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Microscopic Structure of the Low-Lying Collective States in ^{152}Sm

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Abstract. The proton-neutron symplectic shell-model approach is used to determine the microscopic structure of the low-lying collective states of ground and first few excited bands of positive and negative parity in ^{152}Sm. For this purpose, the model Hamiltonian is diagonalized in a U(6)-coupled basis, restricted to state space spanned by the fully symmetric U(6) vectors of the lowest even and odd irreducible representations of Sp(12,R). In this way, the positive- and negative-parity collective bands are treated on equal footing within the framework of the microscopic symplectic-based shell-model scheme without the introduction of additional degrees of freedom inherent to other approaches to odd-parity nuclear states. A good description of the excitation energies of the considered bands is obtained. The microscopic structure of low-lying collective states in ^{152}Sm shows that practically there are no admixtures from the higher shells and hence the presence of a very good U(6) dynamical symmetry. Additionally, the structure of the collective states under consideration shows also the presence of a good SU(3) quasi-dynamical symmetry. The intraband ground band $B(E2)$ and interband $B(E1)$ transition strengths between the states of ground and $K^\pi = 0^-_1$ bands are reproduced without the use of an effective charge, which can be considered as a significant achievement of the present symplectic-based proton-neutron shell-model approach.

KEY WORDS: dynamical symmetry, the proton-neutron symplectic model, quasi-dynamical symmetry, Sp(12,R) dynamical algebra

1 Introduction

Experimental spectra in heavy nuclei show the emergence of simple collective patterns represented primarily by the nuclear collective rotation. In this way, the low-lying spectra of well deformed even-even nuclei consist of different rotational bands − ground state, β, γ, etc. − of positive parity, which properties are described very successfully within the macroscopic nuclear structure physics theories, like the Bohr-Mottelson (BM) [1] and the Interacting Boson Model (IBM) [2] ones.

In some mass regions several bands of negative parity are also observed in the low-lying nuclear spectra in even-even nuclei, like $K^{\pi} = 0^{-}$, 1^{-} and 2^{-} bands [3, 4]. The most well-studied of them is the $K^{\pi} = 0^{-}$ band, usually interpreted as an octupole vibrational band, connected to the ground state band by enhanced $E1$ transitions. Sometimes, as suggested to be the case of static octupole deformation [4], the ground state and $K^{\pi} = 0^{-}$ bands intertwine into a single alternating parity band. Ussually, the negative-parity states have been described within different approaches mainly by inclusion of additional octupole or/and dipole (cluster) degrees of freedom.

From other side, it is well known that within the microscopic shell-model approach with the increase of the number of valence particles or/and the single-particle states available, the model space dimensionalities grows rapidly and rule out the use of standard shell-model theory. As a consequence, different approaches have been proposed to truncate the many-particle configuration space. Among them, the algebraic models based on the symmetries, exact or approximate, provide elegant and very efficient methods for reducing the model space in manageable size. Something more, different truncation schemes have been used to reorganize the available shell-model spaces in such a way that their symmetry-adapted bases already capture the essential features of the observed nuclear properties.

Recently, the fully microscopic proton-neutron symplectic model (PNSM) of nuclear collective motion with Sp(12,R) dynamical algebra was introduced by considering the symplectic geometry and possible collective flows in the two-component many-particle nuclear system [5]. It was shown that the dynamical group of possible collective excitations in the two-component nuclear systems is provided by the non-compact Sp(12,R) symplectic group. The latter, in shell-model terms, contains the valence shell proton-neutron degrees of freedom, containing the basic SU(3) rotational and low-energy shape vibrational, as well as the high-energy $2\hbar\omega$ giant resonance vibrational degrees of freedom. That is, in the PNSM we have the valence $0\hbar\omega$ $0p$-$0h$ and many-particle $2n$ $\hbar\omega$ np-nh shell model excitations with oscillator energy $N_0 + 2n$. Symplectic generators change the number of oscillator quanta by two units, hence they preserve the parity and permutational symmetry. Further, it is well known that the symplectic groups admit two types of irreducible representations: even and odd ones which contain all the harmonic oscillator shell-model states with even and odd number of excitation quanta, respectively. This allows to account for both the positive- and negative-parity states without the introduction of additional degrees of freedom that are inherent to other approaches to odd-parity nuclear states.

