

Four-Dimensional Gauss-Bonnet Gravity Without Gauss-Bonnet Coupling to Matter – Spherically Symmetric Solutions, Domain Walls and Spacetime Singularities

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Abstract. We discuss a new extended gravity model in ordinary $D = 4$ space-time dimensions, where an additional term in the action involving Gauss-Bonnet topological density is included without the need to couple it to matter fields unlike the case of ordinary $D = 4$ Gauss-Bonnet gravity models. Avoiding the Gauss-Bonnet density becoming a total derivative is achieved by employing the formalism of metric-independent non-Riemannian spacetime volume-forms. The non-Riemannian volume element triggers *dynamically* the Gauss-Bonnet scalar to become an arbitrary integration constant on-shell. We describe in some detail the class of static spherically symmetric solutions of the above modified $D = 4$ Gauss-Bonnet gravity including solutions with deformed (anti-)de Sitter geometries, black holes, domain walls and Kantowski-Sachs-type universes. Some solutions exhibit physical spacetime singular surfaces *not hidden* behind horizons and bordering whole forbidden regions of space. Singularities can be avoided by pairwise matching of two solutions along appropriate domain walls. For a broad class of solutions the corresponding matter source is shown to be a special form of nonlinear electrodynamics whose Lagrangian $L(F^2)$ is a *non-analytic* function of F^2 (the square of Maxwell tensor $F_{\mu\nu}$), *i.e.*, $L(F^2)$ is *not* of Born-Infeld type.

KEY WORDS: Non-Riemannian spacetime volume forms, Gauss-Bonnet modified gravity, non-linear electrodynamics, spherically symmetric solutions, spacetime singularities, forbidden spacetime regions, (anti-)de Sitter geometries, Kantowski-Sachs universes, non-standard black holes, exotic matter domain walls.

1 Introduction

In the last decade or so a host of problems of primary importance in cosmology (problems of dark energy and dark matter), quantum field theory in curved spacetime (renormalization in higher loops) and string theory (low-energy effective field theories) motivated a very active development of extended gravity theories as alternatives/generalizations of the standard Einstein General Relativity (for detailed accounts, see Refs. [1–4] and references therein).

One possible approach towards alternative/extended theories to General Relativity is to employ the formalism of non-Riemannian spacetime volume-forms (alternative metric-independent generally covariant volume elements or spacetime integration measure densities) in the pertinent Lagrangian actions, defined in terms of auxiliary antisymmetric tensor gauge fields of maximal rank, instead of the canonical Riemannian volume element given by the square-root of the determinant of the Riemannian metric. The systematic geometrical formulation of the non-Riemannian volume-form approach was given in Refs. [5, 6], which is an extension of the originally proposed method [7, 8].

This formalism is the basis for constructing a series of extended gravity-matter models describing unified dark energy and dark matter scenario [9], quintessential cosmological models with gravity-assisted and inflaton-assisted dynamical generation or suppression of electroweak spontaneous symmetry breaking and charge confinement [10–12], and a novel mechanism for the supersymmetric Brout-Englert-Higgs effect in supergravity [5].

Let us recall that in standard generally-covariant theories (with actions of the form $S = \int d^D x \sqrt{-g} \mathcal{L}$) the standard Riemannian spacetime volume-form ω is defined through the “D-bein” (frame-bundle) canonical one-forms $e^A = e_\mu^A dx^\mu$ ($A = 0, \dots, D-1$), related to the Riemannian metric ($g_{\mu\nu} = e_\mu^A e_\nu^B \eta_{AB}$, $\eta_{AB} \equiv \text{diag}(-1, 1, \dots, 1)$):

$$\omega = e^0 \wedge \dots \wedge e^{D-1} = \det \|e_\mu^A\| dx^{\mu_1} \wedge \dots \wedge dx^{\mu_D} ,$$

so that the Riemannian volume element reads:

$$\Omega \equiv \det \|e_\mu^A\| d^D x = \sqrt{-\det \|g_{\mu\nu}\|} d^D x . \quad (1)$$

Instead of $\sqrt{-g}$ we will employ below a different alternative *non-Riemannian* volume element given by a non-singular *exact D-form* $\omega = d\mathcal{C}$ where:

$$\mathcal{C} = \frac{1}{(D-1)!} C_{\mu_1 \dots \mu_{D-1}} dx^{\mu_1} \wedge \dots \wedge dx^{\mu_{D-1}} , \quad (2)$$

so that the *non-Riemannian* volume element becomes:

$$\Omega \equiv \Phi(C) d^D x = \frac{1}{(D-1)!} \varepsilon^{\mu_1 \dots \mu_D} \partial_{\mu_1} C_{\mu_2 \dots \mu_D} d^D x . \quad (3)$$

Four-Dimensional Gauss-Bonnet Gravity

Here $C_{\mu_1 \dots \mu_{D-1}}$ is an auxiliary rank $(D - 1)$ antisymmetric tensor gauge field. $\Phi(C)$ is in fact the density of the dual of the rank D field-strength $F_{\mu_1 \dots \mu_D} = \frac{1}{(D-1)!} \partial_{[\mu_1} C_{\mu_2 \dots \mu_D]} = -\varepsilon_{\mu_1 \dots \mu_D} \Phi(C)$. Like $\sqrt{-g}$, $\Phi(C)$ similarly transforms as scalar density under general coordinate reparametrizations.

Now, we observe that if we replace the usual Riemannian volume element density $\sqrt{-g}$ with a non-Riemannian one $\Phi(C)$ (3) in the Lagrangian action integral over the 4-dimensional Gauss-Bonnet scalar $\int d^4x \Phi(C) R_{\text{GB}}^2$ (cf. Eqs. (4)–(5) below), then the latter will cease to be a total derivative in $D = 4$. In this way we will avoid the necessity to couple R_{GB}^2 in $D = 4$ directly to matter fields or to use nonlinear functions of R_{GB}^2 unlike the usual $D = 4$ Einstein-Gauss-Bonnet gravity. For reviews of the latter, see Refs. [13, 14]; for recent discussions of Gauss-Bonnet cosmology, see Refs. [15]– [24], and references therein.

Our non-standard $D = 4$ Gauss-Bonnet gravity with a Gauss-Bonnet action term $\int d^4x \Phi(C) R_{\text{GB}}^2$ has the following principal properties:

- The equation of motion w.r.t. auxiliary tensor gauge field $C_{\mu_1 \dots \mu_{D-1}}$ defining $\Phi(C)$ (3) dynamically triggers the Gauss-Bonnet scalar R_{GB}^2 to be on-shell an arbitrary integration constant (Eq. (15) below).
- Now the composite field $\chi = \frac{\Phi(C)}{\sqrt{-g}}$ appears as an additional physical field degree of freedom related to the geometry of spacetime. Let us note that the latter is in sharp contrast w.r.t. other extended gravity-matter models constructed in terms of (one or several) non-Riemannian volume-forms [5, 6, 9–12], where we start within the first-order (Palatini) formalism and where composite fields of the type of χ (ratios of non-Riemannian to Riemannian volume element densities) turn out to be (almost) pure gauge (non-propagating) degrees of freedom, the only remnants being the appearance of some further free integration constants.

The dynamically triggered constancy of R_{GB}^2 in our modified $D = 4$ Gauss-Bonnet gravity has several interesting implications for cosmology [25], in particular, the additional degree of freedom χ absorbing completely the effect of the matter dynamics within the Friedmann-Lemaître-Robertson-Walker formalism. The above properties are the most significant differences of the present approach w.r.t. the approach in several recent papers [26–28], which extensively study static spherically symmetric solutions in gravitational theories in the presence of a constant Gauss-Bonnet scalar. In the latter papers the constancy of the Gauss-Bonnet scalar is *imposed as an additional condition on-shell* beyond the standard equations of motion resulting from an action principle. Therefore, the full set of equations (equations of motion plus the *ad hoc* imposed constancy of R_{GB}^2) in the latter papers is *not equivalent* to the full set of equations of motion in the present modified $D = 4$ Gauss-Bonnet gravity based on the non-Riemannian spacetime volume-form formalism.

The plan of the paper is as follows. After presenting in Section 2 the basics of the non-Riemannian volume-form formulation of modified $D = 4$ Gauss-Bonnet gravity, in Section 3 we describe the general properties of the whole class of static spherically symmetric solutions for the various values of the pertinent free integration constants.

In Section 4 we analyze in some detail the domains of definition of the static spherically symmetric metrics and the locations of physical spacetime singularities, domain walls and horizons. The spacetime singularities of the modified $D = 4$ Gauss-Bonnet gravity are constant $r = r_*$ surfaces bordering whole forbidden space regions of finite or infinitely large extent where the metric becomes complex. Most of these spacetime singularities are *not hidden* behind horizons. They resemble the so called branch singularities at finite r of static spherically symmetric solutions in higher-dimensional ($D \geq 5$) Einstein-Maxwell-Gauss-Bonnet gravity [29] where the higher-dimensional quadratic curvature invariants exhibit the same singular behaviour near r_* as in the present case (cf. Eq. (42) below).

Section 5 contains the graphical representations of the whole class of static spherically symmetric solutions. In Section 6 we briefly illustrate how to avoid spacetime singularities via pairwise matching of two solutions along appropriate domain wall. In the last discussion Section 7 we add some comments and conclusions.

