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Evolution of Viscous Dark Energy Models in Brans-Dicke Theory of Gravitation

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Abstract. The present study is based on Bianchi type-V universe filled with dark energy and viscous fluid in the framework of Brans-Dicke theory of gravitation. We have discussed two models: the special form of deceleration parameter model and hybrid law model. It is found that due to viscosity, the effective equation of state parameter of dark energy models behaves like a phantom dark energy. An estimate of the dimensionless jerk and snap parameters are also studied for the dark energy models. It is observe that in the investigated models jerk and snap parameter support the recent observational data.

KEY WORDS: Bianchi type-V universe, Brans-Dicke theory, dark energy, dark matter, jerk and snap parameters.

1 Introduction

The Brans-Dicke (BD) theory is an alternative to general theory relativity (GR) proposed by Brans and Dicke [1] in 1961 by absorbing Mach's principle into gravity. In BD theory, the gravitational constant G of GR is replaced with the inverse of a time-dependent BD scalar field φ. The BD theory describes the recent accelerated expansion of the universe and accommodates the observational data as well [2–4]. Mak and Harko [5] have studied the cosmological model with bulk viscosity in BD theory. Bianchi type string cosmological models with bulk viscosity are studied by Rao and Sireesha [6] in BD Theory of gravitation.

On other hand the study of bulk viscosity has attracted the attention of many researchers. Some of the physicists [7, 8] have been examined that if we assumed ideal cosmic fluid, i.e. non-viscous fluid, give raise to the occurrence of a big rip singularity of the universe in the far future. The singularity problem can be modified or soften by two methods. The first is the effect of quantum corrections due to the conformal anomaly [9–11] and second is to assuming the bulk viscosity of the cosmic fluid [12]. The standard theory of bulk viscosity was first proposed by Eckart [13] in 1940 and later on Landau and Lifshitz [14] modified it in 1987.

In recent years, Brevik *et al.* [15] have studied the DE model in presence of viscosity. Cataldo *et al.* [16] have shown that the negative pressure generated due to bulk viscosity. Amirhashchi *et al.* [17] have investigated that due to viscosity the universe crossing the phantom region in non-interacting and interacting cases. Sharif *et al.* [18] have studied the bulk viscosity in $f(T)$ gravity. Brevik *et al.* [19] have studied the DE cosmological models in viscous fluid cosmology. Amirhashchi [20, 21] has studied the behavior of viscous DE in general theory relativity.

The motive behind the paper is to investigate Bianchi type-V space-time with viscous dark energy in BD theory. We have discussed two models: the special form of deceleration parameter model and hybrid law model. The paper is arranged as follows. In Section 2, the metric and field equations are described. The solutions of the field equations are obtained in Section 3. In Section 4, we have discussed the physical and geometrical properties of the derived models. We have studied the jerk and snap parameters in Section 5. The last Section 6 is dedicated to conclusions.

2 The Metric and Brans-Dicke Field Equations

In this paper we consider the homogeneous and anisotropic Bianchi type-V line element in the form

$$ds^2 = -dt^2 + A^2 dx^2 + e^{2\alpha x}\left[B^2 dy^2 + C^2 dz^2\right], \tag{1}$$

where the scale factors A, B, C are functions of time t only and α is non-zero constant.

The BD field equations are given by

$$R_{ij} - \frac{1}{2}g_{ij}R + \bar{\omega}\varphi^{-2}\Big(\varphi_{,i}\varphi_{,j} - \frac{1}{2}g_{ij}\varphi^{,k}\varphi_{,k}\Big)\varphi^{-1} + \Big(\varphi_{i;j} - g_{ij}\varphi^{;k}_{;k}\Big) = -\varphi^{-1}(T_{ij}) \tag{2}$$

and

$$\varphi^{k}_{;k} = \frac{T}{3+2\bar{\omega}}, \tag{3}$$

where φ is the BD scalar field, $\bar{\omega}$ is the BD parameter and T is the trace of energy momentum tensor T_{ij}.

The energy momentum tensor is defined as

$$T^i_j = T^{(m)i}_j + T^{(de)i}_j, \tag{4}$$

where $T^{(m)i}_j$ and $T^{(de)i}_j$ are energy momentum tensors of dark matter (DM) and DE respectively. These are given by

$$T^{(m)i}_j = \text{diag}\left[-\rho^m, p^m, p^m, p^m\right] = \text{diag}\left[-1, \omega^m, \omega^m, \omega^m\right]\rho^m \tag{5}$$

and

$$T_j^{(de)i} = \text{diag}\left[-\rho^{de}, p^{de}, p^{de}, p^{de}\right] = \text{diag}\left[-1, \omega^{de}, \omega^{de}, \omega^{de}\right]\rho^{de}\,, \tag{6}$$

where ρ^m, ρ^{de} are energy densities and p^m, p^{de} are the pressure of DM and DE respectively, while $\omega^m = p^m/\rho^m$ and $\omega^{de} = p^{de}/\rho^{de}$ are the corresponding EoS parameters of DM and DE.

