

Toroidal States of the ^{12}C Nucleus*

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Abstract.

Among the states of ^{12}C , there is an important subset of $K = 0$ and $K = I$ planar intrinsic states in which the intrinsic motion of the nucleons are confined in the planar region defined by the three-alpha cluster or by their generated toroid. The intrinsic nuclear densities of these states are toroidal in nature. We study these ^{12}C toroidal states from the generator-coordinate viewpoints in both the alpha cluster model and the toroidal shell model. Numerical solutions in the toroidal mean field approximation are examined to pave the way for future extensions and refinements.

KEY WORDS: ^{12}C Hoyle state, toroidal ^{12}C states, generator coordinate method

1 Introduction

John Wheeler suggested that under an “extreme behavior”, the nuclear fluid may assume a toroidal shape. He encouraged his students and fellow physicists to think about where such extreme behavior might occur and to look for it [1, 2]. Possible extreme behavior might occur in the presence of (i) a large Coulomb energy [1, 3], (ii) a strong nuclear shell effect [3], and/or (iii) a large angular momentum [4]. There is a strong shell effect of a doubly closed-shell for ^{12}C in a toroidal potential and the extrapolation from heavier nuclei points to a possible low-lying toroidal state in ^{12}C [3], suggesting that the ^{12}C nucleus may be a favorable candidate for a toroidal configuration [5]. The study of light toroidal nuclei gains additional impetus recently because the effects of strong toroidal shells and large angular momenta lead to toroidal high-spin isomers in many light alpha-conjugate nuclei, as shown in different theoretical calculations [6–11] and in possible experimental observations of highly excited states in ^{28}Si .

For the ^{12}C nucleus in particular, Wheeler’s model of a triangular 3α cluster [12] has been studied extensively [13–20]. However, Wheeler’s other concept of a possible toroidal nucleus [1,2] has up to now not been applied to the ^{12}C nucleus. Wheeler’s two different concepts should play their separate and important roles

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under different probes of the nucleus. In matters of 3α decay and the escape through the external Coulomb barrier, the 3α cluster description is clearly the simpler description. However, because nucleons can traverse azimuthal orbitals in a toroidal nucleus with low energies and can be excited to higher orbitals with low excitation energies, a description in terms of a ^{12}C nucleus in toroidal doubly-closed shells may be an efficient description in matters associated with particle-hole excitations and in the density of particle-hole multiplet states. Furthermore, the toroidal concept provides a novel geometrical insight, organizes useful correlations, helps guide our intuition, and may find many applications involving the ^{12}C nucleus. It is therefore useful to develop the toroidal concepts for the ^{12}C nucleus.

2 Generator Coordinate Equation for the Planar Alpha Cluster Model

We envisage that among the states of ^{12}C , there is an important subset of oblate $K = 0$ and $K = I$ planar intrinsic states in which the intrinsic motion of the nucleons are confined in the planar region defined by the cluster of three alpha particle or by the toroid generated by the planar rotation of the 3α cluster. States in this subset are characterized by the angular momentum component quantum number K normal to the plane, and appear as the band heads of collective rotational states when the states are projected into the full three-dimensional space. States of this kind can be described in many different ways, depending on an additional assumption on how the intrinsic shape is generated. For example, we can follow Wheeler [12] and assume the intrinsic shape to be generated by an even more basic triangular cluster of three alphas particles, $|\phi_j(\Delta)\rangle$, and consider as in Griffin, Hill, and Wheeler [14, 15] a trial wave function $|\Psi^K\rangle$ as a coherent sum of these antisymmetrized triangular three-alpha cluster wave functions $|\phi_j(\Delta)\rangle$ of different triangular orientations on the plane

$$|\Psi^K\rangle = \int d\gamma e^{-iK\gamma} e^{i\gamma\hat{L}_z} \sum_j f_j^K |\phi_j(\Delta)\rangle. \quad (1)$$