2 Gauss-Bonnet Gravity in $D = 4$ with a Non-Riemannian Volume Element

We propose the following self-consistent action of $D = 4$ Gauss-Bonnet gravity without the need to couple the Gauss-Bonnet scalar to some matter fields (for simplicity we are using units with the Newton constant $G_N = 1/16\pi$):

$$S = \int d^4x \sqrt{-g} [R + L_{\text{matter}}] + \int d^4x \Phi(C) R_{\text{GB}}^2 . \quad (4)$$

Here the notations used are as follows:

- R_{GB}^2 denotes the Gauss-Bonnet scalar:

$$R_{\text{GB}}^2 \equiv R^2 - 4R_{\mu\nu}R^{\mu\nu} + R_{\mu\nu\kappa\lambda}R^{\mu\nu\kappa\lambda} . \quad (5)$$

- $\Phi(C)$ denotes a non-Riemannian volume element density defined as a scalar density of the dual field-strength of an auxiliary antisymmetric tensor gauge field of maximal rank $C_{\mu\nu\lambda}$:

$$\Phi(C) = \frac{1}{3!} \epsilon^{\mu\nu\kappa\lambda} \partial_\mu C_{\nu\kappa\lambda} . \quad (6)$$

Let us particularly stress that, although we stay in $D = 4$ spacetime dimensions and although we *don't couple* the Gauss-Bonnet scalar (5) to

the matter fields, the last term in (4) thanks to the presence of the non-Riemannian volume element (6) is non-trivial (*not* a total derivative as with the ordinary Riemannian volume element $\sqrt{-g}$) and yields a non-rivial contribution to the Einstein equations (Eqs. (10) below).

- As we will see in what follows, the specific form of the matter Lagrangian L_{matter} in (4) will depend on the specific class of static spherically symmetric (SSS) solutions we are looking for:

- (i) For a broad class of SSS solutions specified below (see Eqs. (36)–(38) below) L_{matter} will be required by consistency of the equations of motion to be a Lagrangian of a *nonlinear electrodynamics* $L(F^2)$:

$$L_{\text{matter}} = L(F^2), \quad F^2 \equiv F_{\mu\nu}F_{\kappa\lambda}g^{\mu\kappa}g^{\nu\lambda}, \quad F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu. \quad (7)$$

An important property of $L(F^2)$ we prove below is that the latter must be a *non-analytic* function of F^2 .

- (ii) A special narrow class of SSS solutions (Eq. (29) below) will require “hedgehog” scalar field matter source with a Lagrangian of a $O(3)$ nonlinear sigma-model (λ being a Lagrange multiplier) with a “hedgehog” solution [30]:

$$L_{\text{matter}} = -\frac{1}{2}g^{\mu\nu}\partial_\mu\vec{\phi}\cdot\partial_\nu\vec{\phi} - \lambda(\vec{\phi}\cdot\vec{\phi} - v^2) \quad , \quad \vec{\phi} = \pm v\hat{r}. \quad (8)$$

with \hat{r} – unit radial vector.

- (iii) Another set of SSS solutions require matter sources lacking explicit Lagrangian action formulation, including additional thin-shell (brane) ones describing domain walls (Eqs. (44)–(47) below).

We now have three types of equations of motion resulting from the action (4):

- Einstein equations w.r.t. $g^{\mu\nu}$ where we employ the definition for a composite field:

$$\chi \equiv \frac{\Phi(C)}{\sqrt{-g}} \quad (9)$$

representing the ratio of the non-Riemannian to the standard Riemannian volume element densities:

$$\begin{aligned} R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R &= \frac{1}{2}T_{\mu\nu} - \frac{1}{2}g_{\mu\nu}\chi R_{\text{GB}}^2 + 2R\nabla_\mu\nabla_\nu\chi \\ &+ 4\Box\chi(R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R) - 4R_\mu^\rho\nabla_\rho\nabla_\nu\chi - 4R_\nu^\rho\nabla_\rho\nabla_\mu\chi \\ &+ 4g_{\mu\nu}R^{\rho\sigma}\nabla_\rho\nabla_\sigma\chi - 4g^{\kappa\rho}g^{\lambda\sigma}R_{\mu\kappa\nu\lambda}\nabla_\rho\nabla_\sigma\chi, \end{aligned} \quad (10)$$

where $T_{\mu\nu} = g_{\mu\nu}L_{\text{matter}} - 2\frac{\partial}{\partial g^{\mu\nu}}L_{\text{matter}}$ is the relevant standard matter energy-momentum tensor. In particular, for the nonlinear electrodynamics:

$$T_{\mu\nu} = g_{\mu\nu}L(F^2) - 4L'(F^2)F_{\mu\kappa}F_{\nu\lambda}g^{\kappa\lambda}, \quad (11)$$

where $L'(F^2) \equiv \frac{\partial L}{\partial F^2}$, and for the scalar ‘‘hedgehog’’ field $\vec{\phi}$:

$$T_{\mu\nu} = \partial_\mu \vec{\phi} \cdot \partial_\nu \vec{\phi} - \frac{1}{2} g_{\mu\nu} g^{\kappa\lambda} \partial_\kappa \vec{\phi} \cdot \partial_\lambda \vec{\phi}. \quad (12)$$

- The equations of motion w.r.t. scalar ‘‘hedgehog’’ field $\vec{\phi}$ and the nonlinear gauge field have the standard form (they are not affected by the presence of the Gauss-Bonnet term):

$$\square \vec{\phi} - \lambda \vec{\phi} = 0, \quad \vec{\phi} \cdot \vec{\phi} - v^2 = 0, \quad (13)$$

$$\nabla_\nu (L'(F^2) F^{\mu\nu}) = 0. \quad (14)$$

- The crucial new feature are the equations of motion w.r.t. auxiliary non-Riemannian volume element tensor gauge field $C_{\mu\nu\lambda}$:

$$\partial_\mu R_{\text{GB}}^2 = 0 \quad \longrightarrow \quad R_{\text{GB}}^2 = 24M = \text{const}, \quad (15)$$

where M is an *arbitrary dimensionful integration constant* and the numerical factor 24 in (15) is chosen for later convenience.

The dynamically triggered constancy of the Gauss-Bonnet scalar (15) comes at a price as we see from the generalized Einstein Eqs. (10) – namely, now the composite field $\chi = \frac{\Phi(C)}{\sqrt{-g}}$ appears as an *additional* physical field degree of freedom.

In what follows we will see that when considering SSS solutions we can consistently ‘‘freeze’’ the composite field $\chi = \text{const}$ so that all terms on the r.h.s. of (10) with derivatives of the composite field χ will vanish. Thus, we are left with an *overdetermined* system of equations:

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = \frac{1}{2} T_{\mu\nu} - g_{\mu\nu} 12\chi M, \quad R_{\text{GB}}^2 = 24M, \quad (16)$$

plus the matter field equations of motion (13)–(14) determining $T_{\mu\nu}$.

3 Static Spherically Symmetric Solutions with a Dynamically Constant Gauss-Bonnet Scalar – General Properties

Let us now consider the system (16) with a static spherically symmetric (SSS) ansatz for the metric:

$$ds^2 = -A(r)dt^2 + \frac{dr^2}{A(r)} + r^2(d\theta^2 + \sin^2\theta d\phi^2). \quad (17)$$

Inserting (17) in (15) we have:

$$R_{\text{GB}}^2 = 24M \quad \longrightarrow \quad \frac{2}{r^2} \frac{d^2}{dr^2} [(A(r) - 1)^2] = 24M, \quad (18)$$

Four-Dimensional Gauss-Bonnet Gravity

which yields the following general solution for $A(r)$ already noted in Ref. [27]:

$$A(r) = 1 \pm \sqrt{P_4(r)}, \quad P_4(r) \equiv Mr^4 + c_1r + c_0 \quad (19)$$

with $c_{0,1}$ together with M representing three *a priori* arbitrary integration constants. Let us stress that the gravity solution (19) is not affected by the matter sources.

For the SSS ansatz (17) the Ricci tensor components and the scalar curvature read:

$$R_0^0 = R_r^r = -\frac{1}{2r^2} \partial_r (r^2 \partial_r A), \quad R_j^i = -\delta_j^i \left[\frac{1}{r^2} (A-1) + \frac{1}{r} \partial_r A \right], \quad (20)$$

$$R = -2 \left[\frac{1}{r^2} (A-1) + \frac{1}{r} \partial_r A \right] - \frac{1}{r^2} \partial_r (r^2 \partial_r A), \quad (21)$$

whereupon the Einstein equations in (16) become:

$$\frac{1}{r^2} (A-1) + \frac{1}{r} \partial_r A = \frac{1}{2} T_0^0 - 12\chi M, \quad (22)$$

$$\frac{1}{2r^2} \partial_r (r^2 \partial_r A) = \frac{1}{2} T_\theta^\theta - 12\chi M. \quad (23)$$

Consistency of SSS Einstein equations (22)–(23) requires for the components of the matter energy momentum tensor:

$$T_0^0 = T_r^r, \quad T_\theta^\theta = T_\phi^\phi, \quad \text{rest} = 0. \quad (24)$$

Conditions (24) are fulfilled for the SSS solutions in nonlinear electrodynamics ($F_{0r} = F_{0r}(r)$ being the only surviving component of $F_{\mu\nu}$; $F^2 = -2F_{0r}^2$):

$$T_0^0 = T_r^r = L(F^2) + 4F_{0r}^2 L'(F^2), \quad T_\theta^\theta = T_\phi^\phi = L(F^2), \quad (25)$$

where Eqs. (14) reduce to:

$$\partial_r (r^2 F_{0r} L'(F^2)) = 0 \quad \longrightarrow \quad F_{0r} L'(F^2) = \frac{q}{16\pi r^2}, \quad (26)$$

q indicating the electric charge.