The BD field Eqs. (2) and (3) for the Bianchi type-V metric are

$$\frac{\dot{A}\dot{B}}{AB} + \frac{\dot{B}\dot{C}}{BC} + \frac{\dot{A}\dot{C}}{AC} - \frac{3\alpha^2}{A^2} - \frac{\bar{\omega}}{2}\frac{\dot{\varphi}^2}{\varphi^2} + \Big(\frac{\dot{A}}{A} + \frac{\dot{B}}{B} + \frac{\dot{C}}{C}\Big)\frac{\dot{\varphi}}{\varphi} = \frac{\rho^m + \rho^{de}}{\varphi}, \tag{7}$$

$$\frac{\ddot{B}}{B} + \frac{\ddot{C}}{C} + \frac{\dot{B}\dot{C}}{BC} - \frac{\alpha^2}{A^2} + \frac{\bar{\omega}}{2}\frac{\dot{\varphi}^2}{\varphi^2} + \frac{\ddot{\varphi}}{\varphi} + \Big(\frac{\dot{B}}{B} + \frac{\dot{C}}{C}\Big)\frac{\dot{\varphi}}{\varphi} = \frac{1}{\varphi}[-\omega^m\rho^m - \omega^{de}\rho^{de}], \tag{8}$$

$$\frac{\ddot{A}}{A} + \frac{\ddot{C}}{C} + \frac{\dot{A}\dot{C}}{AC} - \frac{\alpha^2}{A^2} + \frac{\bar{\omega}}{2}\frac{\dot{\varphi}^2}{\varphi^2} + \frac{\ddot{\varphi}}{\varphi} + \Big(\frac{\dot{A}}{A} + \frac{\dot{C}}{C}\Big)\frac{\dot{\varphi}}{\varphi} = \frac{1}{\varphi}[-\omega^m\rho^m - \omega^{de}\rho^{de}], \tag{9}$$

$$\frac{\ddot{A}}{A} + \frac{\ddot{B}}{B} + \frac{\dot{A}\dot{B}}{AB} - \frac{\alpha^2}{A^2} + \frac{\bar{\omega}}{2}\frac{\dot{\varphi}^2}{\varphi^2} + \frac{\ddot{\varphi}}{\varphi} + \Big(\frac{\dot{A}}{A} + \frac{\dot{B}}{B}\Big)\frac{\dot{\varphi}}{\varphi} = \frac{1}{\varphi}[-\omega^m\rho^m - \omega^{de}\rho^{de}], \tag{10}$$

$$\frac{\dot{B}}{B} + \frac{\dot{C}}{C} = \frac{2\dot{A}}{A} \tag{11}$$

and the wave equation is

$$\ddot{\varphi} + \Big(\frac{\dot{A}}{A} + \frac{\dot{B}}{B} + \frac{\dot{C}}{C}\Big)\dot{\varphi} = \frac{\rho^m(1-3\omega^m) + \rho^{de}(1-3\omega^{de})}{3+2\bar{\omega}}, \tag{12}$$

where an overhead dot indicates differentiation with respect to time t.

The energy conservation equation $T^{ij}_{;j} = 0$ is $T^{(m)}{}^{ij}_{;j} + T^{(de)}{}^{ij}_{;j} = 0$ and is obtained as

$$\dot{\rho}^m + 3H(1+\omega^m)\rho^m + \dot{\rho}^{de} + 3H(1+\omega^{de})\rho^{de} = 0\,, \tag{13}$$

where

$$H = \frac{1}{3}\Big(\frac{\dot{A}}{A} + \frac{\dot{B}}{B} + \frac{\dot{C}}{C}\Big)$$

is the mean Hubble parameter.

3 Solutions of Field Equations

Integrating Eq. (11), we get

$$A^2 = k_0 BC\,, \tag{14}$$

where k_0 is a constant of integration and we assume its value as a unity without loss of any generality. So that we obtain

$$A^2 = BC\,. \tag{15}$$

The field Eqs. (7)–(12) are coupled system of highly non-linear differential equations and hence we need additional conditions to solve it. In order to solve the field equations, we consider the well accepted power-law relation [22] between the average scale factor a and scalar field φ in the following form:

$$\varphi = \varphi_0 a^{\beta}\,, \tag{16}$$

where φ_0, $\beta > 0$ are constants and $a = (ABC)^{1/3}$ is the average scale factor.