The PNSM has been applied for the simultaneous description of the microscopic structure of the ground, γ and β bands in ^{166}Er [6], using the $SU_p(3) \otimes SU_n(3)$ symmetry-adapted basis which is appropriate for the case of generic non-fully symmetric irreducible representations of Sp(12,R). The PNSM was further used to study the microscopic structure of the low-lying collective states of the ground, β and γ bands in ^{154}Sm [7] and ^{238}U [8], using the more general U(6)-coupled basis, restricted to the symplectic state space spanned by the fully symmetric U(6) vectors. Recently, the results for the microscopic structure of negative-parity states of the lowest $K^\pi = 0_1^-$ and $K^\pi = 1_1^-$ bands in ^{154}Sm and ^{238}U were reported [9, 10], including the low-energy $B(E1)$ interband transition strengths between the states of the ground band and $K^\pi = 0_1^-$ band [10] for these two nuclei. Here we present the results for the microscopic structure of the low-lying collective states of the first few positive- and negative-parity bands in one more nucleus, namely ^{152}Sm.

2 The Proton-Neutron Symplectic Model

Collective observables of the proton-neutron symplectic model, which span the Sp(12,R) algebra, are given by the following one-body operators [5]:

$$Q_{ij}(\alpha,\beta) = \sum_{s=1}^{m} x_{is}(\alpha)x_{js}(\beta), \tag{1}$$

$$S_{ij}(\alpha,\beta) = \sum_{s=1}^{m}\Big(x_{is}(\alpha)p_{js}(\beta) + p_{is}(\alpha)x_{js}(\beta)\Big), \tag{2}$$

$$L_{ij}(\alpha,\beta) = \sum_{s=1}^{m}\Big(x_{is}(\alpha)p_{js}(\beta) - x_{js}(\beta)p_{is}(\alpha)\Big), \tag{3}$$

$$T_{ij}(\alpha,\beta) = \sum_{s=1}^{m} p_{is}(\alpha)p_{js}(\beta), \tag{4}$$

where $i, j = 1, 2, 3$; $\alpha, \beta = p, n$ and $s = 1, \ldots, m = A - 1$. In Eqs.(1)$-$(4), $x_{is}(\alpha)$ and $p_{is}(\alpha)$ denote the coordinates and corresponding momenta of the translationally-invariant Jacobi vectors of the m-quasiparticle two-component nuclear system and A is the number of protons and neutrons.

The PNSM dynamical algebra Sp(12,R) has many subalgebra chains, which roughly can be divided on two type of chains, the collective model and the shell model chains. The form (1)$-$(4) of the symplectic algebra Sp(12,R) is naturally adapted to the collective model chain which reveals the dynamical content of symplectic symmetry. Among the subalgebras of this chain are, for example, the general collective motion in six dimensions $GCM(6)$ and the coupled two-rigid rotor model $ROT_p(3) \otimes ROT_n(3) \supset ROT(3)$ Lie algebras. The $GCM(6)$ algebra introduces the $SO(6)$ intrinsic vortex degrees of freedom which coupled to the giant resonances allows for the continuous range of rotational dynamics from rigid to irrotational flow. For more details about the dynamical content of the PNSM, we refer the reader to Ref. [5].

The shell-model chain of Sp(12,R) algebra relates the PNSM to the shell-model nuclear theory and thus providing a connection to the microscopic fermion physics. It provides a shell-model coupling scheme and a basis for detailed microscopic shell-model calculations. The shell-model chain is naturally expressed in terms of the harmonic oscillator creation and annihilation operators

$$\begin{aligned} b^{\dagger}_{i\alpha,s} &= \sqrt{\frac{m_\alpha \omega}{2\hbar}}\Big(x_{is}(\alpha) - \frac{i}{m_\alpha \omega} p_{is}(\alpha)\Big), \\ b_{i\alpha,s} &= \sqrt{\frac{m_\alpha \omega}{2\hbar}}\Big(x_{is}(\alpha) + \frac{i}{m_\alpha \omega} p_{is}(\alpha)\Big). \end{aligned} \tag{5}$$

Then the symplectic generators take an alternative form as all bilinear combinations of the harmonic oscillator raising and lowering operators that are $O(m)$ invariant [11]:

$$F_{ij}(\alpha,\beta) = \sum_{s=1}^{m} b^{\dagger}_{i\alpha,s} b^{\dagger}_{j\beta,s}, \tag{6}$$

$$G_{ij}(\alpha,\beta) = \sum_{s=1}^{m} b_{i\alpha,s} b_{j\beta,s}, \tag{7}$$

$$A_{ij}(\alpha,\beta) = \frac{1}{2}\sum_{s=1}^{m} (b^{\dagger}_{i\alpha,s} b_{j\beta,s} + b_{j\beta,s} b^{\dagger}_{i\alpha,s}). \tag{8}$$