Conditions (24) are fulfilled as well as for the SSS “hedgehog” solution of (8):

$$T_0^0 = T_r^r = -v^2/r^2, \quad T_\theta^\theta = T_\phi^\phi = 0. \quad (27)$$

In the case of nonlinear electrodynamics source, combining (22)–(23) with (25)–(26) we obtain an exact expression for the radial electric field ($E_r \equiv -F_{0r}$) in terms of the metric function (19):

$$F_{0r} = -\frac{4\pi r^2}{q} \left[\partial_r^2 A - \frac{2}{r^2} (A-1) \right]. \quad (28)$$

Let us note the following two obvious well-defined non-trivial solutions for $A(r)$ (19) satisfying (22)–(23):

- For $(M > 0, c_0 = c_1 = 0)$ (19) becomes the standard (anti)-de Sitter solution $A(r) = 1 \pm \sqrt{M}r^2$, where $T_0^0 = T_\theta^\theta = 0$ and $\chi = \mp \frac{1}{4\sqrt{M}}$.

- $A(r)$ (19) becomes for:

$$(M = 0, c_1 = 0, c_0 = v^4/4) \longrightarrow A(r) = 1 - v^2/2 = \text{const}, \quad (29)$$

the (minus) 00-component of the metric generated by the SSS energy-momentum tensor (27) of the ‘‘hedgehog’’ scalar field (8).

Before proceeding let us stress that:

- The solutions $A(r)$ (19) will have well-defined large r asymptotics only for $M > 0$ (see (30) below), or for $(M = 0, c_1 > 0)$. In the case $M < 0$ or $(M = 0, c_1 < 0)$ the large r region will be inaccessible (forbidden region) since $A(r)$ becomes complex there, *i.e.* spacetime does not exist there.
- Similarly, for all solutions $A(r)$ (19) with $c_0 < 0$ the region of small r , where $A(r)$ becomes complex, will be inaccessible (forbidden region).

Asymptotically, for large r (cf. [27]) $A(r)$ (19) with $M > 0$ becomes:

$$A(r) \simeq 1 \pm \left(\sqrt{M}r^2 + \frac{c_1}{2\sqrt{M}r} + \frac{c_0}{2\sqrt{M}r^2} \right) + O(r^{-4}), \quad (30)$$

so that asymptotically (30) can be viewed as Reissner-Nordström-(anti)-de Sitter metric $A_{\text{RN-(A)DS}}(r) = 1 \pm \frac{\Lambda}{3}r^2 - \frac{2m}{r} + \frac{q^2}{(8\pi)^2 r^2}$ upon the following identification of the signs and values of the free integration constants (M, c_1, c_0) :

$$\sqrt{M} = \Lambda/3, \quad c_1 = \mp 4m\sqrt{M}, \quad c_0 = \pm \frac{q^2}{32\pi^2}\sqrt{M}, \quad (31)$$

where the upper/lower signs in (31) and below refer to anti-de Sitter/de Sitter asymptotics.

Now, let us insert (30) in (23) with a nonlinear electrodynamics source:

$$L(F^2) - 24\chi M = \partial_r^2 A + \frac{2}{r}\partial_r A \simeq \pm \left(6\sqrt{M} + \frac{c_0}{\sqrt{M}r^4} - \frac{3c_1^2}{2M^{3/2}r^6} + O(r^{-7}) \right), \quad (32)$$

and compare with the large r asymptotics of F_{0r} upon inserting (30) in (28):

$$F_{0r} \simeq \mp \frac{4\pi}{q} \left(\frac{2c_0}{\sqrt{M}r^2} - \frac{9c_1^2}{4M^{3/2}r^4} + O(r^{-5}) \right) \quad (33)$$

(noting that for SSS configurations $F^2 = -2F_{0r}^2$). Thus, we obtain again $\chi = \mp \frac{1}{4\sqrt{M}}$ as for the pure (anti)-de Sitter solution, but more importantly,

Four-Dimensional Gauss-Bonnet Gravity

we explicitly find that for weak electromagnetic fields the nonlinear electro-dynamics Lagrangian is a *non-analytic* function of F^2 :

$$L(F^2) = -\frac{1}{4}F^2 + c_2\sqrt{-F^2}(-F^2) + O((-F^2)^2), \quad (34)$$

$$c_2 \equiv \pm \frac{3\sqrt{2}m^2}{\sqrt{M}} \left(\frac{4\pi}{q}\right)^3 \quad (35)$$

using the parameter identification (31), in other words $L(F^2)$ is *not of Born-Infeld type*.

Let us consider again the system of three equations (19), (32) and (28):

$$A(r) = 1 \pm \sqrt{Mr^4 + c_0 + c_1r}, \quad (36)$$

$$F_{0r} = -\frac{4\pi r^2}{q} \left[\partial_r^2 A - \frac{2}{r^2}(A-1) \right], \quad (37)$$

$$\begin{aligned} L(F^2) &= 24\chi M + \partial_r^2 A + \frac{2}{r}\partial_r A \\ &= 24\chi M - \frac{q}{4\pi r^2}F_{0r} + 2 \left[\frac{1}{r}\partial_r A + \frac{1}{r^2}(A-1) \right]. \end{aligned} \quad (38)$$

In principle (36)–(38) allows to determine the full nonlinear, and non-analytic as we proved in (34), functional expression of $L(F^2)$ by first expressing r as implicit function of F_{0r} from (36)–(37) and then substituting in (38) using (36) (recall $F^2 = -2F_{0r}^2$).

Now, an important remark regarding the matter sources in (16) is in order.

- According to (31) and (32) only metrics (17) with $A(r)$ (19), whose parameters are of the form ($M > 0, c_0 > 0$) for anti-de Sitter asymptotics (upper sign in (19)) or ($M > 0, c_0 < 0$) for de Sitter-like asymptotics (lower sign in (19)), will have nonlinear electro-dynamical matter source.
- In what follows we will concentrate on solutions for the SSS metric $A(r)$ (19) with de Sitter-like asymptotics. Thus, while being generated by nonlinear electro-dynamics source (for ($M > 0, c_0 < 0$)) the pertinent $A(r)$ (19) will become complex for small r as already pointed out above. Therefore, for all SSS solutions (19) with de Sitter-like large r asymptotics, whose matter source is nonlinear electro-dynamics, the region of small r will be a forbidden one. In particular, in this case ($M > 0, c_0 < 0$) there are *no* black hole solutions. For ($M > 0, c_0 < 0$) there is only one SSS solution with a horizon – with de Sitter-like geometry outside the forbidden small r region (see Figure 10 below). The above results conform to the non-existence theorems of Ref. [31] (see also [32]) stating that for nonlinear electro-dynamics source with $L(F^2) \sim F^2$ at weak fields (as in (34)) the SSS electrically charged solutions cannot have a regular center at $r = 0$.

- On the other hand, when $(M > 0, c_0 > 0)$ the matter source for $A(r)$ (19) with de Sitter-like asymptotics will be formally again nonlinear electro-dynamics but with a purely imaginary electric charge q – recall from (31) $c_0 = -\frac{q^2}{32\pi^2}\sqrt{M}$ and compare with the large r asymptotics (32)–(33) with the lower signs. This is similar to the formal electromagnetic source producing the Riessner-Nordström-like metric with a negative charge-squared ($q^2 < 0$) in Einstein-Rosen’s classic 1935 paper [33]. In this latter case $(M > 0, c_0 > 0)$ there exist black hole solutions with de Sitter large r asymptotics for $A(r)$ (19) – see Figures 6, 7, 9 below.

4 Domains of Definition and Horizons for the Metric Function $A(r)$

The defining domain of $A(r)$ (19), i.e., for those r for which $A(r)$ is real-valued, is given by the condition on the 4-th order polynomial $P_4(r)$ under the square root in (19):

$$P_4(r) \equiv Mr^4 + c_1r + c_0 \geq 0. \quad (39)$$

The intervals of r where $P_4(r) < 0$ are forbidden regions (spacetime does not exist there since $A(r) = 1 \pm \sqrt{P_4(r)}$ becomes complex-valued).

- Simple positive roots $r_* > 0$ of $P_4(r)$ (where $P_4(r) \simeq (r - r_*)P_4'(r_*) + O((r - r_*)^2)$):

$$P_4(r_*) = 0 \longrightarrow \partial_r A = \pm \frac{P_4'(r)}{2\sqrt{P_4(r)}} \simeq \pm \frac{\sqrt{|P_4'(r_*)|}}{2|r - r_*|^{1/2}} \rightarrow \pm\infty \quad (40)$$

for $r \rightarrow r_*$, signify the existence of a physical spacetime singularity – e.g., the scalar curvature R (21) and the quadratic curvature invariants diverge there:

$$R \sim \mp \frac{\sqrt{|P_4'(r_*)|}}{4|r - r_*|^{3/2}} \rightarrow \mp\infty, \quad (41)$$

$$\text{quadratic curvature invariants} = O(|r - r_*|^{-3}). \quad (42)$$

Similarly, also the electric field (37) has the same singularity at $r = r_*$ as in (41).

- When $P_4'(r_*) > 0$ the forbidden region (for de Sitter asymptotics – lower sign in (40)) is a finite-extent internal one ($0 < r < r_*$) – see Figure 10 below where $(M > 0, c_1 \text{ any}, c_0 < 0)$ and Figure 21 below where $(M = 0, c_1 > 0, c_0 < 0)$.
- When $P_4'(r_*) < 0$ the forbidden region (for de Sitter asymptotics) is an infinite-extent external one ($r_* < r < \infty$) – see Figures 11, 12, 13, 14 below where $(M < 0, c_1 \text{ any}, c_0 > 0)$, and Figure 19 below where $(M = 0, c_1 < 0, c_0 > 1)$.