From Eqs. (8)–(10), we obtain

$$\frac{\ddot{A}}{A} - \frac{\ddot{B}}{B} + \frac{\dot{C}}{C}\Big(\frac{\dot{A}}{A} - \frac{\dot{B}}{B}\Big) + \Big(\frac{\dot{A}}{A} - \frac{\dot{B}}{B}\Big)\frac{\dot{\varphi}}{\varphi} = 0\,, \tag{17}$$

$$\frac{\ddot{B}}{B} - \frac{\ddot{C}}{C} + \frac{\dot{A}}{A}\Big(\frac{\dot{B}}{B} - \frac{\dot{C}}{C}\Big) + \Big(\frac{\dot{B}}{B} - \frac{\dot{C}}{C}\Big)\frac{\dot{\varphi}}{\varphi} = 0\,, \tag{18}$$

$$\frac{\ddot{A}}{A} - \frac{\ddot{C}}{C} + \frac{\dot{B}}{B}\Big(\frac{\dot{A}}{A} - \frac{\dot{C}}{C}\Big) + \Big(\frac{\dot{A}}{A} - \frac{\dot{C}}{C}\Big)\frac{\dot{\varphi}}{\varphi} = 0\,. \tag{19}$$

Integrating equations (17)–(19), we get

$$\frac{\dot{A}}{A} - \frac{\dot{B}}{B} = \frac{k_1}{\varphi_0 a^{3+\beta}}\,, \tag{20}$$

$$\frac{\dot{B}}{B} - \frac{\dot{C}}{C} = \frac{k_2}{\varphi_0 a^{3+\beta}}\,, \tag{21}$$

$$\frac{\dot{A}}{A} - \frac{\dot{C}}{C} = \frac{k_3}{\varphi_0 a^{3+\beta}}\,, \tag{22}$$

where k_1, k_2 and k_3 are constants of integration.

Eqs. (20)–(22) further reduces to

$$\frac{A}{B} = d_1 \exp\left(k_1 \int \frac{dt}{\varphi_0 a^{3+\beta}}\right), \tag{23}$$

$$\frac{B}{C} = d_2 \exp\left(k_2 \int \frac{dt}{\varphi_0 a^{3+\beta}}\right), \tag{24}$$

$$\frac{A}{C} = d_3 \exp\left(k_3 \int \frac{dt}{\varphi_0 a^{3+\beta}}\right), \tag{25}$$

where d_1, d_2 and d_3 are constants of integration.

Using Eqs. (23)–(25), we can write the metric functions A, B and C explicitly as

$$A\,(t) = a\,, \tag{26}$$

$$B\,(t) = Xa\exp\left(L\int\frac{dt}{\varphi_0 a^{3+\beta}}\right), \tag{27}$$

$$C\,(t) = \frac{1}{X}a\exp\left(-L\int\frac{dt}{\varphi_0 a^{3+\beta}}\right), \tag{28}$$

where $X = (d_2d_3)^{-1/3}$, $L = (k_2+k_3)/3$, $d_2 = d_1^{-1}$, $k_2 = -k_1$.

Hence the model (1) reduced to

$$ds^2 = -dt^2 + a^2\Big[dx^2 + e^{2\alpha x}\Big(X^2 e^{2L\int\frac{dt}{\varphi_0 a^{3+\beta}}}dy^2 + \frac{1}{X^2}e^{-2L\int\frac{dt}{\varphi_0 a^{3+\beta}}}dz^2\Big)\Big]. \tag{29}$$

It is interesting to note that the result of our Bianchi type-V viscous DE model in BD theory resembles to Amirhashchi [21] in GR when the scalar field $\varphi \to 1$.

Using Eqs. (26)–(28) in Eqs. (7)–(10), we get

$$\beta+2)\frac{\ddot a}{a} + \Big[\Big(\frac{2\bar\omega+3}{3}\Big)\beta^2+\beta+1\Big]\frac{\dot a^2}{a^2} + \frac{L^2}{\varphi_0^2 a^{2(\beta+3)}} - \frac{\alpha^2}{a^2} = \frac{1}{\varphi_0 a^\beta}(-p^m - p^{de}), \tag{30}$$

$$\Big(3(\beta+1) - \frac{\bar\omega}{2}\beta^2\Big)\frac{\dot a^2}{a^2} - \frac{L^2}{\varphi_0^2 a^{2(\beta+3)}} - \frac{3\alpha^2}{a^2} = \frac{1}{\varphi_0 a^\beta}(\rho^m + \rho^{de}). \tag{31}$$

Eqs. (30) and (31) further reduce to

$$(\beta+2)\frac{\ddot a}{a} + \Big(\frac{2\bar\omega+3}{3}\beta^2\Big)\frac{\dot a^2}{a^2} + \frac{4L^2}{\varphi_0^2 a^{2(\beta+3)}} = \frac{-1}{\varphi_0 a^\beta}(3p+\rho t) \tag{32}$$

and

$$\Big(3(\beta+1) - \frac{\bar\omega}{2}\beta^2\Big)\frac{\dot a^2}{a^2} - \frac{L^2}{\varphi_0^2 a^{2(\beta+3)}} - \frac{3\alpha^2}{a^2} = \frac{1}{\varphi_0 a^\beta}\rho\,, \tag{33}$$

where $p = p^m + p^{de}$ and $\rho = \rho^m + \rho^{de}$ are the total pressure and the total energy density respectively.