An Sp(12,R) unitary irreducible representation is characterized by the U(6) quantum numbers $\sigma = [\sigma_1, \ldots, \sigma_6]$ of its lowest-weight state $|\sigma\rangle$, i.e. $|\sigma\rangle$ satisfies

$$\begin{aligned} &G_{ab}|\sigma\rangle = 0; \\ &A_{ab}|\sigma\rangle = 0, \quad a < b; \\ &A_{aa}|\sigma\rangle = \Big(\sigma_a + \frac{m}{2}\Big)|\sigma\rangle \end{aligned} \tag{9}$$

for the indices $a \equiv i\alpha$ and $b \equiv j\beta$ taking the values $1, \ldots, 6$. If we introduce the U(6) tensor product operators $P^{(n)}(F) = [F \times \ldots \times F]^{(n)}$, where $n = [n_1, \ldots, n_6]$ is a partition with even integer parts, then by an U(6) coupling of these tensor products to the lowest-weight state $|\sigma\rangle$, one constructs the whole basis of states for an Sp(12,R) irrep

$$|\Psi(\sigma n \rho E \eta)\rangle = [P^{(n)}(F) \times |\sigma\rangle]^{\rho E}_{\eta}, \tag{10}$$

where $E = [E_1, \ldots, E_6]$ indicates the U(6) quantum numbers of the coupled state, η labels a basis of states for the coupled U(6) irrep E and ρ is a multiplicity index. In this way we obtain a basis of Sp(12,R) states that reduces the subgroup chain $Sp(12,R) \supset U(6)$. To fix the basis η one has to consider further the reduction of the U(6) to the 3-dimensional rotational group $SO(3)$. Thus, in order to completely classify the basis states, we use the following reduction chain [7, 11]:

$$\begin{array}{ccccccccc} Sp(12,R) & \supset U(6) & \supset & SU_p(3)\otimes SU_n(3) & \supset & SU(3) & \supset SO(3) & \supset & SO(2) \\ \sigma & n\rho \;\; E & \gamma & (\lambda_p,\mu_p)\;\;(\lambda_n,\mu_n) & \varrho & (\lambda,\mu) & K \;\; L & & M \end{array} \tag{11}$$

which defines a shell-model coupling scheme. Under different subgroups in (11) are given the corresponding quantum numbers that characterize their irreducible representations plus some multiplicity indices. In this way, the symplectic bandhead ($0\hbar\omega$ space) is determined by the σ quantum number. The core collective excitations (np-nh shell-model configurations) are determined by the n and E quantum numbers, whereas the proton/neutron and total deformations are determined by $(\lambda_\alpha,\mu_\alpha)$ ($\alpha = p,n$) and (λ,μ) respectively. Finally, the rotation is given by the standard angular momentum quantum number L and its third projection M.

3 Application

We apply the theory to the well deformed nucleus ^{152}Sm. Since in heavy mass regions the spin-orbit interaction is strong, we use the pseudo-SU(3) scheme to determine the relevant irreducible representations of Sp(12,R). Actually, the lowest even and odd Sp(12,R) irreps relevant to the low-lying positive- and negative-parity states in ^{152}Sm coincide with those of ^{154}Sm. Thus, for more details concerning the shell-model classification of the collective states, see Refs. [7, 9]. We remember that in the pseudo-SU(3) scheme, usually the particles occupying the unique-parity orbitals are assumed to be in the seniority-zero configurations and are considered as passive spectators which addiabatically follow the rotational motion of their normal-parity partners. Their contribution to the structure of low-lying states could be taken into account by appropriate scalling factors (see, e.g., Ref. [12]). This is obviously so for the low-lying possitive-parity states, at least for low angular momenta below the backbending region. For negative-parity states, a particle from the normal-parity orbitals can be excited to the unique-parity ones, giving rise to the negative-parity states too. Thus the intruder particles might play an important role in the description of the low-lying negative-parity states in the strongly deformed heavy nuclei. In particular, since in the present application of the PNSM the neutron subsystem is considered to be unexcited (cf. Table II of Ref. [9]), the $h_{11/2}$ together with the normal-parity orbit configurations will account for the arising of the low-lying negative-parity states in ^{152}Sm.