Here, \hat{L}_z is the rotation operator about the body-fixed z -axis on the triangular plane which does not involve the spin contribution, γ is the corresponding Euler angle for such a rotation, and f_j^K are the Griffin-Hill-Wheeler generator coordinate amplitude for the state $|\Psi^K\rangle$, and the operator $\int d\gamma e^{-iK\gamma} e^{i\gamma\hat{L}_z}$ projects out states of good K quantum number. The generator coordinate projection sum of the cluster wave function $|\phi_i(\Delta)\rangle$ on the triangular plane over all the orientations specified by γ leads to wave functions with toroidal characteristics, which we can label as $|\phi_j^K(\text{toroid})\rangle$,

$$|\phi_j^K(\text{toroid})\rangle = \int d\gamma e^{-iK\gamma} e^{i\gamma\hat{L}_z} |\phi_j(\Delta)\rangle. \quad (2)$$

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In terms of these toroidal wave functions $|\phi_i^K(\text{toroid})\rangle$ projected from the basic three-alpha cluster state $|\phi_i(\Delta)\rangle$, the trial wave function becomes

$$|\Psi^K\rangle = \sum_i f_j^K |\phi_j^K(\text{toroid})\rangle. \quad (3)$$

Quantization of the system can be carried out by minimizing the energy with respect to the variation of the trial wave function $|\Psi^K\rangle$ under the constraint of a fixed normalization, resulting in the Griffin-Hill-Wheeler equation for f_j^K ,

$$\sum_j [H_{ij} - EB_{ij}] f_j^K = 0, \quad (4a)$$

where $H_{ij} = \langle \phi_i^K(\text{toroid}) | H | \phi_j^K(\text{toroid}) \rangle,$ (4b)

$$B_{ij} = \langle \phi_i^K(\text{toroid}) | \phi_j^K(\text{toroid}) \rangle. \quad (4c)$$

The generator-coordinate method involves introducing a two-body interaction, solving the above Griffin-Hill-Wheeler equation, and projecting the intrinsic toroidal solution onto the full three-dimensional laboratory space for states with angular momentum quantum numbers I and M afterwards.

3 Generator Coordinate Equation for the Toroidal Shell Model

The expansion of the physical wave function $|\Psi^K\rangle$ in Eq. (3) in terms of the 3α cluster wave functions $|\phi_j^K(\Delta)\rangle$ of Eq. (2) is useful in problems where the cluster properties manifest themselves. However, it is not the only way to construct a trial wave function $|\Psi^K\rangle$. Because nucleons can traverse azimuthal orbitals in a toroidal nucleus with low energies and they can scatter from good-Lambda orbitals just below the Fermi level to orbitals just above the Fermi level with low excitation energies, a description in terms of a ^{12}C nucleus in toroidal configurations may be an efficient description in matters associated with particle-hole excitations. For such problems, the presence of the toroidal degree of freedom allows an alternative trial wave function for the state with quantum number K ,

$$|\Psi^K\rangle = \sum_j g_j^K |\Phi_j^K(\text{toroid})\rangle, \quad (5)$$

where $|\Phi_j^K(\text{toroid})\rangle$ is chosen to be those obtained in a toroidal shell model, either self-consistently from a mean-field approximation or non-self-consistently from a single-particle model with an assumed toroidal shape. The Griffin-Hill-Wheeler equation for the amplitude g_j^K is

$$\sum_j [H_{ij} - EB_{ij}] g_j^K = 0, \quad (6a)$$

where
$$H_{ij} = \langle \Phi_i^K(\text{toroid}) | H | \Phi_j^K(\text{toroid}) \rangle, \quad (6b)$$

$$B_{ij} = \langle \Phi_i^K(\text{toroid}) | \Phi_j^K(\text{toroid}) \rangle. \quad (6c)$$

The quantities H_{ij} and B_{ij} in (6b) and (6c) are now defined in terms of $|\Phi_j^K\rangle$ from a toroidal shell model in place of $|\phi_j^K\rangle$ from the projection of three-alpha cluster states. Different choices of the trial wave functions from either Eq. (2) (and (3)) or Eq. (5) should converge to the same physical state $|\Psi^K\rangle$, provided they have been chosen efficiently with a sufficient number of components in the expansion.