We have considered that DE and DM with $\omega^m = 0$ do not interact with each other. Therefore, the energy conservation Eq. (13) becomes [23]

$$\dot\rho^m + 3H\rho^m = 0 \tag{34}$$

and

$$\dot\rho^{de} + 3H\left(1+\omega^{de}\right)\rho^{de} = 0\,. \tag{35}$$

After integrating Eq. (34), we get

$$\rho^m = \rho_0^m a^{-3}\,. \tag{36}$$

With the help of Eqs. (32), (33) and (36), the energy density and pressure of non-viscous DE are given by

$$\begin{aligned}\rho^{de} =& \varphi_0 a^\beta \Big\{ \Big(3(\beta+1) - \frac{\bar{\omega}}{2}\beta^2\Big)H^2 - L^2\varphi_0^{-2}a^{-2(\beta+3)} \\ & - 3\alpha^2 a^{-2}\Big\} - \rho_0^m a^{-3}\end{aligned} \tag{37}$$

and

$$\begin{aligned}p^{de} =& \varphi_0 a^\beta \Big\{ -(\beta+2)\frac{\ddot{a}}{a} - \Big[\Big(\frac{\bar{\omega}}{2}+1\Big)\beta^2 + (\beta+1)\Big]H^2 \\ & - L^2\varphi_0^{-2}a^{-2(\beta+3)} + \alpha^2 a^{-2}\Big\}.\end{aligned} \tag{38}$$

The EoS parameter of non-viscous DE obtained as

$$\begin{aligned}\omega^{de} = \frac{p^{de}}{\rho^{de}} = & \Bigg[\varphi_0 a^\beta \Big\{ (\beta+2)q - \Big[\Big(\frac{\bar{\omega}}{2}+1\Big)\beta^2 + (\beta+1)\Big] \\ & - L^2\varphi_0^{-2}a^{-2(\beta+3)}H^{-2} + \alpha^2 a^{-2}H^{-2}\Big\}\Bigg] \\ & \times \Bigg[\varphi_0 a^\beta \Big\{ \Big(3(\beta+1) - \frac{\bar{\omega}}{2}\beta^2\Big) - L^2\varphi_0^{-2}a^{-2(\beta+3)}H^{-2} \\ & - 3\alpha^2 a^{-2}H^{-2}\Big\} - 3\Omega_0^m a^{-3}\Bigg]^{-1},\end{aligned} \tag{39}$$

where $q = \left(\frac{-\ddot{a}}{a}\right)/H^2$ is the deceleration parameter and $\Omega_0^m = \frac{\rho_0^m}{3H^2}$ is the current value of dark matter energy density. In this paper we have assume the following expression for the pressure and EoS parameter of the viscous fluid [13]:

$$p_{\text{eff}}^{de} = p^{de} + \Pi \qquad \text{and} \qquad \omega_{\text{eff}}^{de} = \frac{p_{\text{eff}}^{de}}{\rho^{de}}\,, \tag{40}$$

where $\Pi = -\xi(\rho^{de})u^i_{;i}$ is the viscous pressure and $H = \frac{1}{3}u^i_{;i}$ is the Hubble parameter. Here $u^i = (1,0,0,0)$ is the four velocity vector satisfying $u^i u_j = -1$. Here ξ has to be positive value. In general, $\xi\left(\rho^{de}\right) = \xi_0\left(\rho^{de}\right)^\tau$, where $\xi_0\,(>0)$ and τ are constant parameter. In this paper we show that there is a transition from quintessence to phantom region if viscosity is considered.

With the help of Eq. (40) in Eq. (39), the effective EoS parameter of viscous DE is obtained as

$$\omega_{\text{eff}}^{de} = \Bigg[\varphi_0 a^\beta \Big\{(\beta+2)q - \Big[\Big(\frac{\bar{\omega}}{2}+1\Big)\beta^2 + (\beta+1)\Big) - L^2\varphi_0^{-2}a^{-2(\beta+3)}H^{-2} + \alpha^2 a^{-2}H^{-2}\Big\}\Bigg] \times \Bigg[\varphi_0 a^\beta \Big\{\Big(3(\beta+1) - \frac{\bar{\omega}}{2}\beta^2\Big) - L^2\varphi_0^{-2}a^{-2(\beta+3)}H^{-2} - 3\alpha^2 a^{-2}H^{-2}\Big\} - 3\Omega_0^m a^{-3} - \frac{3\xi_0 H^{1-2\eta}}{(3\Omega^{de})^\eta}\Bigg]^{-1}, \quad (41)$$

where $\Omega^{de} = \dfrac{\rho^{de}}{3H^2}$ and $\eta = 1-\tau$.