3.1 The energy spectra

We use the following model Hamiltonian

$$H = N\hbar\omega - \frac{1}{2}\chi\big[Q_p \cdot Q_n - (Q_p \cdot Q_n)_{TE}\big] - (\xi + \xi_{sym})C_2[SU(3)] + aL^2 + \epsilon(N_{b.h.} - N_0), \quad (12)$$

which is diagonalized in a U(6)-coupled basis that is restricted to the space spanned by the fully symmetric U(6) irreps. In our considerations we include 20 harmonic oscillator major shells in the model space. In Eq.(12) $N = N_p + N_n$ and $Q_\alpha \equiv Q(\alpha, \alpha)$ with $\alpha = p, n$ are the full major-shell mixing quadrupole tensor operators and are given by Eq.(1). The trace-equivalent part $(Q_p \cdot Q_n)_{TE}$ [13] is subtracted from the collective potential in order to preserve the mean-field shell structure [12, 14, 15] under the action of the proton-neutron quadrupole-quadrupole interaction. The SU(3) second-order Casimir operator $C_2[SU(3)]$ splits energetically different SU(3) multiplets and in this way determines the bandhead energies of excited bands with respect to the ground band. The term $\epsilon(N_{b.h.} - N_0)$ is introduced in the model to take into account the energy difference between the even and odd symplectic bandheads. $N_{b.h.}$ is the number operator of the symplectic bandhead which eigenvalues for the $0\hbar\omega$ and $1\hbar\omega$ shell-model subspaces are given by N_0 and $N_0' = N_0 + 1$, respectively. N_0 denotes the minimum number of oscillator quanta allowed by the Pauli principle [11]. Without this term, the negative-parity states would appear at energy $\sim 1\hbar\omega$. The term aL^2, which represents a residual rotor part, allows the experimentally observed moment of inertia to be reproduced without altering the wave functions. In order to account for experimentally observed different moments of inertia of the negative-parity bands with respect to the positive-parity ones in ^{152}Sm, we use the following parametrization for the inertia parameter a: $a = a_0/(1 + 0.63\langle\Delta N_0\rangle)$, where $\langle\Delta N_0\rangle$ is the eigenvalue of the operator $\Delta N_0 = (N_{b.h.} - N_0)$ described above. We also introduce an additional parameter ξ_{sym} which role is to shift the SU(3) irreps with either λ or μ odd relative to those with λ and μ both even [16], for which ξ_{sym} is zero, as the former belongs to different symmetry types (B_α, $\alpha = 1, 2, 3$, rather than A) of the intrinsic Vierergruppe [17, 18], D_2.

The results for the low-lying energy levels of the ground, β and γ bands together with the $K^\pi = 0_1^-$ and $K^\pi = 1_1^-$ bands, are compared with experiment [3] in Figure 1. The model parameters χ, ξ, a are determined by fitting to the low-lying positive-parity states of the ground, β, and γ bands. Their values in MeV, respectively, are: $\chi = 0.0033$, $\xi = 0.0036$, and $a_0 = 0.016$. The adopted values of the last two Hamiltonian parameters, related to the negative parity bands, are $\epsilon = -6.733$ and $\xi_{sym} = 0.001$.

In Figure 2, the SU(3) decomposition of the wave functions (probability distribution) of ground state and 1^- state of the $K^\pi = 0_1^-$ band is shown. From the

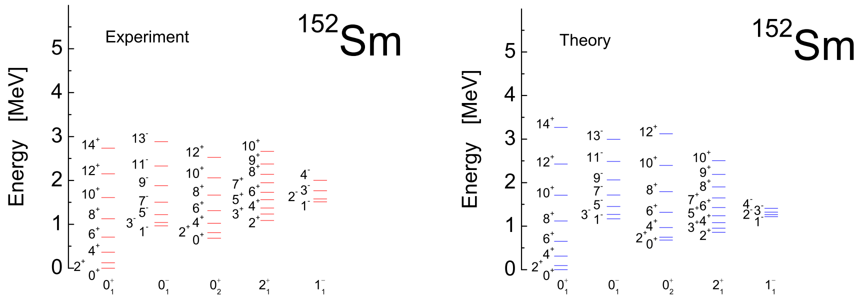

Figure 1. Comparison of experimental energy levels with the theory for the low-lying positive-parity ground, β and γ bands and negative-parity $K^{\pi} = 0^{-}_{1}$ and $K^{\pi} = 1^{-}_{1}$ bands in ^{152}Sm.