Four-Dimensional Gauss-Bonnet Gravity

- Two simple positive roots $r_{1*} > 0$ and $r_{2*} > 0$ of $P_4(r)$, $r_{1*} < r_{2*}$, with two physical spacetime singularities there (cf. (40)–(42)).
 - (i) For $P'_4(r_{1*}) < 0$ and $P'_4(r_{2*}) > 0$ the forbidden region (for de Sitter asymptotics) is a finite-extent intermediate one ($r_{1*} < r < r_{2*}$) – see Figures 8, 9 below where ($M > 0, c_1 < -4M(c_0/3M)^{3/4}, c_0 > 0$).
 - (ii) For $P'_4(r_{1*}) > 0$ and $P'_4(r_{2*}) < 0$ there are two forbidden regions (for de Sitter asymptotics): a finite-extent internal ($0 < r < r_{1*}$) and an infinite-extent external ($r_{2*} < r < \infty$). See Figures 15, 16, 17 below where ($M < 0, c_1$ any, $c_0 < 0$).
- Double positive root $r_{\text{DW}} \equiv (c_0/3M)^{1/4}$ of $P_4(r)$:

$$P_4(r) = (r - r_{\text{DW}})^2 M [6r_{\text{DW}}^2 + 4r_{\text{DW}}(r - r_{\text{DW}}) + (r - r_{\text{DW}})^2] \quad (43)$$

yield spacetime geometry (17) with:

$$A(r) = 1 \pm |r - r_{\text{DW}}| \sqrt{M [6r_{\text{DW}}^2 + 4r_{\text{DW}}(r - r_{\text{DW}}) + (r - r_{\text{DW}})^2]} \quad (44)$$

which contains a *domain wall* located at $r = r_{\text{DW}} \equiv (c_0/3M)^{1/4}$ (see Figure 2 and Figure 7 below where ($M > 0, c_1 = -4M(c_0/3M)^{3/4}, c_0 < 0$)) since while $A(r)$ is continuous there, its derivative $\partial_r A$ has a discontinuity. Therefore, the second derivative gets a delta-function contribution plus an additional discontinuity at $r = r_{\text{DW}} \equiv (c_0/3M)^{1/4}$:

$$\partial_r^2 A = -\sqrt{24M} r_{\text{DW}} \delta(r - r_{\text{DW}}) - \sqrt{\frac{8M}{3}} \text{sign}(r - r_{\text{DW}}) + \text{regular} . \quad (45)$$

Eq. (45) through the SSS Einstein equation (23) and Eq. (28) indicates the presence of a surface stress-energy tensor S_ν^μ of an additional static charged thin-shell (brane) matter source located at $r = r_{\text{DW}} \equiv (c_0/3M)^{1/4}$ so that (23) is modified as:

$$L(F^2) = 24\chi M + \partial_r^2 A + \frac{2}{r} \partial_r A - T_\theta^\theta \Big|_{\text{brane}} , \quad (46)$$

$$T_\nu^\mu \Big|_{\text{brane}} = S_\nu^\mu \delta(r - r_{\text{DW}}), \quad S_0^0 = S_r^r = 0, \quad S_\theta^\theta = S_\phi^\phi, \quad (47)$$

whereas (26) is modified as:

$$F_{0r} L'(F^2) = \frac{q}{16\pi r^2} + \frac{1}{8} j_{\text{brane}}^0 \text{sign}(r - r_{\text{DW}}), \quad (48)$$

with j_{brane}^0 being the surface brane charge density. Relations (47) comply with the general formalism for thin-shell domain walls developed in Ref. [34].

Similarly, there appear the same brane stress-energy and surface charge contributions in the modification of (28):

$$F_{0r} \left[\frac{q}{16\pi r^2} + \frac{1}{2} j_{\text{brane}}^0 \text{sign}(r - r_{\text{DW}}) \right] = -\frac{4\pi r^2}{q} \left[\partial_r^2 A - \frac{2}{r^2} (A - 1) - T_\theta^\theta \Big|_{\text{brane}} \right]. \quad (49)$$

Choosing the value of the brane pressure:

$$S_\theta^\theta = -\sqrt{24M} r_{\text{DW}}, \quad (50)$$

we exactly cancel the delta-function part in $L(F^2)$ (46) and F_{0r} (49) due to the delta-function singularity in $\partial_r^2 A$ (45), whereas the discontinuous term on the l.h.s. of (49) is matched by the discontinuity in $\partial_r^2 A$ (45). Let us note that Eq. (50) together with (47) implies that the thin-shell matter forming the domain wall is an *exotic matter* (violating null energy condition).

- Another class of SSS solutions for $A(r)$ with de Sitter-like asymptotics is when $P_4(r) > 1$ for all r , i.e., $A(r) = 1 - \sqrt{P_4(r)} < 0$ for all r , which means that r becomes timelike whereas $t \equiv z$ becomes “radial-like” spacelike. In this case the SSS metric (17) acquires the following form upon introducing a new “cosmological” time coordinate ξ instead of the timelike r :

$$ds^2 = -d\xi^2 + |A(r(\xi))| dz^2 + r^2(\xi) (d\theta^2 + \sin^2 \theta d\phi^2), \quad (51)$$

$$\frac{dr}{d\xi} = \sqrt{|A(r(\xi))|}.$$

This describes the geometry of a particular type of Kantowski-Sachs universe [35] – contracting, expanding or bouncing – depending of the values of the free integration constants. See Figures 3, 4, 5 below, where ($M > 0, c_1 \geq -4M(c_0/3M)^{3/4}, c_0 > 1$) and Figure 18 below, where ($M = 0, c_1 > 0, c_0 > 1$).

On the other hand, for de Sitter-like asymptotics (lower sign in (19)) there might exist one or two horizons r_0 of $A(r)$ provided there are (one or two) positive roots r_0 of the related polynomial $Q_4(r)$:

$$Q_4(r) \equiv P_4(r) - 1 = Mr^4 + c_1 r + c_0 - 1, \quad (52)$$

$$Q_4(r_0) = 0 \rightarrow A(r_0) = 1 - \sqrt{1 + Q_4(r_0)} = 0. \quad (53)$$

The various types of horizons are as follows. For one positive root r_0 of $Q_4(r) \equiv P_4(r) - 1$ (52)–(53):

Four-Dimensional Gauss-Bonnet Gravity

- The single horizon $r_0 \equiv r_{\text{Schw}}$ is of Schwarzschild type (see Figure 14 below, where $(M < 0, c_1 \text{ any}, C_0 > 1)$, and Figure 19 below, where $(M = 0, c_1 < 0, c_0 > 1)$) for:

$$A(r_0) = 0, \quad \partial_r A(r_0) = -\frac{1}{2}P'_4(r_0) > 0. \quad (54)$$

- The single horizon $r_0 \equiv r_{\text{dS}}$ is of de Sitter type (see Figures 1, 2, 8, 10, 20, 21 below) for:

$$A(r_0) = 0, \quad \partial_r A(r_0) = -\frac{1}{2}P'_4(r_0) < 0. \quad (55)$$

For two positive roots $r_0^{(1)} < r_0^{(2)}$ of $Q_4(r) \equiv P_4(r) - 1$ (52)–(53):

- The two horizons are of the same type as for the Schwarzschild-de Sitter black hole when:

$$\begin{aligned} A(r_0^{(1,2)}) = 0, \quad \partial_r A(r_0^{(1)}) > 0, \quad \partial_r A(r_0^{(2)}) < 0; , \\ \text{i.e. } P'_4(r_0^{(1)}) < 0, \quad P'_4(r_0^{(2)}) > 0, \end{aligned} \quad (56)$$

(see Figures 6, 7 below, where $(M > 0, -4M((c_0)/3M)^{3/4} \leq c_1 < -4M((c_0 - 1)/3M)^{3/4}, c_0 > 1)$. In the case $(M > 0, c_1 < -4M(c_0/3M)^{3/4}, c_0 > 1)$ (see Figure 9 below) there are again two horizons of the same type as for the Schwarzschild-de Sitter black hole, however they are separated by an intermediate forbidden region.

- The two horizons are of the same type as for the Reissner-Nordström black hole for:

$$\begin{aligned} A(r_0^{(1,2)}) = 0, \quad \partial_r A(r_0^{(1)}) < 0, \quad \partial_r A(r_0^{(2)}) > 0, \\ \text{i.e. } P'_4(r_0^{(1)}) > 0, \quad P'_4(r_0^{(2)}) < 0, \end{aligned} \quad (57)$$

however, in this case the black hole exists only in a finite-extent space region (see Figure 12 and Figure 16 where $(M < 0, c_1 > 4|M|\left(\frac{1-c_0}{3|M|}\right)^{3/4}, c_0 < 1)$).

- In the case of coalescence of the two roots of $A(r)$:

$$A(r_0^{(1,2)}) = 0, \quad r^{(1)} = r^{(2)}, \quad \partial_r A(r_0^{(1,2)}) = 0, \quad \text{i.e. } P'_4(r_0^{(1,2)}) = 0, \quad (58)$$

the horizon is of extremal black hole type (see Figure 13 and Figure 17 where $(M < 0, c_1 = 4|M|\left(\frac{1-c_0}{3|M|}\right)^{3/4}, c_0 < 1)$).

5 Graphical Representations of the Class of SSS Solutions of Modified $D = 4$ Gauss-Bonnet Gravity

In what follows we will graphically illustrate the various possible classes of solutions for $A(r)$ (19), focusing on de Sitter-like asymptotics of the latter, with or without physical singularities, including with or without domain walls, as well as with or without horizons (black hole type or de Sitter cosmological type) depending on the values of the free integration constants (M, c_1, c_0) .