It has been observed that the cosmos have a transition from deceleration in past to the acceleration at the present, the deceleration parameter shows the signature flipping nature and therefore the deceleration parameter is time dependent not a constant in general. Hence we consider the scale factor as in special form of deceleration parameter and hybrid expansion law.

3.1 Model-I (special form of deceleration parameter)

By analyzing three SNe type Ia samples Cunha and Lima [24] favors early deceleration and late time acceleration. In order to match this observation, we assume well motivated ansatz defined by Singha and Debnath [25] as

$$q = -\frac{\ddot{a}a}{\dot{a}} = -1 + \frac{k}{1+a^k}\,, \quad (42)$$

where $k\ (>0)$ is a constant.

Solving Eq. (42), we obtain mean scale factor a and volume V as

$$a = V^{1/3} = \left(e^{\gamma k t} - 1\right)^{1/k}, \quad (43)$$

where γ is the constant of integration.

Using Eq. (43) for $\gamma = 1$, in Eqs. (26) to (28) we obtained

$$A(t) = a = \left(e^{kt} - 1\right)^{\frac{1}{k}}\ , \tag{44}$$

$$B(t) = X\left(e^{kt} - 1\right)^{\frac{1}{k}} \exp\left\{L\Big[\frac{-1}{(\beta+3)}\left(1 - e^{-kt}\right)^{\frac{(\beta+3)}{k}}\left(e^{kt} - 1\right)^{\frac{-(\beta+3)}{k}} \times {}_2F_1\Big(\frac{\beta+3}{k}, \frac{\beta+3}{k}; \frac{k+\beta+3}{k}; e^{-kt}\Big)\Big]\right\}, \tag{45}$$

$$C(t) = \frac{1}{X}\left(e^{kt} - 1\right)^{\frac{1}{k}} \exp\left\{-L\Big[\frac{-1}{(\beta+3)}\left(1 - e^{-kt}\right)^{\frac{(\beta+3)}{k}}\left(e^{kt} - 1\right)^{\frac{-(\beta+3)}{k}} \times {}_2F_1\Big(\frac{\beta+3}{k}, \frac{\beta+3}{k}; \frac{k+\beta+3}{k}; e^{-kt}\Big)\Big]\right\}, \tag{46}$$

where ${}_2F_1\,(l, m; n; t)$ is the hypergeometric function.

The geometrical and physical parameters are given by

$$\theta = 3H = 3\frac{\dot{a}}{a} = \frac{3e^{kt}}{(e^{kt} - 1)}, \tag{47}$$

$$\Delta = \frac{1}{3}\sum_{i=1}^{3}\Big(\frac{H_i - H}{H}\Big)^2 = \frac{2L^2\left(e^{kt} - 1\right)^{\frac{2(k-\beta-3)}{k}}}{3\varphi_0^2 e^{2kt}}, \tag{48}$$

$$\sigma^2 = \frac{3}{2}\Delta H^2 = \frac{L^2\left(e^{kt} - 1\right)^{\frac{-2(\beta+3)}{k}}}{\varphi_0^2} \tag{49}$$

and

$$q = -1 + ke^{-kt}\,. \tag{50}$$

The non-vanishing component of conformal curvature tensor for the model (29) using Eq. (43) are given by

$$C_{12}^{12} = C_{34}^{34} = \frac{1}{2}\frac{L\,(\beta+2)\,e^{kt}}{\varphi_0\left(e^{kt} - 1\right)^{\frac{\beta+3}{k}+1}} + \frac{1}{3}\frac{L^2}{\varphi_0^2\left(e^{kt} - 1\right)^{\frac{2(\beta+3)}{k}}}, \tag{51}$$

$$C_{13}^{13} = C_{24}^{24} = -\frac{1}{2}\frac{L\,(\beta+2)\,e^{kt}}{\varphi_0\left(e^{kt} - 1\right)^{\frac{\beta+3}{k}+1}} + \frac{1}{3}\frac{L^2}{\varphi_0^2\left(e^{kt} - 1\right)^{\frac{2(\beta+3)}{k}}}, \tag{52}$$

$$C_{14}^{14} = C_{23}^{23} = -\frac{2}{3}\frac{L^2}{\varphi_0^2\left(e^{kt} - 1\right)^{\frac{2(\beta+3)}{k}}}. \tag{53}$$

Adding Eqs. (51) to (53), we obtain

$$C_{12}^{12} + C_{13}^{13} + C_{14}^{14} = 0. \tag{54}$$

Hence our model is conformally flat for large values of t.