4 Toroidal Mean Field Approximation

The simplest expansion in Eq. (5) consists of a single term $|\Phi_{i=1}^K\rangle$ in the mean-field approximation in which we describe the ^{12}C nucleus with a single Slater determinant that takes into account the dominant correlations with a mean-field interaction. Consider as an example the case of $K = 0$. With an intrinsic axial symmetry about the z -axis, the ^{12}C nucleus with $K = 0$ can be represented by the single Slater determinant of neutrons and protons occupying the lowest single-particle $|\Lambda, \Omega_z\rangle$ states, where $\Lambda = |\Lambda_z|$, Λ_z is the z -component of the orbital angular momentum, $\Omega_z = \Lambda_z + s_z$, and s_z is the spin component along the z axis. Limiting our attention on the state with the lowest oscillation quanta in the ρ - and z - directions for ^{12}C , the occupied single-particles states are $|0, \pm 1/2\rangle$, $|1, \pm 3/2\rangle$, and $|1, \pm 1/2\rangle$. Assuming the same set of wave functions for neutrons and protons and neglecting the spin-orbit interaction, we write down the variational spatial wave functions of the occupied single-particle states in terms of variational parameters (R, d, a_2)

$$\Psi_{\Lambda_z \Omega_z}(\rho, z, \phi) = \mathcal{R}_\Lambda(\rho) Z(z) [\Phi_{\Lambda_z}(\phi) \chi_{s_z}]^{\Omega_z}, \quad (7a)$$

where
$$\mathcal{R}_\Lambda(\rho) = N_\Lambda \rho^\Lambda \exp \left\{ -\frac{(\rho - R)^2}{2(d^2 e^{2a_2} / \ln 2)} \right\}, \quad \Lambda = 0, 1, \quad (7b)$$

$$Z(z) = N_Z \exp \left\{ -\frac{z^2}{2(d^2 e^{-2a_2} / \ln 2)} \right\}, \quad (7c)$$

$$\Phi_{\Lambda_z}(\phi) = \frac{e^{i\Lambda_z \phi}}{\sqrt{2\pi}}, \quad (7d)$$

with normalization constants N_Z and N_Λ , and $[\Phi_{\Lambda_z}(\phi) \chi_{s_z}]^{\Omega_z}$ to denote the coupling of orbital Λ_z and spin s_z to Ω_z . We utilize the density-dependent Skyrme SkM* interaction [21] that is designed to describe well the surface and bulk properties and large quadrupole deformation properties. The set of variational parameters $(R = 0, d = 1.49 \text{ fm}, a_2 = -0.089)$ gives the ground state of the system. The nuclear density on the x - z plane at $y = 0$ is given in Figure 1. The equidensity surfaces in the low density region are nearly oblate ellipsoids. When the density increases to $n \geq 0.21 \text{ /fm}^3$, the equidensity surfaces turn into

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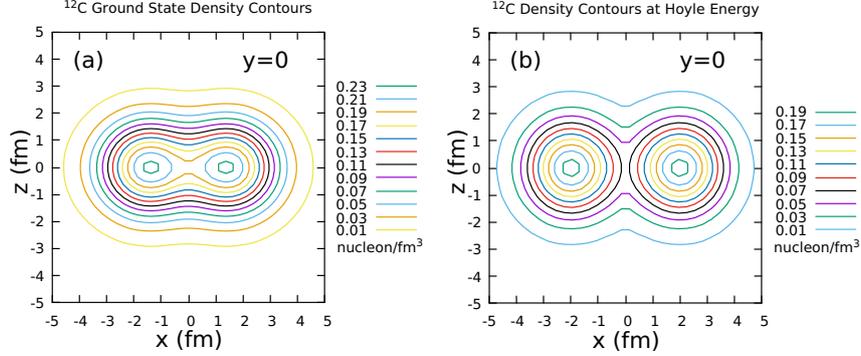


Figure 1. (color online). The nuclear density of the ^{12}C on the $y = 0$ plane for (a) ground state and (b) at the Hoyle energy, obtained with the Skyrme SkM^* interaction in the mean-field approximation.

toroids. The ground state of the ^{12}C nucleus has a dense toroidal core immersed in oblate ellipsoids.