5.1 Metrics without forbidden regions ($M > 0$)

5.1.1 ($M > 0, 0 < c_0 < 1, c_1 > -4M(c_0/3M)^{3/4}$)

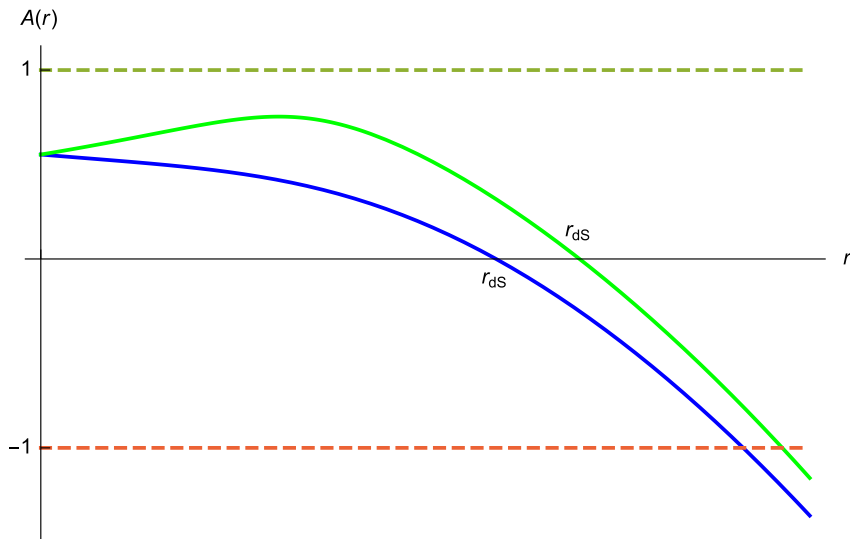


Figure 1. De Sitter-like (deviation from the standard de Sitter) $A(r)$ with a de Sitter-type horizon at r_{dS} . The lower curve corresponds to $c_1 > 0$ and the upper curve corresponds to $-4M(c_0/3M)^{3/4} < c_1 < 0$.

5.1.2 ($M > 0$, $0 < c_0 < 1$, $c_1 = -4M(c_0/3M)^{3/4}$)

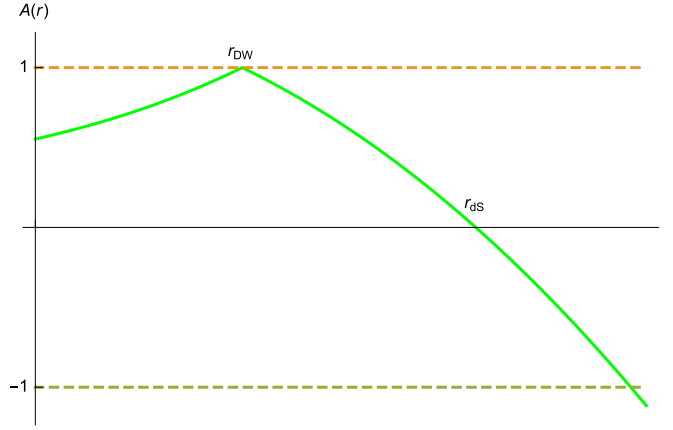


Figure 2. De Sitter-like $A(r)$ with a domain wall at $r = r_{\text{DW}} \equiv \left(\frac{c_0}{3M}\right)^{1/4}$ and a de Sitter-type horizon at r_{dS} . As pointed out above after Eq. (50) the thin-shell matter of the domain wall must be *exotic*.

5.1.3 ($M > 0$, $c_0 > 1$, $c_1 > 0$)

In this case $A(r) = 1 - \sqrt{P_4(r)} < 0$ for all r , so the metric is given by (51), where the solution of $dr/d\xi = \sqrt{|A(r(\xi))|}$ is $r(\xi) \simeq \sqrt{\sqrt{c_0} - 1} \xi$ for small ξ , and $r(\xi) \simeq \exp\{M^{1/4}\xi\}$ for large ξ , and thus (51) describes monotonically expanding Kantowski-Sachs universe.

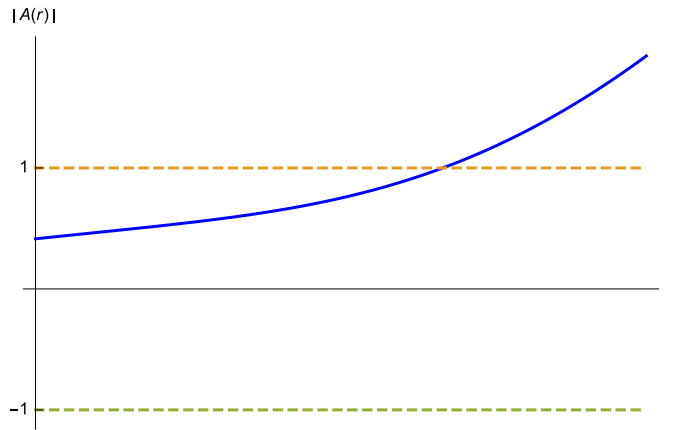


Figure 3. Monotonically expanding Kantowski-Sachs universe; r is timelike coordinate.

5.1.4 ($M > 0$, $c_0 > 1$, $0 > c_1 > -4M((c_0 - 1)/3M)^{3/4}$)

Here again $A(r) = 1 - \sqrt{P_4(r)} < 0$ for all r , but now there is a local minimum of $|A(r)|$ at:

$$r = r_b \equiv \left(\frac{|c_1|}{4M}\right)^{1/3}, \quad (59)$$

as depicted in Figure 4. Therefore, the metric (51) describes a *bouncing* Kantowski-Sachs universe.

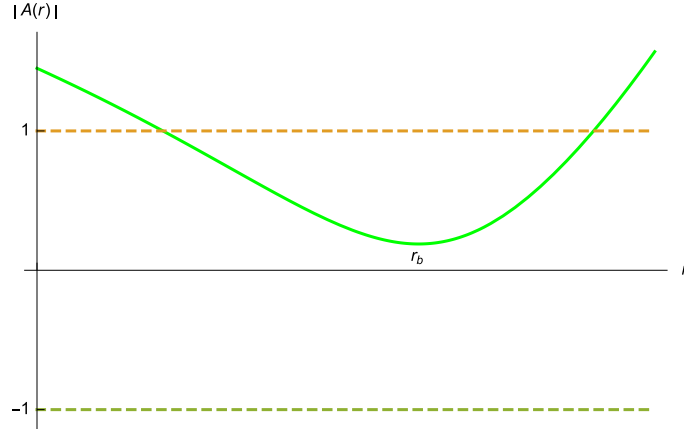


Figure 4. Bouncing Kantowski-Sachs universe at $r = r_b$; r is timelike coordinate.

Namely, the size squared $|A(r(\xi))|$ of the new radial dimension z in (51) starts at a finite value $|A(0)| = \sqrt{\sqrt{c_0} - 1}$ for $\xi = 0$, drops down to a minimum value $|A(r_b)|$ at some finite cosmological time ξ_b and then it expands indefinitely for $\xi > \xi_b$.

5.1.5 ($M > 0$, $c_0 > 1$, $c_1 = -4M((c_0 - 1)/3M)^{3/4}$)

This is a limiting case of the above bouncing Kantowski-Sachs solution, where now the minimum of the scale factor squared $|A(r)|$ vanishes when r reaches r_b :

$$|A(r_b)| = 0, \quad r_b \equiv \left(\frac{c_0 - 1}{3M}\right)^{1/4}. \quad (60)$$

Accordingly we have a very different properties of the pertinent Kantowski-Sachs universe.

In the present case the solution $r(\xi)$ of the second Eq.(51) splits into two branches:

Four-Dimensional Gauss-Bonnet Gravity

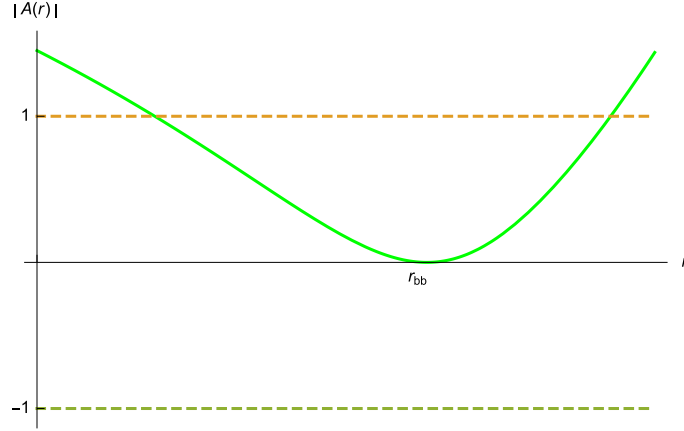


Figure 5. Kantowski-Sachs universe: the left branch describing a contracting universe with a big crunch on the interval $(0 < r < r_b)$, and the right branch describing an expanding universe with a big bang on the interval $(r_b < r < \infty)$; r is timelike coordinate.

- (a) Contracting Kantowski-Sachs universe with a *big crunch* on the interval $0 \leq r \leq r_b$:

$$r(\xi) \simeq \begin{cases} \sqrt{|A(0)|} \xi & \xi \sim 0, r \sim 0 \\ r_b - e^{-3Mr_b^2 \xi} & \xi \rightarrow \infty, r \sim r_b \end{cases} \quad (61)$$

Here the evolution starts at cosmological time $\xi = 0$ with a non-zero scale factor squared $|A(0)| = \sqrt{\sqrt{c_0} - 1}$ (“emergent universe”) and monotonically contracts to a big crunch $|A(r(\xi))| \rightarrow |A(r_b)| = 0$ at $\xi \rightarrow \infty$.

- (b) Expanding Kantowski-Sachs universe with a *big bang* on the interval $r_b \leq r < \infty$:

$$r(\xi) \simeq \begin{cases} r_b + e^{3Mr_b^2 \xi} & \xi \rightarrow -\infty, r \sim r_b \\ e^{M^{1/4} \xi} & \xi \rightarrow \infty, r \rightarrow \infty \end{cases} \quad (62)$$

Here evolution starts with a big bang at $\xi \rightarrow -\infty$ where the scale factor squared $|A(r(\xi))| \rightarrow |A(r_b)| = 0$ and then monotonically expands indefinitely for $\xi \rightarrow +\infty$.

