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Viscous Dark Energy Cosmological Models in Brans-Dicke Theory of Gravitation

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Abstract. Bianchi type-I cosmological models in Brans-Dicke theory of gravitation has been investigated in terms of viscous dark energy. The exact solutions of the field equations are obtained by the assumptions of constant deceleration parameter and power law relation between scalar field φ and scale factor a. The physical properties of the models are also discussed in detail.

KEY WORDS: Bianchi type-I universe, Brans-Dicke theory, dark energy.

1 Introduction

Many observational data sets (Riess et al. [1]; Perlmutter et al. [2]) has confirmed the accelerating expansion of the Universe. According to the observations, an exotic energy dubbed as 'Dark Energy' (DE), is responsible for the accelerating expansion of the universe and it constitutes near about 73% of the total universe. DE has been conventionally characterized by the equation of state (EoS) parameter $\omega = p/\rho$ which is not necessarily constant. A variety of DE candidates are cosmological constant $(\omega = -1)$ (Carroll [3]), quintessence $(\omega > -1)$ (Caldwell et. al. [4]), phantom $(\omega < -1)$ (Caldwell et al. [5]), chaplygin gas (Kamenshchik et al. [6], Avelino et al. [7]) etc. From SN Ia data (Knop et al. [8]) and SN Ia data collaborated with CMBR anisotropy and galaxy clustering statistics (Tegmark et al. [9]), the other constraints are $-1.67 < \omega < -0.62$ and $-1.33 < \omega < -0.79$ respectively.

Scalar tensor theories such as Brans-Dicke theory [10], Barber's second self-creation theory [11], Saez-Ballester theory [12] etc. and modified theories of gravitation such as $f(R)$ (Nojiri and Odintsov [13], Caroll et al. [14]), $f(T)$ (Ferraro and Fiorini [15]), $f(R,T)$ (Harko et al. [16]) etc. are attracting more and more attention to explain the accelerating expansion of the universe and dark energy. Brans-Dicke theory is a generalization of general relativity which is more consistent with Mach's principle and involves a violation of the strong principle of equivalence, but the weak principle of equivalence is satisfied. A dynamical scalar field φ is introduced in Brans-Dicke theory which corresponds to the variation in gravitational constant with respect to cosmic time, i.e. $\varphi \approx G^{-1}$ and coupled to gravity with coupling parameter ϖ. Accelerated expansion of the Universe and accommodation of the observational data is very well explained by this theory in last decade (Bertolami and Martins [17], Kim [18], Clifton and Barrow [19].

The viscosity theory of relativistic fluids was first suggested by Eckart [20], Landau and Lifshitz [21]. The effect of viscosity on the evolution of the universe is studied by Misner [22, 23]. Padmanabhan and Chitre [24] explained the possibility of a viscosity dominated late epoch of the Universe with accelerated expansion of the universe. The role of viscous pressure as an agent have been extensively worked out by the relativists Balakin et al. [25] and Zimdahl et al. [26] that drives the present acceleration of the universe. Several physicsts [27–40] have studied the effect of bulk viscosity on the background expansion of universe. The validity of the generalized second law of thermodynamics is examined by Setare and Sheykhi [41] for a non-flat universe in the presence of viscous dark energy. Chen et al. [42] investigated the evolution of the viscous dark energy (VDE) interacting with the dark matter (DM) in the Einstein cosmology model. The general Bianchi type I with dark energy and modified Chaplygin gas with variable G and Λ and bulk viscosity have been investigated by Fayaz et al. [43]. Friedmann-Robertson-Walker model filled with barotropic fluid and bulk viscous stresses has been investigated by Amirhashchi et al. [44]. Phantom dark energy as an effect of bulk viscosity has been investigated by Velten et al. [45]. Raut et al. [46] have studied LRS Bianchi type II model filled with barotropic fluid and bulk viscous dark energy. Khurshudyan [47] discussed viscous holographic dark energy universe with Nojiri-Odintsov cut-off. Mostaghel et al. [48] considered bulk viscosity for dark fluid in a spatially flat two-component Universe. Amirhashchi [49] compared the dark energy equation of state (EoS) parameter for viscous and non-viscous dark energy in the scope of anisotropic Bianchi type V space-time. Singh and Shrivastava [50] studied a flat Friedmann–Robertson–Walker universe filled with dark matter and viscous new holographic dark energy. Mostaghel et al. [51] examine linear perturbation theory to evaluate the contribution of viscosity coefficient in the context of the bulk viscous dark energy model.