In the mean-field approximation, we find that the case of ($R = 1.16$ fm, $d = 1.370$ fm, $a_2 = -0.007$) corresponds to an excitation energy of the Hoyle energy at $E_x = 7.65$ MeV with the corresponding nuclear density on the x - z plane at $y = 0$ shown in Figure 1(b). One observes that for this state at the Hoyle energy, the equidensity surfaces with density $n \geq 0.07/\text{fm}^3$ appear as separated toroids, while those with lower densities as spindle toroids. The meridian cross sections are nearly circular.

In the mean-field approximation with a single determinant, the ground state with the density shown in Figure 1(a) is a state of lowest energy minimum, but the state at the Hoyle energy with density shown in Figure 1(b) lies on an energy slope as a function of R , even though it is stable against variations in d and a_2 . It is therefore unstable against the contraction of the radius parameter R in the mean-field approximation. We nonetheless call it a provisional toroidal state at the Hoyle energy, on account of its toroidal density shape, pending further investigation of its stability against R variations beyond the mean field. Pending modifications beyond the mean field may modify slightly the R location and the shape of the density distribution but will not likely change its toroidal characteristics.

5 Phenomenological Study of the ^{12}C Toroidal Configuration

In this first exploration of its kind to examine the toroidal degree of freedom of the ^{12}C nucleus, it is appropriate to study the problem from both the microscopic and phenomenological points of view. In the phenomenological study as presented in [5], we can search for the signature of the ^{12}C nucleus in a toroidal

configuration so as to facilitate its identification. The toroidal intrinsic shape shows up as a bunching of single-particle states into “ Λ -shells” whose spacing is intimately tied to the size of the toroidal major radius. This set of single-particle shells will generate a distinct pattern of particle-hole multiplet excitations between one toroidal single-particle shell to another. From such a signature and experimental data, we find phenomenologically that the Hoyle state and many of its higher excited states may be tentatively attributed to those of the ^{12}C nucleus in a toroidal configuration [5]:

- the matching of the gross structure of the low-lying spectrum with the toroidal signature,
- the approximate equality of the strengths of the excitation function in the $^{11}\text{B}(^3\text{He},\text{d})^{12}\text{C}^* \rightarrow 3\alpha$ reaction for toroidal states within a multiplet,
- at low excitation energies, the presence of the underlying structure of unresolved ^{12}C states in the $^{11}\text{B}(^3\text{He},\text{d})^{12}\text{C}^* \rightarrow 3\alpha$ reaction and the presence of as yet unidentified members of the multiplet states,
- at high excitation energies, the presence of a large excitation function in $^{10}\text{B}(^3\text{He},\text{p})^{12}\text{C}^* \rightarrow 3\alpha$ reactions and the presence of a large number of toroidal particle-hole states as a function of energy.

There are however many items that need to be further investigated to confirm the presence of such a toroidal configuration. Our suggested description contains future proposed tests that may be able to shed more lights on the proper description of the states of ^{12}C [5].