5.1.6 ($M > 0$, $c_0 > 1$, $-4M((c_0 - 1)/3M)^{3/4} > c_1 > -4M(c_0/3M)^{3/4}$)

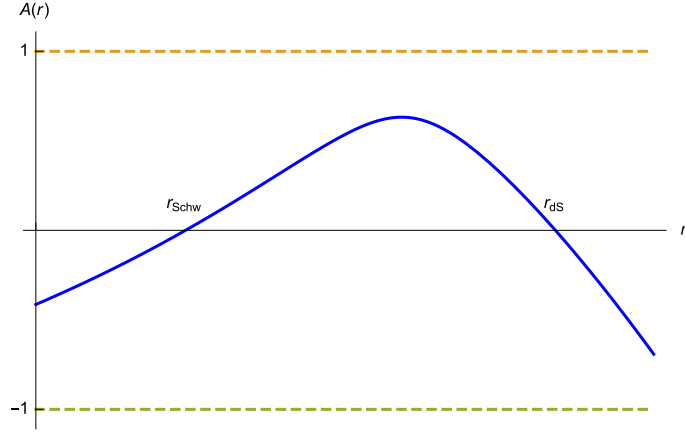


Figure 6. Blackhole with two horizons – internal Schwarzschild-type at r_{Schw} and external de Sitter-type at r_{dS} . Unlike the standard Schwarzschild-de Sitter blackhole here $A(r)$ is finite at $r = 0$ ($A(0) = -(\sqrt{c_0} - 1)$).

5.1.7 ($M > 0$, $c_0 > 1$, $c_1 = -4M(c_0/3M)^{3/4}$)

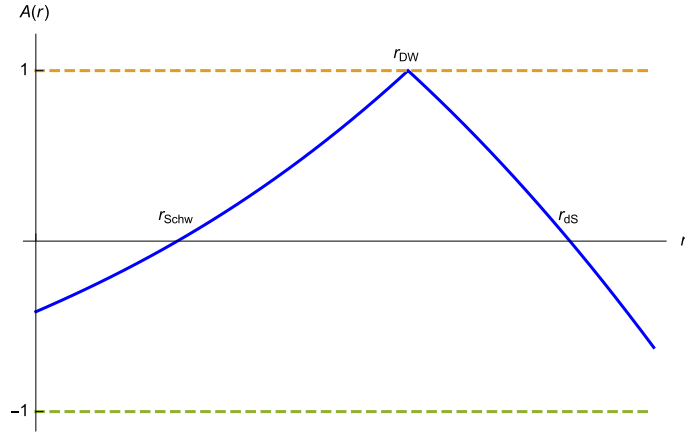


Figure 7. Blackhole with two horizons – internal Schwarzschild-type at r_{Schw} and external de Sitter-type at r_{dS} , and with an additional domain wall at $r_{\text{DW}} \equiv (c_0/3M)^{1/4}$ between the two horizons. As in subsection 5.1.2, in particular, an additional thin-shell brane *exotic* matter source must be present to match the delta-function singularities in the corresponding Einstein equations at $r = r_{\text{DW}}$.

5.2 Metrics with one forbidden region ($M > 0$)

5.2.1 ($M > 0$, $0 < c_0 < 1$, $c_1 < -4M(c_0/3M)^{3/4}$)

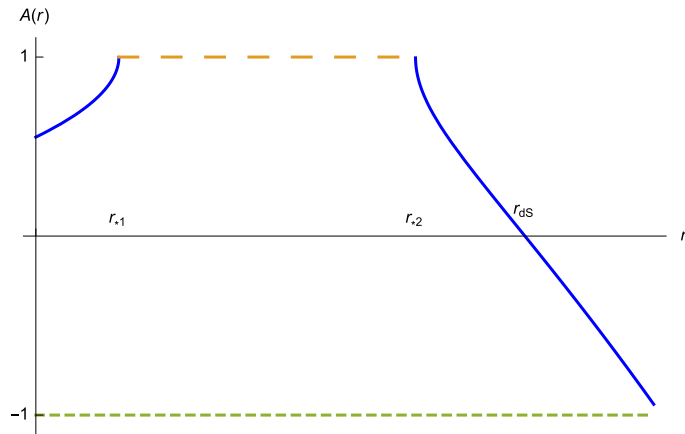


Figure 8. De Sitter-like geometry with one intermediate finite-extent forbidden region ($r_{1*} < r < r_{2*}$), and a de Sitter-type horizon at r_{dS} .

5.2.2 ($M > 0$, $c_0 > 1$, $c_1 < -4M(c_0/3M)^{3/4}$)

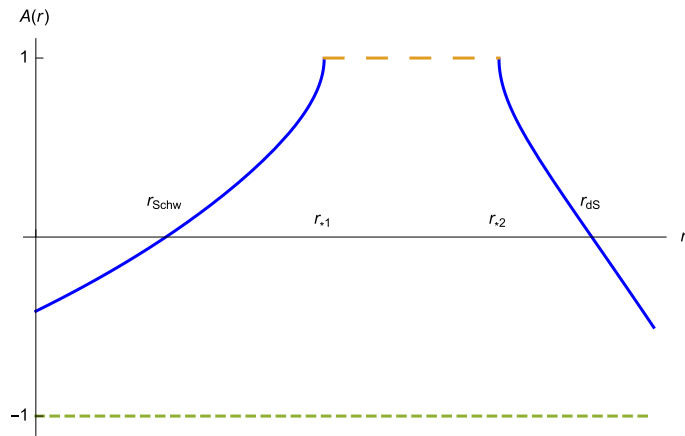


Figure 9. Black Hole with two horizons – a Schwarzschild-type at r_{Schw} and a de Sitter-type at r_{dS} separated by an intermediate finite-extent forbidden region ($r_{1*} < r < r_{2*}$).

5.2.3 ($M > 0$, $c_0 < 0$, c_1 any)

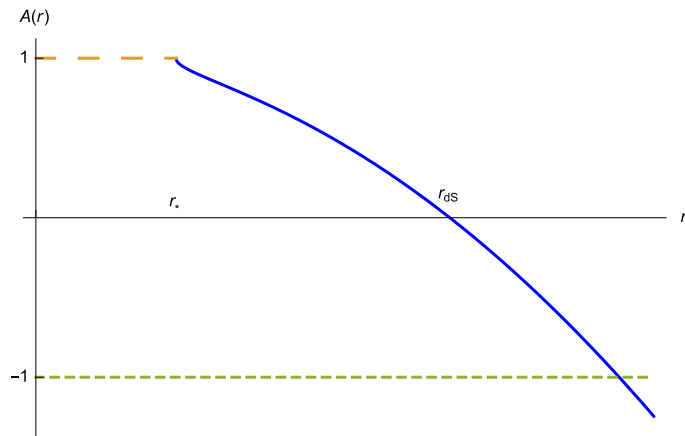


Figure 10. De Sitter-like geometry with a de Sitter-type horizon at r_{dS} , and one internal finite-extent forbidden region ($0 < r < r_*$).

5.3 Metrics with one forbidden region ($M < 0$)

5.3.1 ($M < 0$, $0 < c_0 < 1$, $c_1 < 4|M|((1 - c_0)/3|M|)^{3/4}$)

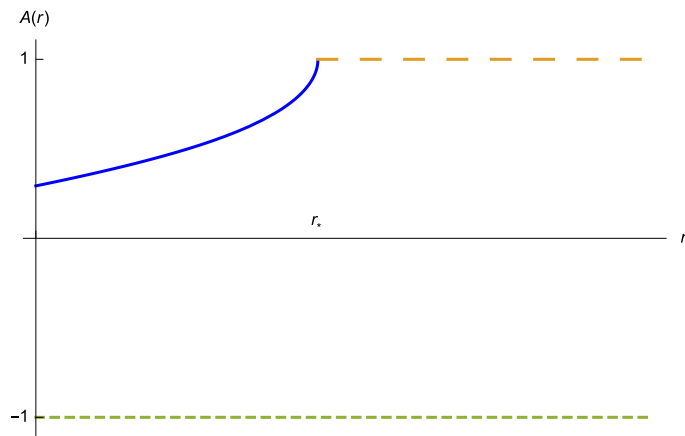


Figure 11. Finite-extent internal region of regular geometry ($0 < r < r_*$), and one external infinite forbidden region ($r_* < r < \infty$).

Four-Dimensional Gauss-Bonnet Gravity

5.3.2 ($M < 0$, $0 < c_0 < 1$, $c_1 > 4|M|((1 - c_0)/3|M|)^{3/4}$)

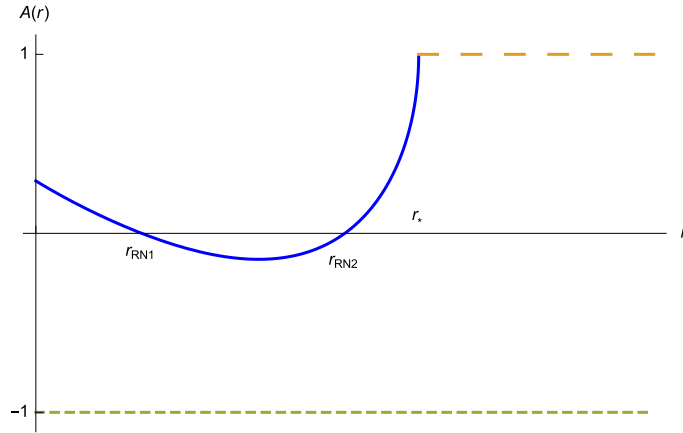


Figure 12. Black hole with two horizons of Reissner-Nordström-type at r_{RN1} and r_{RN2} within a finite extent internal space region ($0 < r < r_*$), and an infinite external forbidden region ($r_* < r < \infty$).