Motivated with the situation discussed above, in this paper, the behaviour of viscous dark energy for LRS Bianchi type-I cosmological models in BD theory of gravitation has been studied. This paper is organized as follows. In Section 2, the field equations of Brans-Dicke theory for the Bianchi type-I metric are obtained. Section 3 is devoted to the solution of the field equations, using the assumption of constant deceleration parameter. The physical and kinematical properties of the models are also discussed. Section 4 contains some concluding remarks.

2 The Metric and Field Equations

The LRS Bianchi type-I space-time is given by

$$ds^2 = dt^2 - A^2(t)dx^2 - B^2(t)(dy^2 + dz^2), \tag{1}$$

where A and B are functions of cosmic time t only. BD field equations for the combined scalar and tensor fields are given by

$$R_{ij} - \frac{1}{2}g_{ij}R - \bar{\omega}\varphi^{-2}\Big(\varphi_{,i}\varphi_{,j} - \frac{1}{2}g_{ij}\varphi^{,k}\varphi_{,k}\Big) - \varphi^{-1}(\varphi_{i;j} - g_{ij}\varphi^{;k}_{;k}) = \varphi^{-1}(T_{ij}), \tag{2}$$

where R is the Ricci scalar, R_{ij} is the Ricci tensor, φ is the Brans-Dicke scalar field, $\bar{\omega}$ is the dimensionless constant and T_{ij} is the energy momentum tensor. Here $8\pi G = c = 1$.

The scalar field satisfy the following equation:

$$\varphi^{;k}_{;k} = \frac{T_{ij}}{3+2\bar{\omega}}. \tag{3}$$

The field equations (2) and (3) for the model (1) can be written as

$$\frac{2\dot{A}\dot{B}}{AB} + \frac{\dot{B}^2}{B^2} - \frac{\bar{\omega}}{2}\frac{\dot{\varphi}^2}{\varphi^2} + \left(\frac{\dot{A}}{A} + \frac{2\dot{B}}{B}\right) = \frac{T_{00}}{\varphi}\ , \tag{4}$$

$$\frac{2\ddot{B}}{B} + \frac{\dot{B}^2}{B^2} + \frac{\bar{\omega}}{2}\frac{\dot{\varphi}^2}{\varphi^2} + \frac{2\dot{B}\dot{\varphi}}{B\varphi} + \frac{\ddot{\varphi}}{\varphi} = -\frac{T_{11}}{\varphi}\ , \tag{5}$$

$$\frac{\ddot{A}}{A} + \frac{\ddot{B}}{B} + \frac{\dot{A}\dot{B}}{AB} + \frac{\bar{\omega}}{2}\frac{\dot{\varphi}^2}{\varphi^2} + \left(\frac{\dot{A}}{A} + \frac{\dot{B}}{B}\right)\frac{\dot{\varphi}}{\varphi} + \frac{\ddot{\varphi}}{\varphi} = -\frac{T_{22}}{\varphi}\ , \tag{6}$$

and the wave equation is

$$\ddot{\varphi} + \left(\frac{\dot{A}}{A} + \frac{2\dot{B}}{B}\right)\dot{\varphi} = \frac{T}{3+2\bar{\omega}}\ , \tag{7}$$

where an overhead dot denotes differentiation with respect to t.

The energy momentum tensor

$$T^i_j = T^{(m)i}_j + T^{(X)i}_j\ , \tag{8}$$

where $T^{(m)i}_j$ and $T^{(X)i}_j$ are energy momentum tensors of perfect fluid and viscous dark energy respectively. These are given by

$$T^{(m)i}_j = \text{diag}\,[-\rho^m, p^m, p^m, p^m] = \text{diag}\,[-1, \omega^m, \omega^m, \omega^m]\,\rho^m \tag{9}$$

$$T^{(X)i}_j = \text{diag}\,\left[-\rho^X, p^X, p^X, p^X\right] = \text{diag}\,\left[-1, \omega^X, \omega^X, \omega^X\right]\rho^X, \tag{10}$$

where ρ^m and p^m are energy density and pressure of perfect fluid component while $\omega^m = p^m/\rho^m$ is its EoS parameters. Similarly, ρ^X and p^X are respectively the energy density and pressure of the viscous DE component while $\omega^X = p^X/\rho^X$ is the corresponding EoS parameters. In this paper, the expression for the pressure of the viscous fluid (Eckart [20]) is taken as

$$p^X_{\text{eff}} = p^X + \Pi\,, \tag{11}$$

where $\Pi = -\xi(\rho^X)u^i_{;i}$ is viscous pressure and $H = u^i_{;i}/3$ is the Hubble parameter. Here $u^i = (1,0,0,0)$ is the four velocity vector satisfying $u^i u_j = -1$. On thermodynamical ground, in conventional physics ξ has to be positive. This is a consequence of the positive sign of the change in entropy as an irreversible process (Landau and Lifshitz [21]). In general, $\xi(\rho^X) = \xi_0(\rho^X)^\tau$, where $\xi_0 > 0$ and τ are constant parameter.