6 Residual Octupole-Octupole Interactions in the Mean-Field Approximation

Even though we do not find a toroidal local energy minimum as a function of R in the mean-field approximation with a single-determinant for $K = 0$, we are however motivated to continue the theoretical search for an energy minimum in view of the many pieces of experimental evidence supporting the tentative identification the Hoyle state and many of its excited states as toroidal ^{12}C states, as discussed in [5]. From intuitive viewpoints, we are further encouraged by the small energy separation between the Hoyle state and its excited states, by the large number of both the identified and the un-identified broad excited states, by the close average energy spacing between the states, and by their predominance in their decay into three alpha particles. These characteristics suggest that the Hoyle state is intrinsically a spatially extended object that is capable of possessing a complex particle-hole excitation structure. A local toroidal energy minimum description is consistent with such a suggestion. Theoretically we find it promising that the ground state nuclear density in the mean-field approximation as exhibited in Figure 1 already shows a toroidal structure in its core and the provisional state at the Hoyle energy in Figure 1 exhibits prominent toroidal

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characteristics. We also note with interest that many previous microscopic models with alpha clusters have been successful in describing both the ground state and the Hoyle state [16–20]. We are therefore motivated to continue the search for a local energy minimum at the Hoyle energy.

We shall first continue our search within the limitation of a single determinant, we would like to explore the effects of residual interactions in the mean-field theory. We have reasons to infer that the quadrupole correlations of the nucleons leading to the large quadrupole deformation is properly taken into account by the Skyrme SkM* interaction. Indeed, the toroidal feature of the ^{12}C nucleus of the ground state as exhibited in Figure 1(a) is in agreement with earlier results from earlier Hartree-Fock calculations [22] and the resonating group method [17]. In the sense of the multipole expansion of the nucleon-nucleon interaction, there is the possible residual octupole-octupole interaction that may lead to octupole deformation of the triangular type. We are further guided by results obtained in previous calculations [16–20] where there are cluster solutions at the Hoyle energy and these solutions have prominent triangular cluster characteristics at the Hoyle state. Accordingly, we postulate the presence of an octupole-octupole interaction in conjunction with the SkM* interaction:

$$V = \frac{1}{2} \sum_i \sum_{j \neq i}' V_{\text{SkM}^*}(i, j) - \frac{\chi}{2} \sum_i \sum_{j \neq i}' Q_{33}(i) Q_{33}(j), \quad (8)$$

where in cylindrical coordinates

$$Q_{33}(i) = \frac{1}{4} \sqrt{\frac{35}{4\pi}} (\rho_i / \overset{\circ}{R}_0)^3 \cos(3\phi_i), \quad (9)$$

$\overset{\circ}{R}_0 = r_0 A^{1/3}$, and $r_0 = 1.2$ fm. We have included an $1/\overset{\circ}{R}_0^3$ scale factor to make the multipole moment Q_{33} dimensionless and χ in MeV.

To study the additional octupole-octupole interaction, we modify the azimuthal wave function in Eq. (7d) to be

$$\Phi_{\Lambda}(\rho, \phi) = \frac{[1 + \sigma_3 (\rho / \overset{\circ}{R}_0)^3 \cos(3\phi)] e^{i\Lambda\phi}}{\sqrt{2\pi}} \quad (10)$$

where σ_3 is the dimensionless 'sausage' deformation parameter of the $\lambda = 3$ order that makes the toroidal nucleus thicker in three sections and thinner in three others along the toroidal rim [3]. We can now study the energy surface of ^{12}C as a function of the variational parameters (R, d, a_2, σ_3) for various magnitudes of the octupole-octupole interaction.

Upon fixing the (d, a_2) values to be those that give an energy minimum for the case of $\chi = 0$, we study then the variations of the energy surface on the (R, σ_3) plane. In Figure 2(a) for $\chi = 0$ without the octupole-octupole interaction, the energy surface at the ground state at the ground state ($R = 0$) is an energy