With the help of Eq. (43) in Eq. (36), we obtain the energy density of DM as

$$\rho^{m} = \rho_0^{m}\left(e^{kt}-1\right)^{-3/k}. \tag{55}$$

Using Eq. (43) in Eqs. (37) and (38), the energy density and pressure of non-viscous DE are given by

$$\begin{aligned}\rho^{de} &= \varphi_0(e^{kt}-1)^{\beta}\Big\{\Big(3(\beta+1)-\frac{\bar{\omega}}{2}\beta^2\Big)\frac{e^{2kt}}{(e^{kt}-1)^2} \\ &\quad - L^2\varphi_0^{-2}(e^{kt}-1)^{\frac{-2(\beta+3)}{k}} - 3\alpha^2(e^{kt}-1)^{\frac{-2}{k}}\Big\} - \rho_0^{m}(e^{kt}-1)^{\frac{-3}{k}}\end{aligned} \tag{56}$$

and

$$\begin{aligned}p^{de} &= \varphi_0(e^{kt}-1)^{\frac{\beta}{k}}\Big\{-(\beta+2)\Big(\frac{(1-k)e^{2kt}}{(e^{kt}-1)^2}+\frac{ke^{2kt}}{(e^{kt}-1)}\Big) \\ &\quad - \Big[\Big(\frac{\bar{\omega}}{2}+1\Big)\beta^2+(\beta+1)\Big]\frac{e^{2kt}}{(e^{kt}-1)^2} \\ &\quad - L^2\varphi_0^{-2}(e^{kt}-1)^{\frac{-2(\beta+3)}{k}} + \alpha^2(e^{kt}-1)^{\frac{-2}{k}}\Big\}.\end{aligned} \tag{57}$$

The EoS parameter of non-viscous DE is obtained as

$$\begin{aligned}\omega^{de} &= \Big[\varphi_0(e^{kt}-1)^{\frac{\beta}{k}}\Big\{(\beta+2)(ke^{-kt}-1)-\Big(\frac{\bar{\omega}}{2}+1\Big)\beta^2+(\beta+1) \\ &\quad - L^2\varphi_0^{-2}e^{-2kt}(e^{kt}-1)^{\frac{2(k-\beta-3)}{k}} + \alpha^2e^{-2kt}(e^{kt}-1)^{\frac{2(k-1)}{k}}\Big\}\Big] \\ &\quad \times\Big[\varphi_0(e^{kt}-1)^{\frac{\beta}{k}}\Big\{\Big(3(\beta+1)-\frac{\bar{\omega}}{2}\beta^2\Big) - L^2\varphi_0^{-2}e^{-2kt}(e^{kt}-1)^{\frac{2(k-\beta-3)}{k}} \\ &\quad - 3\alpha^2e^{-2kt}(e^{kt}-1)^{\frac{2(k-1)}{k}}\Big\} - 3\Omega_0^{m}(e^{kt}-1)^{\frac{-3}{k}}\Big]^{-1}.\end{aligned} \tag{58}$$

The effective EoS parameter of viscous DE is obtained as

$$\begin{aligned}\omega_{\text{eff}}^{de} &= \Big[\varphi_0(e^{kt}-1)^{\frac{\beta}{k}}\Big\{(\beta+2)(ke^{-kt}-1)-\Big(\frac{\bar{\omega}}{2}+1\Big)\beta^2+(\beta+1) \\ &\quad - L^2\varphi_0^{-2}e^{-2kt}(e^{kt}-1)^{\frac{2(k-\beta-3)}{k}} + \alpha^2e^{-2kt}(e^{kt}-1)^{\frac{2(k-1)}{k}}\Big\}\Big] \\ &\quad \times\Big[\varphi_0(e^{kt}-1)^{\frac{\beta}{k}}\Big\{3(\beta+1)-\frac{\bar{\omega}}{2}\beta^2 - L^2\varphi_0^{-2}e^{-2kt}(e^{kt}-1)^{\frac{2(k-\beta-3)}{k}}\end{aligned}$$

$$- 3\alpha^2 e^{-2kt}(e^{kt}-1)^{\frac{2(k-1)}{k}}\Big\} - 3\Omega_0^m (e^{kt}-1)^{\frac{-3}{k}}\Bigg]^{-1}$$
$$- \frac{3\xi_0\left(\dfrac{e^{kt}}{(e^{kt}-1)}\right)^{1-2\eta}}{(3\Omega^{de})^\eta}. \tag{59}$$

3.2 Model-II (hybrid law)

In this section we use the hybrid expansion law [26] for the scale factor a as

$$a\,(t) = nt^{\alpha_1}e^{\alpha_2 t}, \tag{60}$$

where $n > 0$ and $\alpha_1, \alpha_2 \geq 0$ are constants. Eq. (60) gives the exponential law when $\alpha_1 = 0$ and the power-law when $\alpha_2 = 0$.