Using equations (9),(10),(11), the field equations (4) to (7) for viscous dark energy can be written as

$$(2m+1)\frac{\dot{B}^2}{B^2} - \frac{\bar{\omega}}{2}\frac{\dot{\varphi}^2}{\varphi^2} + (m+2)\frac{\dot{B}\dot{\varphi}}{B\varphi} = \frac{\rho^m + \rho^X}{\varphi}\,, \tag{12}$$

$$\frac{2\ddot{B}}{B} + \frac{\dot{B}^2}{B^2} + \frac{\bar{\omega}}{2}\frac{\dot{\varphi}^2}{\varphi^2} + 2\frac{\dot{B}\dot{\varphi}}{B\varphi} + \frac{\ddot{\varphi}}{\varphi} = \frac{1}{\varphi}[-\omega^m\rho^m - \omega^X_{\text{eff}}\rho^X + \Pi]\,, \tag{13}$$

$$(m+)\frac{\ddot{B}}{B} + m^2\frac{\dot{B}^2}{B^2} + \frac{\bar{\omega}}{2}\frac{\dot{\varphi}^2}{\varphi^2} + (m+1)\frac{\dot{B}\dot{\varphi}}{B\varphi} + \frac{\ddot{\varphi}}{\varphi} = \frac{1}{\varphi}[-\omega^m\rho^m - \omega^X_{\text{eff}}\rho^X + \Pi], \tag{14}$$

and the wave equation is

$$\ddot{\varphi} + (m+2)\frac{\dot{B}}{B}\dot{\varphi} = \frac{\rho^m(1-3\omega^m) + \rho^X(1-3\omega^X_{\text{eff}}) + 3\Pi}{3+2\bar{\omega}}\,, \tag{15}$$

where

$$A = B^m\,. \tag{16}$$

Throne [52] elaborated the motive behind assuming equation (16), the observations of the velocity red-shift relation for extragalatic sources suggest that Hubble expansion of the universe is isotropy today within $\approx 30\%$ (Kantowski and Sachs [53]; Kristian and Sachs [54]). For spatially homogeneous metric, the normal congruence to the homogeneous expansion satisfies that the condition σ/θ is constant have been pointed out by Collin et al. [55]. Most precisely, red-shift studies place the limit $\sigma/H \leq 0.3$ on the ratio of shear σ to Hubble constant H in the neighbourhood of our galaxy today.

The energy conservation equation $T^{ij}_{;j} = 0$, leads to

$$\dot{\rho}^m + 3(1+\omega^m)\rho^m H + \dot{\rho}^X + 3(1+\omega^X)\rho^X H = 0\,, \tag{17}$$

where H is the Hubble parameter.

Let us assume that two-fluid do not interact with each other. Therefore, the general form of conservation equation (17), leads to

$$\dot{\rho}^m + 3(1+\omega^m)\rho^m H = 0\,, \tag{18}$$

$$\dot{\rho}^X + 3(1+\omega^X)\rho^X H = 0\,. \tag{19}$$

The spatial volume for LRS Bianchi type-I metric is given by

$$V^3 = AB^2\,. \tag{20}$$

The average scale factor a of LRS Bianchi type-I metric is defined as

$$a = (AB^2)^{1/3}\,. \tag{21}$$

The directional Hubble parameter in the direction of x, y and z axes respectively, are

$$H_x = \frac{\dot{A}}{A}\,, \qquad H_y = H_z = \frac{\dot{B}}{B}\,. \tag{22}$$

The mean Hubble parameter H is given by

$$H = \frac{\dot{a}}{a} = \frac{1}{3}\left(\frac{\dot{A}}{A} + \frac{2\dot{B}}{B}\right). \tag{23}$$

The physical quantities, i.e. the scalar expansion (θ), mean anisotropy parameter (A_m), shear scalar (σ) and deceleration parameter (q) are expressed as

$$\theta = 3H\,, \tag{24}$$

$$A_m = \frac{1}{3}\sum_{i=1}^{3}\left(\frac{\Delta H_i}{H}\right)^2, \tag{25}$$

$$\sigma^2 = \frac{1}{2}\left(\sum_{i=1}^{3} H_i^2 - 3H^2\right) = \frac{3}{2}A_m H^2\,, \tag{26}$$

$$q = -\frac{a\,\ddot{a}}{\dot{a}^2}\,, \tag{27}$$

where $\Delta H_i = H_i - H \quad (i = 1, 2, 3)$.

