+	$-96.67(-95.31)$	-65.95	-4.63	26.1
4_1^+	$-123.04(-123.12)$	-82.61	-7.02	33.41
6_1^+	$-135.34(-135.43)$	-88.65	-9.66	37.03
8_1^+	$-142.45(-142.58)$	-90.30	-12.91	39.24
10_1^+	$-147.09(-147.21)$	-89.60	-16.86	40.63

J	$B(E2; J \rightarrow J-2)\ e^2fm^4$			
	$(-,-)$	$(+,-)$	$(-,+)$	$(+,+)$
2_1^+	2273.96(2276.93)	1069.13	4.61	164.89
4_1^+	3225.34(3229.59)	1503.04	7.43	233.99
6_1^+	3506.56(3511.13)	1608	9.98	254.61
8_1^+	3601.29(3605.99)	1613.12	13.44	261.78
10_1^+	3605.81(3610.51)	1565.64	18.32	262.44

-1404 with less than 1% deviation; $SU(3)$ irrep for the ground band is $(36, 0)$ generated by the irreps $(18, 0)$ for protons and neutrons. Energies of the yrast J states are also same for the four H_Q's and they are also within 1% deviation from the $3J(J+1)/4$ law. Turning to $Q(J)$ and $B(E2)$'s, in the calculations used are $e^p_{\text{eff}} = 1.5e$, $e^n_{\text{eff}} = 0.5e$ and $b^2 = A^{1/3}\ fm^2$ with $A = 92$. Here, again Eq. (5) with Eq. (7) will apply for the states from $H_Q^{(-,-)}$ and for this, as shown in Table 2, DSM results agree with the $SU(3)$ formulas. However, the results from the other three H_Q's are quite different as in the previous $(sdg)^{6_p}$ and $(sdg)^{6p,2n}$ examples. Again, it is seen from Table 2 that the results for $Q(J)$ and $B(E2)$'s from $H_Q^{(+,-)}$ are closer to those from $H_Q^{(-,-)}$ and the $B(E2)$'s from $H_Q^{(-,+)}$ are much smaller in magnitude. Moreover, $H_Q^{(-,-)}$ generates prolate shape and $H_Q^{(+,+)}$ oblate as in the previous examples. Thus, the results in Tables 1 and 2 (also those in Subsection 3.2) are generic results for the four H_Q's.

4 Test of Shell Model Results Using sdgIBM

In the interacting boson models with the bosons in an oscillator shell η, again there will be $2^{[\eta/2]}$ number of $SU(3)$ algebras just as in the shell model. The eight operators $(L^1_q, Q^2_q(\boldsymbol{\alpha}))$ for these follow from Eq. (2) by dropping the factor 2 and replacing the fermion $(a^\dagger, a)$ operators by boson $(b^\dagger, b)$ operators. It is important to stress that $\alpha_{\ell,\ell+2} = +1$ for all ℓ values is the standard choice in

sdIBM and sdgIBM. For example, in sdgIBM with $\eta = 4$ there will be four $SU^{\boldsymbol{\alpha}}(3)$ algebras generated by,

$$\begin{aligned} L^1_\mu &= \sqrt{10}\left(d^\dagger \tilde{d}\right)^1_\mu + \sqrt{60}\left(g^\dagger \tilde{g}\right)^1_\mu \;, \\ Q^2_\mu(\alpha_{sd}, \alpha_{dg}) &= \sqrt{3}\left\{ -11\sqrt{\frac{2}{21}}\,(d^\dagger \tilde{d})^2_\mu - 2\sqrt{\frac{33}{7}}\,(g^\dagger \tilde{g})^2_\mu \right. \\ &\left. + \alpha_{sd}\, 4\sqrt{\frac{7}{15}}\,\left(s^\dagger \tilde{d} + d^\dagger \tilde{s}\right)^2_\mu + \alpha_{dg}\,\frac{36}{\sqrt{105}}\,\left(d^\dagger \tilde{g} + g^\dagger \tilde{d}\right)^2_\mu \right\} \;, \end{aligned} \tag{8}$$

with $\alpha_{sd} = \pm 1$ and $\alpha_{dg} = \pm 1$. It is easy to see that $-\hat{Q}^2(\boldsymbol{\alpha}) \cdot \hat{Q}^2(\boldsymbol{\alpha}) = C_2(SU^{\boldsymbol{\alpha}}(3)) + \frac{3}{4} L \cdot L$. The shapes generated by this quadrupole-quadrupole interaction can be studied using the three parameter coherent state (CS) in terms of the $(\beta_2, \beta_4, \gamma)$ parameters for a N boson system; see [27, 28] for the explicit form of the CS. The CS expectation value of $-\hat{Q}^2(\boldsymbol{\alpha}) \cdot \hat{Q}^2(\boldsymbol{\alpha})$ gives the energy functional $E_{SU_{sdg}(3)}(N; \beta_2, \beta_4, \gamma)$. Minimizing this with respect to β_2, β_4 and γ will give the equilibrium (ground state) shape parameters (β^0_2, β^0_4, γ^0) and the corresponding equilibrium energy $E^0_{SU_{sdg}(3)}$. Without loss of generality choosing $\gamma^0 = 0°$, we have (β^0_2, β^0_4) = $(\sqrt{20/7}, \sqrt{8/7})$, $(\sqrt{20/7}, -\sqrt{8/7})$, $(-\sqrt{20/7}, -\sqrt{8/7})$ and $(-\sqrt{20/7}, \sqrt{8/7})$ for $(\alpha_{sd}, \alpha_{dg})$ = $(1,1)$, $(1,-1)$, $(-1,1)$ and $(-1,-1)$ respectively. Also, for all the four solutions, the $E^0_{SU_{sdg}(3)} = -16N^2$. This energy value is same as the large N eigenvalue of $-C_2(SU(3))$ in the leading $(4N, 0)$ irrep. The intrinsic structure of the ground $K = 0$ band generated by the $(4N, 0)$ irrep for the four solutions is,