Using Eq. (60) in Eqs. (26) to (28) we obtained

$$A\,(t) = nt^{\alpha_1}e^{\alpha_2 t}\,, \tag{61}$$

$$B\,(t) = Xnt^{\alpha_1}e^{\alpha_2 t}\exp\left\{L\int\frac{dt}{\varphi_0\left(nt^{\alpha_1}e^{\alpha_2 t}\right)^{(\beta+3)}}\right\}, \tag{62}$$

$$C\,(t) = \frac{1}{X}nt^{\alpha_1}e^{\alpha_2 t}\exp\left\{-L\int\frac{dt}{\varphi_0\left(nt^{\alpha_1}e^{\alpha_2 t}\right)^{(\beta+3)}}\right\}. \tag{63}$$

The geometrical and physical parameters are given by

$$\theta = \frac{3\,(\alpha_1+\alpha_2 t)}{t}\,, \tag{64}$$

$$\Delta = \frac{2L^2}{3\varphi_0^2}\left(\frac{t}{\alpha_1+\alpha_2 t}\right)^2\left(nt^{\alpha_1}e^{\alpha_2 t}\right)^{-2(\beta+3)}, \tag{65}$$

$$\sigma^2 = \frac{L^2}{\varphi_0^2}\left(nt^{\alpha_1}e^{\alpha_2 t}\right)^{-2(\beta+3)} \tag{66}$$

and

$$q = -1 + \frac{\alpha_1}{(\alpha_1+\alpha_2 t)^2}\,. \tag{67}$$

The non-vanishing component of conformal curvature tensor for the model (29) using Eq. (60) are given by

$$C_{12}^{12} = C_{34}^{34} = \frac{1}{2}\frac{L(\beta+2)(\alpha_1+\alpha_2 t)}{\varphi_0 t(nt^{\alpha_1}e^{\alpha_2 t})^{(\beta+3)}} + \frac{1}{3}\frac{L^2}{\varphi_0^2(nt^{\alpha_1}e^{\alpha_2 t})^{2(\beta+3)}}, \tag{68}$$

$$C_{13}^{13} = C_{24}^{24} = -\frac{1}{2}\frac{L(\beta+2)(\alpha_1+\alpha_2 t)}{\varphi_0 t(nt^{\alpha_1}e^{\alpha_2 t})^{(\beta+3)}} + \frac{1}{3}\frac{L^2}{\varphi_0^2(nt^{\alpha_1}e^{\alpha_2 t})^{2(\beta+3)}}, \tag{69}$$

$$C_{14}^{14} = C_{23}^{23} = -\frac{2}{3}\frac{L^2}{\varphi_0^2(nt^{\alpha_1}e^{\alpha_2 t})^{2(\beta+3)}}\,. \tag{70}$$

Adding Eqs. (68) to (70), we obtain the result resemble with the result in special form of deceleration parameter model.

With the help of Eq. (56) in Eq. (36), we obtain the energy density of DM as

$$\rho^m = \rho_0^m \left(nt^{\alpha_1}e^{\alpha_2 t}\right)^{-3} . \tag{71}$$

Using Eq. (71) in Eqs. (37) and (38), the energy density and pressure of non-viscous DE are given by

$$\begin{aligned}\rho^{de} &= \varphi_0(nt^{\alpha_1}e^{\alpha_2 t})^{\beta}\bigg\{\Big(3(\beta+1)-\frac{\bar{\omega}}{2}\beta^2\Big)\Big(\frac{\alpha_1+\alpha_2 t}{t}\Big)^2 \\ &\quad - L^2\varphi_0^{-2}(nt^{\alpha_1}e^{\alpha_2 t})^{-2(\beta+3)} - 3\alpha^2(nt^{\alpha_1}e^{\alpha_2 t})^{-2}\bigg\} \\ &\quad - \rho_0^m(nt^{\alpha_1}e^{\alpha_2 t})^{-3}\end{aligned} \tag{72}$$

and

$$\begin{aligned}p^{de} &= \varphi_0(nt^{\alpha_1}e^{\alpha_2 t})^{\beta}\bigg\{-(\beta+2)\Big(\frac{\alpha_1(\alpha_1-1)}{t^2}+\frac{2\alpha_1\alpha_2}{t}+\alpha_2^2\Big) \\ &\quad - \Big[\Big(\frac{\bar{\omega}}{2}+1\Big)\beta^2+(\beta+1)\Big]\Big(\frac{\alpha_1+\alpha_2 t}{t}\Big)^2 \\ &\quad - L^2\varphi_0^{-2}(nt^{\alpha_1}e^{\alpha_2 t})^{-2(\beta+3)} + \alpha^2(nt^{\alpha_1}e^{\alpha_2 t})^{-2}\bigg\}.\end{aligned} \tag{73}$$