3 Solutions of Field Equations

To solve the field equations, consider the power law relation given by

$$\varphi = ba^\alpha\,, \tag{28}$$

where b and α are constants. Shamir and Bhatti [56] used equation (28)to obtain solution of anisotropic dark energy Bianchi type III cosmological model in Brans Dicke theory of gravity.

To get the deterministic solutions , the variation law for generalised Hubble's parameter is applied which yields a constant value of deceleration parameter. Let us consider that the mean Hubble parameter H is related to the average scale factor a by following relation:

$$H = Da^{-n}\,, \tag{29}$$

where $D\ (>0)$ and $n\ (\geq 0)$ are constants.

The above relation had been assumed by Berman [57], Berman and Gomide [58] for solving FRW cosmological models and showed that the constant deceleration parameter models stand adequately for our present view of different phases of the evolution of Universe. Several relativists (Singh et al. [59], Ram et al. [60–62]) have used this assumption for solving Einstein's field equations different contexts.

Using Eqs. (23) and (29), it yields

$$\dot{a} = Da^{-n+1}\,, \tag{30}$$

$$\ddot{a} = -D^2(n-1)a^{-2n+1}\,. \tag{31}$$

Using Eqs. (27), (30) and (31), we obtain

$$q = n - 1\,, \tag{32}$$

which is constant. The deceleration parameter gives a measure of the rate at which the expansion of the universe is taking place. For $n > 1$, we have the positive sign of q which corresponds to standard decelerating model whereas for $0 < n < 1$, we have the negative sign of q which indicates inflation. For the value of $n = 1$, i.e. $q = 0$ corresponds to the expansion of the universe at a constant rate. The current observations of SN Ia (Riess et al. [1]; Perlmutter et al. [2]) and CMBR favour accelerating models, i.e. $q < 0$ and the values of the deceleration parameter lies somewhere in the range $-1 < q < 0$.

Integrating Eq. (30), the law of average scale factor 'a' is obtained as

$$a = (kt + a_0)^{1/n}\,, \qquad \text{when} \ \ n \neq 0\,, \tag{33}$$

$$a = c_1 e^{Dt}\,, \qquad \text{when} \ \ n = 0\,, \tag{34}$$

where $k = nD$ and a_0, c_1 are constant of integration.

From equation (34), for $n \neq 0$, it is clear that the condition for expansion of universe is $n > 0$, i.e. $q + 1 > 0$. Therefore for expansion models of universe the deceleration parameter (q) should be greater than -1 [63].

In the following Sections 3.1 and 3.2, two types of models are considered, i.e. power law model and exponential expansion model.

3.1 Power law model, i.e. when $n \neq 0$

Using equations (28) and (33), the expression for the BD scalar field is obtained as

$$\varphi = b(kt + a_0)^{\alpha/n} . \tag{35}$$

Using Eqs. (21) and (33), the scale factors are obtained as

$$B = (kt + a_0)^{\frac{1}{7}mr} \tag{36}$$

and

$$A = (kt + a_0)^{\frac{1}{7}r} , \tag{37}$$

where $r = \dfrac{n(m+2)}{3m}$.

The metric potentials A and B vanish at point $t = -a_0/k$.

The physical parameters such as Hubble parameter (H), spatial volume (V), anisotropic parameter (A_m), shear scalar (σ) and expansion scalar (θ) of model (1) are respectively given by

$$H = \frac{k}{n(kt + a_0)} , \tag{38}$$

$$V = b(kt + a_0)^{3/n} , \tag{39}$$

$$A_m = \frac{n^2(m^2+2)}{3m^2r^2} - \frac{2n(m+2)}{3mr} + 1 , \tag{40}$$

$$\sigma^2 = \frac{k^2}{(kt + a_0)^2} \left\{ \frac{(m^2+2)}{2m^2r^2} - \frac{(m+2)}{3mnr} + \frac{3}{2n^2} \right\} , \tag{41}$$

$$\theta = \frac{3k}{n(kt + a_0)} . \tag{42}$$

The Hubble parameter H diverges at $t = -a_0/k$ but the spatial volume of the model vanishes here. Thus, it is concluded from these observations that, the model starts its expansion with zero volume at $t = -a_0/k$ and it continues to expand for $0 < n < 1$. Thus, there is a point type singularity in the model at $t = -a_0/k$ (MacCallum [64]). Since σ/θ = constant, the model does not approach isotropy. It can also be seen that, since $A_m \neq 0$, the model remains anisotropic throughout the evolution of the universe.