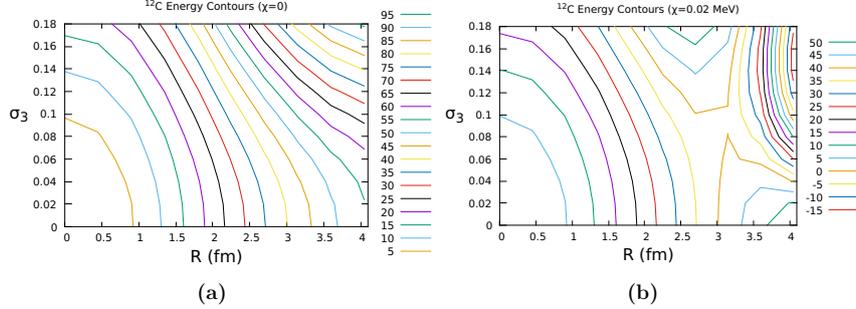


Figure 2. (color online). The energy contour of ^{12}C on the (R, σ_3) plane where the parameters (d, a_2) have been chosen to minimize the energy for each R at $\sigma_3 = 0$. Figure (a) gives the results without the octupole-octupole interaction ($\chi = 0$), and figure (b) the results for the octupole-octupole interaction $\chi = 0.02$ MeV in Eq. (8).

minimum also in σ_3 . At the Hoyle energy ($R = 1.16$ fm), the energy surface rises as a function of σ_3 . In Figure 2(b) for $\chi = 0.02$ MeV with a small octupole-octupole interaction strength, the energy surface has a saddle point at ($R = 3$ fm, $\sigma_3 \sim 0.08$) with a barrier height of about 35-40 MeV. Upon passing over the saddle point beyond $R \gtrsim 3$ fm, the energy surface drops down which reflects the three alpha cluster structure, indicating that there is a high barrier separating the region of the ground state energy minimum and the region of the three-alpha cluster. The general features of the energy surface landscape for other values of χ are similar to those in Figure 2(b). At greater values of χ , the barrier moves lower at a slightly lower value of R . There is however no energy minimum at the Hoyle energy of $E_x = 7.654$ MeV for a linear residual octupole-octupole interaction as given in Eq. (8).

The result indicates that with only a single Slater determinant in the expansion of the generator coordinate sum in Eq. (3) in the toroidal mean-field model and the linear octupole-octupole interaction, there is no energy minimum at the Hoyle energy of $E_x = 7.654$ MeV. The search for a local energy minimum at the Hoyle energy will likely require either a quadratic octupole-octupole interaction or the inclusion of many more Slater determinants.

7 Conclusions and Discussions

We study the intrinsic oblate states of ^{12}C from the generator coordinate viewpoints. We note that under the assumption of planar dominance in which the intrinsic motion of the nucleons is confined in a planar region for the nucleus, the Griffin-Hill-Wheeler equation can be substantially simplified. It is only necessary to solve the equation in the planar region, and the solution of the intrinsic state with quantum number K can be projected out to obtain the rotational band

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of states with angular momentum I and component M in the laboratory frame. For a basic shape in the form of a triangular cluster of three alpha particles under the planar dominance, the generator coordinate sum of the orientations of the cluster triangle lead to toroidal density distributions.

One can take the alternative generator coordinate sum whose elements are toroidal wave functions and obtain a similar Griffin-Hill-Wheeler equation. The case of a single Slater determinant then leads us to the toroidal mean field approximation.

It turns out that a single-determinant mean-field approximation is adequate to describe the ground state of ^{12}C which gives a toroidal core immersed in oblate spheroids. The mean field approximation also gives a toroidal density at the Hoyle state, but the energy of the state lies on a slope as a function of the radial parameter and is unstable with respect to the radial contraction. It shows that to study the stability of the Hoyle state, it will be necessary to go beyond the mean field.

We are however motivated to continue the search for a local toroidal energy minimum for the Hoyle state because phenomenological comparison of the spectrum with the toroidal nucleus signature suggests that the Hoyle state and many of its higher excited states may be tentatively attributed to those of the ^{12}C nucleus in a toroidal configuration. Our next task is to introduce a two-body residual interaction and use the toroidal mean field or the toroidal shell model basis in the Griffin-Hill-Wheeler equation to study the Hoyle state and its associated excited states. Results from such calculations will be of great interest.

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