$$|N; K = 0\rangle = (N!)^{-1/2}\left(x_0 s^\dagger_0 + x_2 d^\dagger_0 + x_4 g^\dagger_0\right)^N |0\rangle \;, \tag{9}$$

where $x_0 = \sqrt{1/5}$, $x_2 = \beta^0_2/\sqrt{5}$ and $x_4 = \beta^0_4/\sqrt{5}$ with $\gamma^0 = 0°$.

For further understanding of the four $SU_{sdg}(3)$ algebras and for comparing with the shell model results, we have examined quadrupole moments and $B(E2)$ values in the ground $K = 0$ band. Using Eq. (9) it is easy to construct the angular momentum projected states $|N; K = 0, L, M\rangle$ and calculate $Q(L)$'s and $B(E2)$'s. The formulation for these, giving results to order $1/N^2$, is available in [29] and this is valid for a general $E2$ transition operator,

$$T^{E2} = \sum_{\ell', \ell} t_{\ell' \ell}\left(b^\dagger_{\ell'} \tilde{b}_\ell\right)^2_q \,, \tag{10}$$

where $t_{\ell', \ell}$ are free parameters. Using the standard choice $T^{E2} = Q^2_q(\alpha_{sd} = +1, \alpha_{dg} = +1)$, as used in IBM studies, the results obtained for $Q(2^+_1)$, $Q(4^+_1)$, $B(E2; 2^+_1 \to 0^+_1)$ and $B(E2; 4^+_1 \to 2^+_1)$ for a 10 boson system are given in Table 3. It is seen that the $SU^{(+,+)}(3)$ and $SU^{(+,-)}(3)$ are closer generating

Table 3. Quadrupole moments and $B(E2)$ values for low-lying states in the ground band for a 10 boson system generated by the four $SU(3)$ algebras in sdgIBM. Note that T^{E2} in Eq. (10) is unit-less and therefore $Q(L)$ and $B(E2)$'s in the table are unit-less. In real applications, T^{E2} need to be multiplied by $e_{\text{eff}}b^2$ where e_{eff} is effective charge.

α_{sd}	α_{dg}	$Q(2_1^+)$	$Q(4_1^+)$	$B(E2; 2_1^+ \to 0_1^+)$	$B(E2; 4_1^+ \to 2_1^+)$
+1	+1	-23.71	-30.18	137.05	194.61
+1	-1	-10.65	-13.67	27.46	37.84
-1	+1	-3.74	-5.16	3.16	4.77
-1	-1	9.32	11.35	21.99	31.54

prolate shape and $SU^{(-,-)}(3)$ generating oblate shape. The $SU^{(-,+)}(3)$ though generates prolate shape, the quadrupole moments are very small. Thus, sdgIBM substantiates the general structures observed in the sdg shell model examples, i.e. one generating prolate shape, one oblate and the other two also generate prolate shape with one of them giving very small quadrupole moments.

5 Conclusions

Multiple $SU(3)$ algebras appearing in shell model (also in IBM) open a new paradigm in the applications of $SU(3)$ symmetry in nuclei. In the first detailed attempt made in this paper, using three (sdg) space examples in SM, we showed that the four $SU(3)$ algebras in this space exhibit quite different properties with regard to quadrupole collectivity as brought out by the quadrupole moments $Q(J)$ and $B(E2)$'s in the ground $K = 0$ band in even-even systems (see Tables 1 and 2). The SM and DSM calculations are restricted to the examples with the leading $SU(3)$ irrep of the type $(\lambda 0)$. The prolate, oblate and intermediate structures from the four $SU(3)$ algebras found using SM and DSM are further substantiated using sdgIBM. Also, the results from $Q(J)$ and $B(E2)$'s for the four $SU(3)$ algebras are consistent with the correlation coefficients between the four different $Q.Q$ operators in the sdg space of SM. Results in Sections 3 may be useful in finding empirical examples for multiple $SU(3)$ algebras in SM spaces. Further SM analysis of multiple $SU(3)$ algebra for example using $sdgi$ space with 8 $SU(3)$ algebras and the application of the H_Q's in Eq. (3) to quantum phase transitions will be reported elsewhere.

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