Using equations (18) and (33), the energy density of matter yields

$$\rho^m = \rho_0(kt + a_0)^{-3(1+\omega^m)/n} , \tag{43}$$

where ρ_0 is an integrating constant.

It is interesting to note that the energy density of matter is the decreasing function of time.

Using Eqs. (12), (32), (43), the DE density is given by

$$\rho^X = \left\{ \frac{2m+1}{m^2r^2} + \frac{\alpha(m+2)}{nmr} - \frac{\bar{\omega}\alpha^2}{2n^2} \right\} k^2 b(kt+a_0)^{\alpha/n-2} - \rho_0 (kt+a_0)^{-3(1+\omega^m)/n} . \quad (44)$$

It is observed that the energy densities of matter and VDE are decreasing function of time.

The pressure and EoS parameter of viscous DE are given by

$$p^X_{\text{eff}} = -\left\{ \frac{3}{m^2r^2} + \frac{2(\alpha-n)}{nmr} + \frac{\bar{\omega}\alpha^2}{2n^2} + \frac{2(\alpha-n)}{n^2} \right\} k^2 b(kt+a_0)^{\alpha/n-2} - \rho_0 \omega^m (kt+a_0)^{-3(1+\omega^m)/n} + \Pi , \quad (45)$$

A negative pressure simulates a repulsive gravity and inflates the universe by overwhelming the usual gravitational effects of matter.

$$\omega^X_{\text{eff}} = \left[-\left\{ \frac{3}{m^2r^2} + \frac{2(\alpha-n)}{nmr} + \frac{\bar{\omega}\alpha^2}{2n^2} + \frac{2(\alpha-n)}{n^2} \right\} k^2 b(kt+a_0)^{\alpha/n-2} - \rho_0 \omega^m (kt+a_0)^{-3(1+\omega^m)/n} + \Pi \right] \left[\left\{ \frac{2m+1}{m^2r^2} + \frac{\alpha(m+2)}{nmr} - \frac{\bar{\omega}\alpha^2}{2n^2} \right\} \times k^2 b(kt+a_0)^{\alpha/n-2} - \rho_0 (kt+a_0)^{-3(1+\omega^m)/n} \right]^{-1} , \quad (46)$$

where

$$\Pi = \frac{-3\xi_0 k}{n(kt+a_0)} \left[\left\{ \frac{2m+1}{m^2r^2} + \frac{\alpha(m+2)}{nmr} - \frac{\bar{\omega}\alpha^2}{2n^2} \right\} k^2 b(kt+a_0)^{\alpha/n-2} - \rho_0 (kt+a_0)^{-3(1+\omega^m)/n} \right]^{\tau} . \quad (47)$$

The behaviour of EoS for DE in terms of cosmic time t is shown in Figure 1. Here the parameter ω^m is taken to be zero. In Figure 1, ξ_0 varies as 0, 0.2 and 0.3, respectively. The EoS of non-viscous DE ($\xi_0 = 0$) is only varying in quintessence region whereas the EoS parameter of viscous DE start with phantom region, crossing PDL and varies in quintessence region. The EoS of both non-viscous and viscous DE approaches to cosmological constant region. This behaviour clearly shows that the phantom phase, i.e. $\omega^X_{\text{eff}} < -1$ is an unstable phase and there is a transition from phantom to the cosmological constant phase at late time. The SN Ia data suggests that $-1.67 < \omega^X_{\text{eff}} < -0.62$ (Knop et

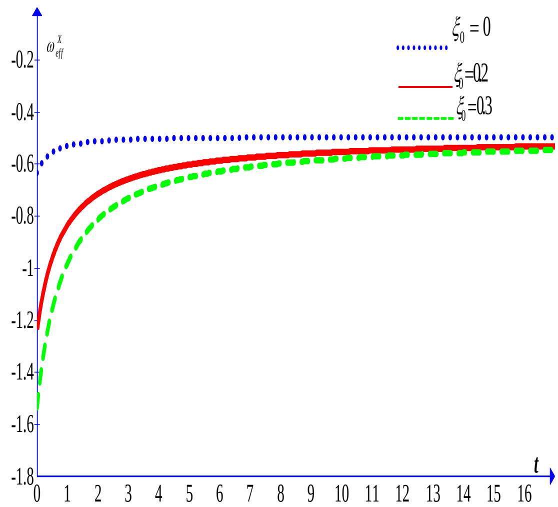

Figure 1. (Color online) The plot of EoS parameter ω^X_{eff} versus time for $\omega^m = 0$, $m = n = k = a_0 = 1$, $\bar{\omega} = 2$, $\rho_0 = 1$, $\alpha = 0.7$.

al. [8]) while the limit imposed on ω^X_{eff} by a combination of SN Ia data (with CMB anisotropy) and galaxy clustering statistics is $-1.33 < \omega^X_{\text{eff}} < -0.79$ (Tegmark [9]). Figure 1 clearly shows that EoS parameter evolves within a range, which is in nice agreement with SNIa and CMB observations.