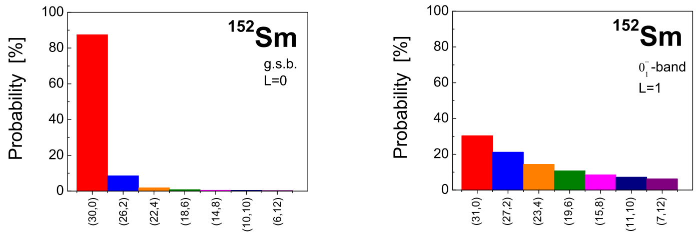

Figure 2. The SU(3) decompositions of the wave functions of the ground state and 1^{-} state of the $K^{\pi} = 0^{-}_{1}$ band.

latter one sees that the SU(3) dynamical symmetry is broken due to the mixing of different irreps. From another side, almost all SU(3) irreducible representations, contributing to the structure, practically belong to a single U(6) irreducible representation, namely that of the symplectic bandhead. More precisely, the U(6) irrep of the lowest-weight state exhausts up to 99.928% and 98.612% of the structure of ground and 1^{-} states, respectively. This indicates that the states under consideration possess a good U(6) dynamical symmetry. Additionally, in Figures 3 and 4, we show the SU(3) decomposition of the wave functions

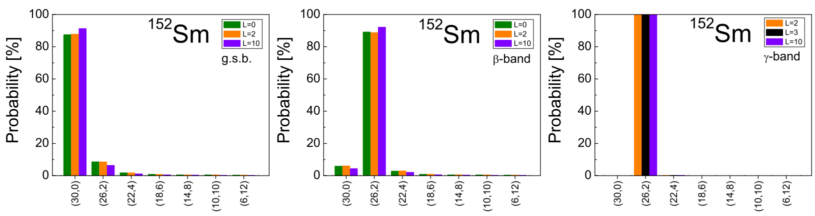

Figure 3. Calculated SU(3) probability distributions for the wave functions of the ground, β, and γ bands for three different angular momentum values.

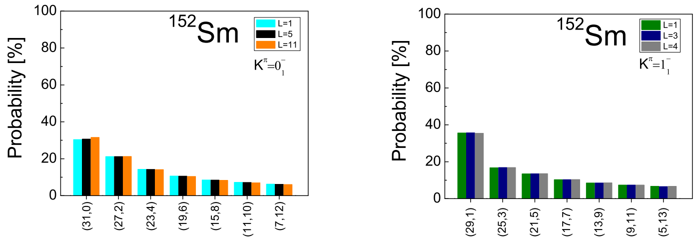

Figure 4. Calculated SU(3) probability distributions for the wave functions of the $K^\pi = 0_1^-$ and $K^\pi = 1_1^-$ bands for three different angular momentum values.

of the ground, β, γ and $K^\pi = 0_1^-$, $K^\pi = 1_1^-$ bands, respectively, for three different values of the angular momentum in each band. From the figures, one sees a highly coherent mixing in which the squared amplitudes are practically L-independent, at least for low angular momenta for which the Coriolis and centrifugal forces are not so strong. The figure thus indicates a new kind of symmetry, called quasi-dynamical symmetry [19]. This symmetry is associated with the mathematical concept of embedded representations [20]. Thus, the results for the microscopic structure of the states of positive- and negative-parity bands under consideration in ^{152}Sm reveal, in addition to the good U(6) symmetry, the presence of a good SU(3) quasi-dynamical symmetry, in the sense given in Refs. [19, 21]. For the γ band, one observes even a very good SU(3) dynamical symmetry.

3.2 Transition probabilities

It is known that the transition probabilities are more sensitive test for each model. With the wave functions obtained we calculate the intraband $B(E2)$ transition strengths between the collective states in the ground band and the lowest $K^\pi = 0_1^-$ band which are a measure of the quadrupole collectivity. The results are shown in Figure 5, compared with experiment [3] for the ground band.

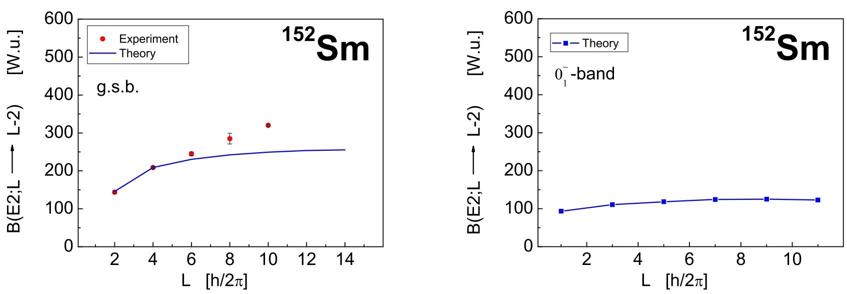

Figure 5. Calculated intraband $B(E2)$ values in Weisskopf units between the states of the ground band and the $K^\pi = 0_1^-$ band in ^{152}Sm. No effective charge is used.