5.3.3 ($M < 0$, $0 < c_0 < 1$, $c_1 = 4|M|((1 - c_0)/3|M|)^{3/4}$)

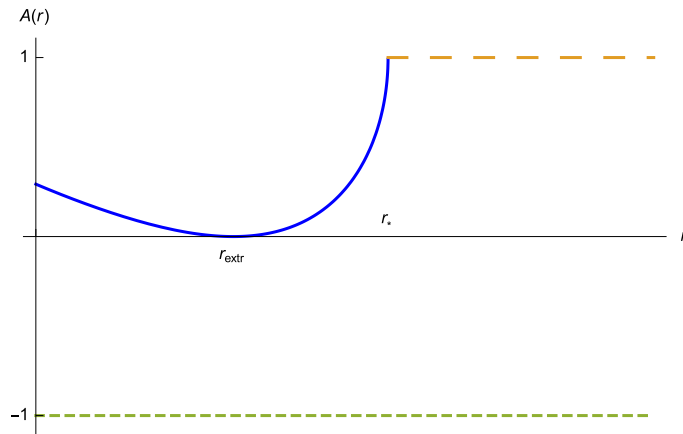


Figure 13. Extremal black hole with two coalescing horizons of Reissner-Nordström-type at r_{extr} within a finite extent internal space region ($0 < r < r_*$), and an infinite external forbidden region ($r_* < r < \infty$).

5.3.4 ($M < 0$, $c_0 > 1$, c_1 any)

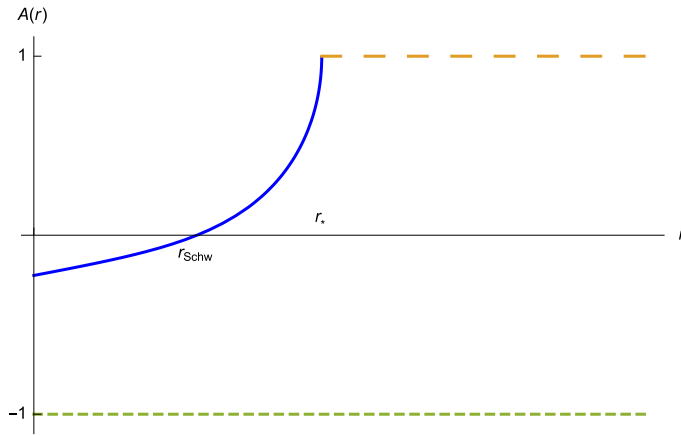


Figure 14. Black hole in a finite-extent internal space region ($0 < r < r_*$ with a Schwarzschild-type horizon at r_{Schw} , and with an infinite external forbidden region ($r_* < r < \infty$).

5.4 Metrics with two forbidden regions ($M < 0$)

5.4.1 ($M < 0$, $c_0 < 0$, $c_1 < 4|M|((1 + |c_0|)/3|M|)^{3/4}$)

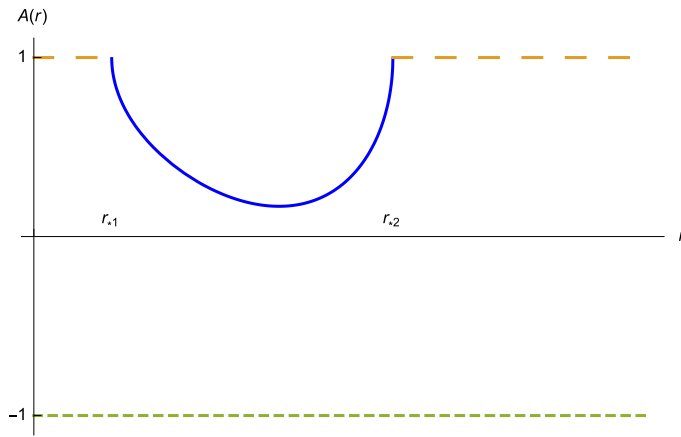


Figure 15. Finite-extent intermediate space region with regular geometry ($r_{1*} < r < r_{2*}$), and with two forbidden regions – one finite-extent internal ($0 < r < r_{1*}$ and one infinite external ($r_{2*} < r < \infty$).

5.4.2 ($M < 0$, $c_0 < 0$, $c_1 > 4|M|((1 + |c_0|)/3|M|)^{3/4}$)

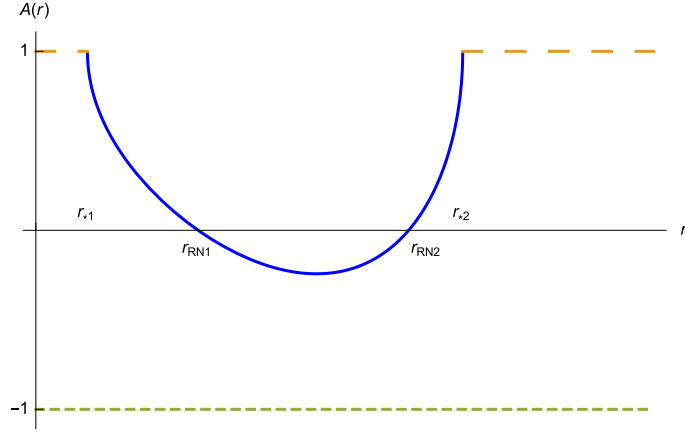


Figure 16. Blackhole with two horizons of Reissner-Nordström-type within a finite-extent intermediate space region ($r_{1*} < r < r_{2*}$), and with two forbidden regions – one finite-extent internal ($0 < r < r_{1*}$ and one infinite external ($r_{2*} < r < \infty$).

5.4.3 ($M < 0$, $c_0 < 0$, $c_1 = 4|M|((1 + |c_0|)/3|M|)^{3/4}$)

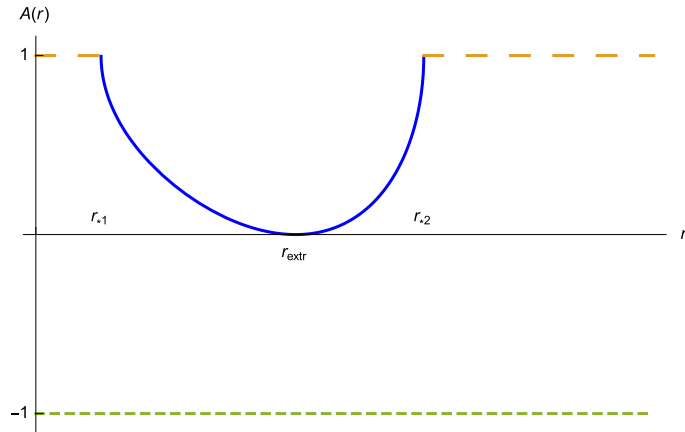


Figure 17. Extremal blackhole with two coalescing horizons of Reissner-Nordström-type within a finite-extent intermediate space region ($r_{1*} < r < r_{2*}$), and with two forbidden regions – one finite-extent internal ($0 < r < r_{1*}$ and one infinite external ($r_{2*} < r < \infty$).

5.5 Metrics with $M = 0$

5.5.1 ($M = 0, c_0 > 1, c_1 > 0$)

Here again $A(r) = 1 - \sqrt{P_4(r)} < 0$ for all r and the metric (51) describes in this case a slowly expanding Kantowski-Sachs universe: $|A(r(\xi))| \sim \xi^{2/3}$ for $\xi \rightarrow \infty$.

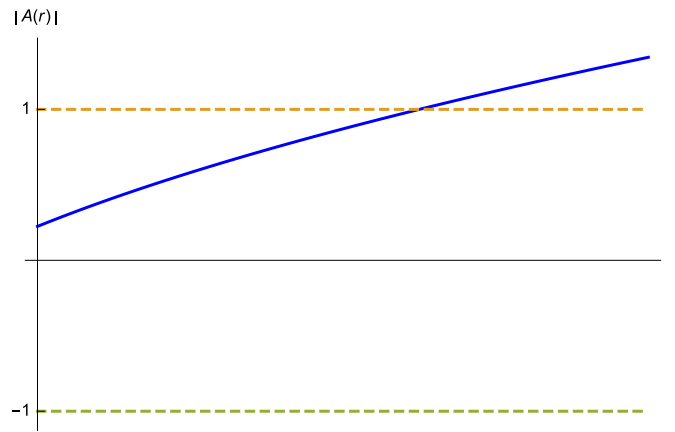


Figure 18. Kantowski-Sachs slowly expanding universe.

5.5.2 ($M = 0, c_0 > 1, c_1 < 0$)

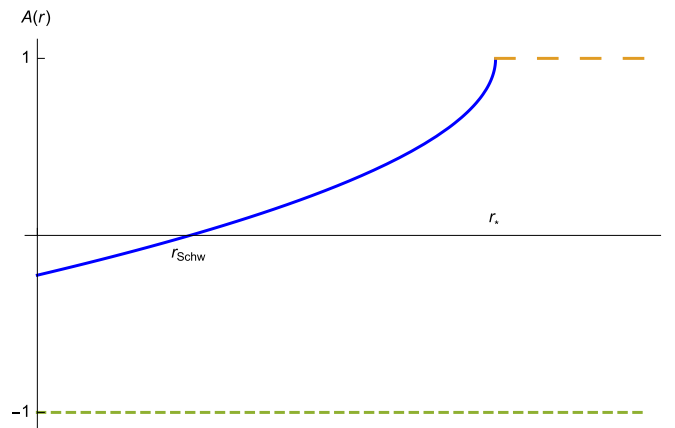


Figure 19. Black hole with a Schwarzschild-type horizon within a finite-extent internal space region ($0 < r < r_*$), and with an infinite external forbidden region ($r_* < r < \infty$).