The matter density parameter $\Omega^{(m)}$ and DE density parameter $\Omega^{(X)}$ are given by

$$\Omega^{(m)} = \frac{\rho^m}{3H^2} = \frac{\rho_0 n^2}{3k^2}(kt + a_0)^{-3(1+\omega^m)/n+2}\,, \tag{48}$$

$$\Omega^{(X)} = \frac{\rho^X}{3H^2} = \left[\frac{(2m+1)}{m^2r^2} + \frac{\alpha(m+2)}{nmr} - \frac{\alpha^2\bar{\omega}}{2n^2}\right]\frac{n^2b}{3}(kt + a_0)^{\alpha/n} - \frac{\rho_0 n^2}{3k^2}(kt + a_0)^{-3(1+\omega^m)/n+2}\,. \tag{49}$$

The overall density parameter (Ω) is given by

$$\Omega = \left[\frac{(2m+1)}{m^2r^2} + \frac{\alpha\,(m+2)}{nmr} - \frac{\alpha^2\bar{\omega}}{2n^2}\right]\frac{n^2b}{3}\,(kt + a_0)^{\alpha/n}\,. \tag{50}$$

From equation (50), it is observed that the overall density parameter (Ω) approaches to 1 at late times. Thus, the derived model predicts a flat Universe at the present epoch. Figure 2 depicts the variation of density parameters versus cosmic time during the evolution of universe. It is observed that initially universe was matter dominated and later on DE dominates the evolution of uni-

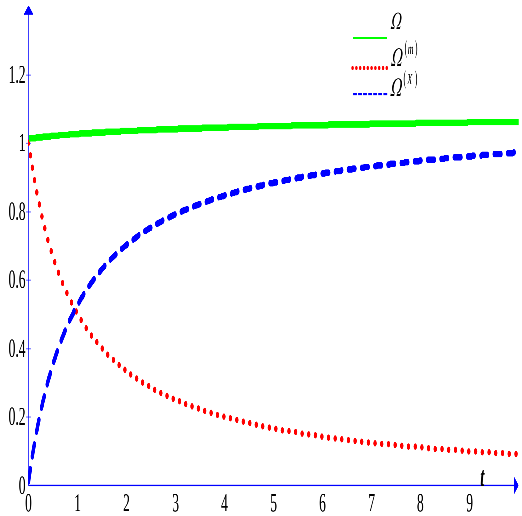

Figure 2. (Color online) The plot of density parameter versus time for $\omega^m = 0$, $m = n = k = a_0 = 1$, $\bar{\omega} = 100$, $\rho_0 = 3$, $\alpha = 0.02$.

verse which is probably responsible for the accelerated expansion of present-day universe.

3.2 Exponential expansion model, i.e. when $n = 0$

Using Eqs.(26) and (33) , the expression for the BD scalar field is obtained as

$$\varphi = c_1 e^{Dnt} . \tag{51}$$

Using Eqs. (19) and (33), the scale factors are given by

$$B = l_0 e^{l_1 t} , \tag{52}$$

where $l_0 = c_1^{3/(m+2)}$ and $l_1 = 3D/(m+2)$, and

$$A = l_2 e^{l_3 t} . \tag{53}$$

The physical parameters such as Hubble parameter (H), spatial volume (V), anisotropic parameter (A_m), shear scalar (σ) and expansion scalar (θ) of model

(26) are respectively given by

$$H = \frac{(m+2)l_1}{3}\,, \tag{54}$$
$$V = c_1 e^{3Dt}\,, \tag{55}$$
$$A_m = 2\left(\frac{m-1}{m+2}\right)^2\,, \tag{56}$$
$$\sigma^2 = \frac{l_1^2(m-1)^2}{3}\,, \tag{57}$$
$$\theta = (m+2)l_1\,. \tag{58}$$

The spatial volume of the model increases exponentially with time which indicates that the universe starts its expansion with zero volume from infinite past. As $t \to \infty$, the scale factors and volume of the universe become infinitely large. Since $\sigma/\theta =$ constant, the model does not approach isotropy. Uniform exponential expansion takes place as the mean Hubble parameter and expansion scalar are constants throughout the evolution. We observe that the model has no initial singularity. For $n = 0$, we get $q = -1$ incidentally this value of deceleration parameter leads to $\frac{dH}{dt} = 0$, which implies the greatest value of Hubble's parameter and the fastest rate of expansion of the universe. Since, the average anisotropy parameter is uniform; the model remains anisotropic throughout the evolution of the universe.