From the figure we see that the $B(E2)$ intraband transitions in the $K^{\pi} = 0^{-}_{1}$ band show a reduced quadrupole collectivity. Unfortunately, there is no experimental data available for these $B(E2)$ transition strengths that would allow to establish the extent to which the quadrupole collectivity in the negative-parity states is captured by the used model Hamiltonian, and possibly, the missing of other important correlations in the structure of the wave functions. Further, in Figure 6 we show the comparison of the interband $B(E1)$ transition strengths between the states of the ground band and $K^{\pi} = 0^{-}_{1}$ band in ^{152}Sm. From the figure one sees a good reproduction of the experimental trend. We want to point out that in the calculations, for both the $B(E2)$ and $B(E1)$ transition strengths, no effective charge is used, i.e. $e = 1$. In this regard, recall that the $E1$ transition operator in sdf-IBM and spdf-IBM contains three adjustable parameters. Thus, the results obtained for the transition probabilities can be considered as a significant achievement of the present approach.

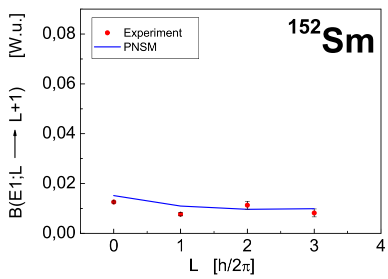

Figure 6. Comparison of the calculated $B(E1)$ values in Weisskopf units between the states of the ground band and $K^{\pi} = 0^{-}_{1}$ band in ^{152}Sm with experiment. No effective charge is used.

4 Conclusions

We remember that in contrast to the conventional shell model which divides the many-body Hilbert space of nuclear systems into different shells (i.e. horizontally), the symplectic models organize the many-particle configuration space vertically, dividing the full Hilbert space into a direct sum of different symplectic slices or vertical cones. Each symplectic slice represents an irreducible collective space for the microscopic collective model and is a small fraction of the full nuclear state space. Each Sp(12,R) subspace of the PNSM, preserving the parity and permutational symmetry, thus contains a set of collective rotational and, in contrast to the Sp(6,R) model, both the low- and high-energy vibrational states of a nucleus, the latter related to the giant resonance degrees of freedom. Such an organization allows to build up the required quadrupole collectivity without the use of an effective charge. It is known also that the symplectic groups admit two types of irreducible representations: even and odd ones which contain all the harmonic oscillator shell-model states with even and odd number of excitation quanta, respectively. This allows to treat the positive- and negative-parity collective bands on equal footing within the framework of the microscopic symplectic-based shell-model scheme simply by considering the even and odd irreducible representations of Sp(12,R), without the introduction of additional degrees of freedom inherent to other approaches to odd-parity nuclear states.

In the present work, the proton-neutron symplectic model with Sp(12,R) dynamical algebra, which naturally involves vertical as well as horizontal mixing of different SU(3) irreducible representations, is applied to the description of the microscopic structure of the low-lying positive- and negative-parity states of the first few bands in ^{152}Sm. In particular, the results for the ground, β and γ bands and the $K^\pi = 0^-_1$ and $K^\pi = 1^-_1$ negative-parity bands are presented. For obtaining the microscopic structure of the low-lying states under consideration, the model Hamiltonian is diagonalized in a U(6)-coupled basis, restricted to state space spanned by the fully symmetric U(6) irreps of the lowest even and odd irreducible representations of Sp(12,R). A good description of the energy levels of the rotational bands under consideration is obtained, except the $K^\pi = 1^-_1$ band which is obtained more compressed than in experiment. The microscopic structure of low-lying collective states with positive and negative parity in ^{152}Sm shows that practically there are no admixtures from the higher shells and hence the presence of a very good U(6) dynamical symmetry. Additionally, the structure of the collective states shows the presence of a good SU(3) quasi-dynamical symmetry. The intraband ground band $B(E2)$ and interband $B(E1)$ transition strengths between the states of ground and $K^\pi = 0^-_1$ bands are reproduced without the use of an effective charge, which can be considered as a significant achievement of the present symplectic-based proton-neutron shell-model approach.

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