5.5.3 ($M = 0$, $0 < c_0 < 1$, $c_1 > 0$)

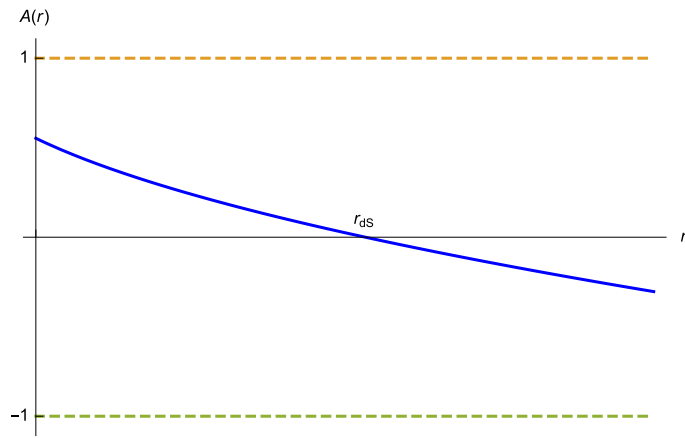


Figure 20. De Sitter-like geometry with a de-Sitter type horizon at r_{dS} .

5.5.4 ($M = 0$, $c_0 < 0$, $c_1 > 0$)

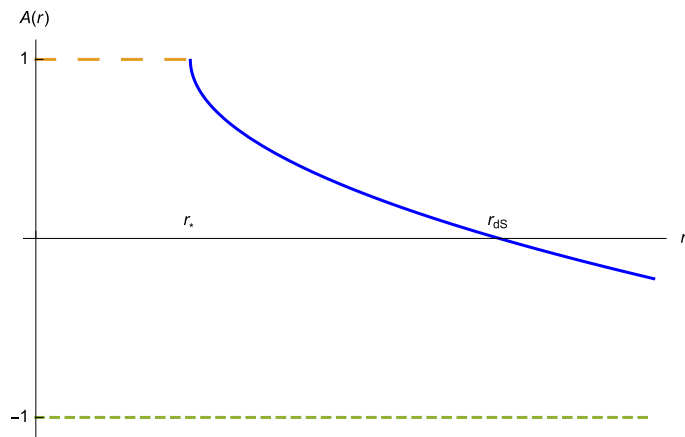


Figure 21. De Sitter-like geometry in an external space region ($r_* < r < \infty$) with a de Sitter type horizon at r_{dS} , and with one finite-extent internal forbidden region ($0 < r < r_*$).

6 Avoiding Spacetime Singularities via Domain Walls

Let us consider two SSS solutions (19) with forbidden regions:

- (a) The one depicted in Figure 9 – describing black hole with two horizons (one Schwarzschild-type and one de Sitter-type) separated by an intermediate finite-extent forbidden region ($r_{*1} < r < r_{*2}$), where $r_{*1,2}$ are two roots of of the 4th-order polynomial under the square root in:

$$A_1(r) = 1 - \sqrt{Mr^4 + c_1^{(1)}r + c_0^{(1)}}, \quad (63)$$

$$(M > 0, c_0^{(1)} > 1, c_1^{(1)} < -4M(c_0/3M)^{3/4}).$$

- (b) The second one graphically depicted on Figure 10 – describing de Sitter-like geometry with a de Sitter-type horizon, and with one internal finite-extent forbidden region ($0 < r < r_*$), where r_* is a root of the 4th-order polynomial under the square root in:

$$A_2(r) = 1 - \sqrt{Mr^4 + c_1^{(2)}r + c_0^{(2)}}, \quad (M > 0, c_0^{(2)} < 0, c_1^{(2)} \text{ any}). \quad (64)$$

Now, we can construct another SSS solution $\widehat{A}(r)$ *without any spacetime singularities* by picking a point $r = \widehat{r}$ with $\widehat{r} < r_{1*}$ (r_{1*} from (63)) and $\widehat{r} > r_*$ (r_* from (64)), and glue together $A_1(r)$ and $A_2(r)$ at $r = \widehat{r}$:

$$\widehat{A}(r) = \begin{cases} A_1(r), & 0 < r \leq \widehat{r} \\ A_2(r), & \widehat{r} \leq r < \infty \end{cases}, \quad (65)$$

so that $\widehat{A}(r)$ is continuous at $r = \widehat{r}$:

$$A_1(\widehat{r}) = A_2(\widehat{r}) \longrightarrow \widehat{r} = \frac{c_0^{(1)} - c_0^{(2)}}{c_1^{(2)} - c_1^{(1)}}, \quad (66)$$

but its first derivative has a discontinuity at $r = \widehat{r}$:

$$[\partial_r \widehat{A}]_{\widehat{r}} \equiv \partial_r A_2(\widehat{r}) - \partial_r A_1(\widehat{r}) = -\frac{c_1^{(2)} - c_1^{(1)}}{2\sqrt{M\widehat{r}^4 + c_1^{(2)}\widehat{r} + c_0^{(2)}}}, \quad (67)$$

and thus the second derivative acquires delta-function contribution at $r = \widehat{r}$:

$$\partial_r^2 \widehat{A} = [\partial_r \widehat{A}]_{\widehat{r}} \delta(r - \widehat{r}) + \dots \quad (68)$$

As in the case of (45) above, Eq. (68) imply presence of thin-shell generated domain wall at $r = \widehat{r}$ where the corresponding brane surface tension matching the delta-function term in (68) is (cf. (47) and (50)):

$$T_\theta^\theta = S_\theta^\theta \delta(r - \widehat{r}), \quad S_\theta^\theta = [\partial_r \widehat{A}]_{\widehat{r}}, \quad (69)$$

Four-Dimensional Gauss-Bonnet Gravity

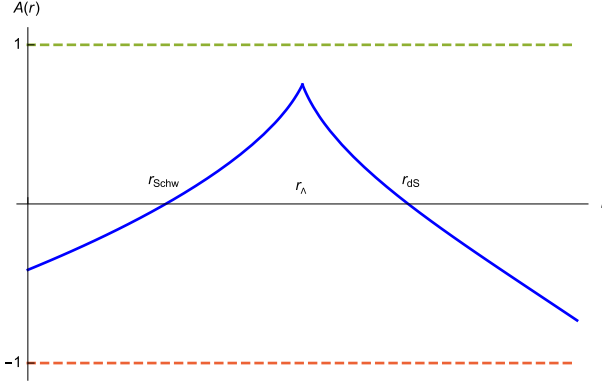


Figure 22. Blackhole with two horizons – internal Schwarzschild-type at r_{Schw} and external de Sitter-type at r_{dS} , and with an additional domain wall at \hat{r} between the two horizons. The section of the graphics for $0 < r < \hat{r}$ is as in Figure 9, whereas the section for $\hat{r} < r < \infty$ is as in Figure 10. Both forbidden regions in Figure 9 and Figure 10 are avoided.

with $[\partial_r \hat{A}]_{\hat{r}}$ as defined in (67). Note that the latter is negative, therefore so is the brane surface tension S_{θ}^{θ} , which confirms the *exotic* nature of the domain wall brane matter, as already pointed out above after Eq. (50).

The graphical representation of (65) (Figure 22) is completely analogous to the case of Figure 7 above, where ($M > 0, c_0 > 1, c_1 = -4M(c_0/3M)^{3/4}$).

The above described “cut and glue together” procedure closely resembles the “cut and paste” formalism for constructing timelike thin-shell wormholes in Ref. [36], ch.15.

7 Discussion and Outlook

In the present paper we have studied in some detail the full class of static spherically symmetric (SSS) solutions in the recently proposed by us new modified Gauss-Bonnet gravity in $D = 4$ based on the formalism of non-Riemannian spacetime volume-forms which avoids the need to couple the Gauss-Bonnet scalar to matter fields or to employ higher powers of the latter as in ordinary $D = 4$ Einstein-Gauss-Bonnet gravity models, where the latter couplings are needed to avoid the ordinary $D = 4$ Gauss-Bonnet density to become total derivative.

The dynamically triggered constancy of the Gauss-Bonnet density due to the equations of motion resulting from the non-Riemannian spacetime volume element by itself completely determines the solutions for the SSS metric component function $g_{00} = -A(r)$ parametrized by three free integration constants. Depending on the signs and values of the latter one finds SSS solutions

with deformed (anti)-de Sitter geometry, black holes of Schwarzschild-de Sitter type, domain walls and Kantowski-Sachs universes (expanding, contracting and bouncing), as well as a multitude of SSS solutions exhibiting physical spacetime singularities *not hidden behind horizons*, which border finite-extent or infinitely large forbidden space regions.

According to the cosmic censorship principle [37] the above class of SSS solutions with naked (visible) spacetime singularities should be ruled out as physically acceptable solutions. However, we showed that it is possible to avoid the singularities by inserting appropriate domain walls and pairwise matching solutions with singularities along the domain wall (a procedure analogous to the construction of timelike thin-shell wormholes in Ref. [36]).

In various cases the field-theoretic Lagrangian actions of the corresponding matter sources for the above SSS gravity solutions are identified – as a complicated nonlinear electrodynamics with a *non-analytic* functional dependence on F^2 (the square of the Maxwell tensor), and in a special case – as the $SO(3)$ nonlinear sigma model (the “hedgehog” scalar field [30]).

An important next task is to study SSS solutions in the more general setting when the composite field $\chi \equiv \frac{\Phi(C)}{\sqrt{-g}}$ (9) will not be “frozen” to a constant, *i.e.*, when one needs to solve the full modified Einstein equations (10) with $\chi = \chi(r)$. Moreover, in the latter case we will need to consider the more general form of SSS metric than (17):

$$ds^2 = -A(r)dt^2 + B(r)dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2), \quad B(r) \neq A^{-1}(r). \quad (70)$$

Inserting the more general SSS ansatz (70) into the system of Eqs. (10) and (15), one gets a very complicated coupled system of highly nonlinear ordinary differential equations of second order which clearly will require numerical treatment.

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