Using equations (18) and (34), the energy density of matter is as follows:

$$\rho^m = \rho_0 (c_1 e^{Dt})^{-3(1+\omega^m)}\,, \tag{59}$$

where ρ_0 is an integrating constant.

Using Eqs. (12), (51) and (58), the DE density is given by

$$\rho^X = \left\{(2m+1)l_1^2 + l_1 D\alpha(m+2) - \frac{\bar{\omega}D^2\alpha^2}{2}\right\} b c_1^{\alpha} e^{D\alpha t} - \rho_0 (c_1 e^{Dt})^{-3(1+\omega^m)}\,. \tag{60}$$

From Eqs. (58) and (59), it is observed that the energy densities of matter and VDE are decreasing function of time.

The pressure and EoS parameter of DE are given by

$$p^X_{\text{eff}} = -\left\{3l_1^2 + 2l_1 D\alpha + \left(1 + \frac{\bar{\omega}}{2}\right) D^2\alpha^2\right\} b c_1^{\alpha} e^{D\alpha t} - \rho_0 \omega^m (c_1 e^{Dt})^{-3(1+\omega^m)} + \Pi\,, \tag{61}$$

$$\omega_{\text{eff}}^{X} = -\Bigg\{\Big[3l_1^2 + 2l_1 D\alpha + \Big(1+\frac{\bar{\omega}}{2}\Big)D^2\alpha^2\Big]bc_1^{\alpha}e^{D\alpha t} - \rho_0\omega^m(c_1e^{Dt})^{-3(1+\omega^m)} + \Pi\Bigg\}\Bigg\{\Big[(2m+1)l_1^2 + l_1 D\alpha(m+2) - \frac{\bar{\omega}D^2\alpha^2}{2}\Big]bc_1^{\alpha}e^{D\alpha t} - \rho_0(c_1e^{Dt})^{-3(1+\omega^m)}\Bigg\}^{-1}, \quad (62)$$

where

$$\Pi = -3\xi_0 l_1(m+2)\Bigg\{\Big[(2m+1)l_1^2 + l_1 D\alpha(m+2) - \frac{\bar{\omega}D^2\alpha^2}{2}\Big]bc_1^{\alpha}e^{D\alpha t} - \rho_0(c_1e^{Dt})^{-3(1+\omega^m)}\Bigg\}^{\tau}. \quad (63)$$

The behaviour of EoS for DE in terms of cosmic time t is shown in Figure 3. Here the parameter ω^m is taken to be zero. In Figure 3, ξ_0 varies as 0, 0.2 and 0.3, respectively. The EoS of non-viscous DE ($\xi_0 = 0$) starts with phantom region and $\omega_{\text{eff}}^{X} \approx -1$ for sufficiently large time represent the vacuum fluid dominated universe, which is mathematically equivalent to cosmological constant

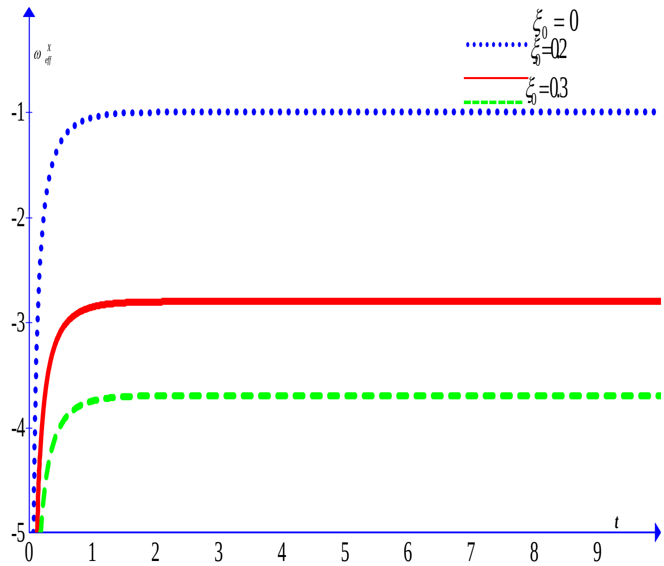

Figure 3. (Color online) The plot of EoS parameter ω_{eff}^{X} versus time for $\omega^m = 0$, $l_1 = c_1 = D = 1$, $\bar{\omega} = 2$, $b = 1$, $\rho_0 = 3$, $\alpha = 0.05$.

whereas the EoS parameter of viscous DE start with phantom region and varies in phantom region.

The matter density parameter $\Omega^{(m)}$ and DE density parameter $\Omega^{(X)}$ are given by

$$\Omega^{(m)} = \frac{\rho^m}{3H^2} = \frac{3\rho_0}{l^2(m+2)^2}(c_1 e^{Dt})^{-3(1+\omega^m)} \tag{64}$$

and

$$\Omega^{(X)} = \frac{\rho^X}{3H^2} = \left\{\left[(2m+1)l_1^2 + l_1 D\alpha(m+2) - \frac{\bar{\omega}\alpha^2 D^2}{2}\right]3bc_1^\alpha e^{D\alpha t} - 3\rho_0(c_1 e^{Dt})^{-3(1+\omega^m)}\right\}\left\{l_1^2(m+2)^2\right\}^{-1}. \tag{65}$$

The overall density parameter Ω is given by

$$\Omega = \left[(2m+1)l_1^2 + l_1 D\alpha(m+2) - \frac{\bar{\omega}\alpha^2 D^2}{2}\right]\frac{3bc_1^\alpha e^{D\alpha t}}{l_1^2(m+2)^2}. \tag{66}$$

From Figure 4, it is observed that for sufficiently large time, the overall density parameter (Ω) approaches to 1. Thus the derived model predicts a flat universe at late time.

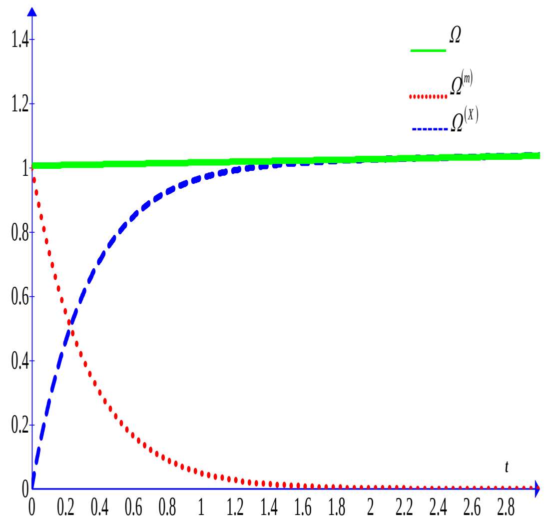

Figure 4. (Color online) The plot of density parameter versus time for $\omega^m = 0$, $m = D = l_1 = 1$, $\bar{\omega} = 100$, $\rho_0 = 3$, $\alpha = 0.01$.

4 Conclusions

In this paper, the LRS Bianchi type-I universe field with viscous dark energy in Brans-Dicke theory of gravitation has been discussed. The power law relation between φ and a is used to find the solutions. The assumption of constant deceleration parameter leads two models, i.e. power law model and exponential model.

For power law model, i.e. when $n \neq 0$, the average scale factor $a = (kt + a_0)^{1/n}$. It has a point type singularity at $t = -a_0/k$. The scale factor A and B vanishes at that point of singularity. The Hubble parameter $H$is infinite but the spatial volume vanishes at $t = -a_0/k$. EoS of non-viscous DE is only varying in quintessence region whereas the EoS parameter of viscous DE start with phantom region, crossing PDL, and varies in quintessence region. The EoS parameter of DE evolves within the range predicted by the observations.

For exponential model of the universe, i.e. when $n = 0$, the average scale factor $a = c_1 e^{Dt}$. It is non-singular. The scale factor A and B are not vanishes for this model. The Hubble parameter H is constant while the spatial volume of this model increases exponentially with time which indicates that the universe starts its expansion with zero volume from infinite past. The EoS of non-viscous DE ($\xi_0 = 0$) starts with phantom region and $\omega_{\text{eff}}^X \approx -1$ for sufficiently large time represent the vacuum fluid dominated universe, which is mathematically equivalent to cosmological constant whereas the EoS parameter of viscous DE start with phantom region and varies in phantom region.

Also it is seen that in both the models $\sigma/\theta =$ constant therefore the proposed models do not approach to isotropy at any late time. From Figures 2, 4 it is observed that the overall density parameter (Ω) approaches to 1 at late times. Thus, the derived model predicts a flat universe at the present epoch.

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