The EoS parameter of non-viscous DE is obtained as

$$\begin{aligned}\omega^{de} &= \bigg\{\varphi_0(nt^{\alpha_1}e^{\alpha_2 t})^{\beta}\bigg[(\beta+2)\Big(-1+\frac{\alpha_1}{(\alpha_1+\alpha_2 t)^2}\Big) \\ &- \Big[\Big(\frac{\bar{\omega}}{2}+1\Big)\beta^2+(\beta+1)\Big] - L^2\varphi_0^{-2}(nt^{\alpha_1}e^{\alpha_2 t})^{-2(\beta+3)}\Big(\frac{t}{\alpha_1+\alpha_2 t}\Big)^2 \\ &+ \alpha^2(nt^{\alpha_1}e^{\alpha_2 t})^{-2}\Big(\frac{t}{\alpha_1+\alpha_2 t}\Big)^2\bigg]\bigg\}\bigg\{\varphi_0(nt^{\alpha_1}e^{\alpha_2 t})^{\beta}\bigg[\Big(3(\beta+1)-\frac{\bar{\omega}}{2}\beta^2\Big) \\ &- L^2\varphi_0^{-2}(nt^{\alpha_1}e^{\alpha_2 t})^{-2(\beta+3)}\Big(\frac{t}{\alpha_1+\alpha_2 t}\Big)^2 \\ &- 3\alpha^2(nt^{\alpha_1}e^{\alpha_2 t})^{-2}\Big(\frac{t}{\alpha_1+\alpha_2 t}\Big)^2\bigg] - 3\Omega_0^m(nt^{\alpha_1}e^{\alpha_2 t})^{-3}\bigg\}^{-1} .\end{aligned} \tag{74}$$

The effective EoS parameter of viscous DE is obtained as

$$\omega_{\text{eff}}^{de} = \Bigg\{ \varphi_0 (nt^{\alpha_1} e^{\alpha_2 t})^\beta \Big\{ (\beta+2) \Big[-1 + \frac{\alpha_1}{(\alpha_1+\alpha_2 t)^2} \Big] - \Big[\Big(\frac{\bar{\omega}}{2} + 1 \Big) \beta^2 + (\beta+1) \Big] - L^2 \varphi_0^{-2} (nt^{\alpha_1} e^{\alpha_2 t})^{-2(\beta+3)} \Big(\frac{t}{\alpha_1+\alpha_2 t} \Big)^2 + \alpha^2 (nt^{\alpha_1} e^{\alpha_2 t})^{-2} \Big(\frac{t}{\alpha_1+\alpha_2 t} \Big)^2 \Big\} \Bigg\} \times \Bigg\{ \varphi_0 (nt^{\alpha_1} e^{\alpha_2 t})^\beta \Big[\Big(3(\beta+1) - \frac{\bar{\omega}}{2} \beta^2 \Big) - L^2 \varphi_0^{-2} (nt^{\alpha_1} e^{\alpha_2 t})^{-2(\beta+3)} \Big(\frac{t}{\alpha_1+\alpha_2 t} \Big)^2 - 3\alpha^2 (nt^{\alpha_1} e^{\alpha_2 t})^{-2} \Big(\frac{t}{\alpha_1+\alpha_2 t} \Big)^2 \Big] - 3\Omega_0^m (nt^{\alpha_1} e^{\alpha_2 t})^{-3} \Bigg\}^{-1} - \frac{3\xi_0 \Big(\frac{\alpha_1+\alpha_2 t}{t} \Big)^{1-2\eta}}{(3\Omega^{de})^\eta}. \tag{75}$$

4 Results

The physical and geometrical behaviours of the special form of deceleration parameter model and Hybrid law model are as follows.

4.1 The spatial volume V and the expansion scalar θ

From Figures 1 and 2, it is observed that for both the models the spatial volume V is zero at $t = 0$ and expand exponentially as time increases whereas the expansion scalar θ start with extremely large value and continue to decrease with the expansion of the universe (which is big-bang scenario).

4.2 The anisotropy parameter Δ

Figure 3 shows the variation of anisotropy parameter Δ versus time t for special form of deceleration parameter and hybrid law models. From figure we observed that as $t \to \infty$ the anisotropy parameter $\Delta \to 0$, i.e. our models have transition from initial anisotropy to isotropy and which is good harmony with current observations.

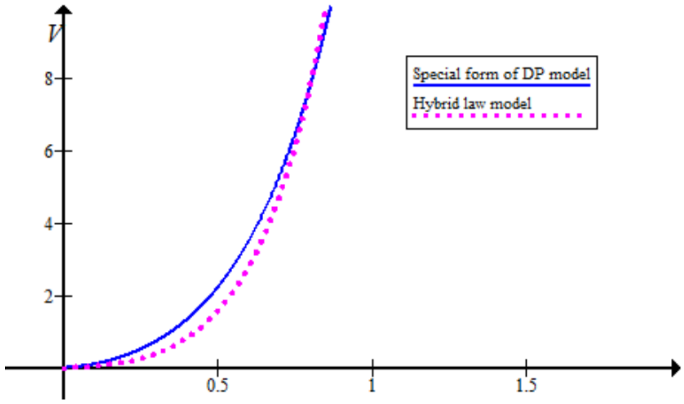

Figure 1. Plot of the volume V of the universe versus time t for $k = 2$, $n = \alpha_2 = 1$, $\alpha_1 = 0.5$.

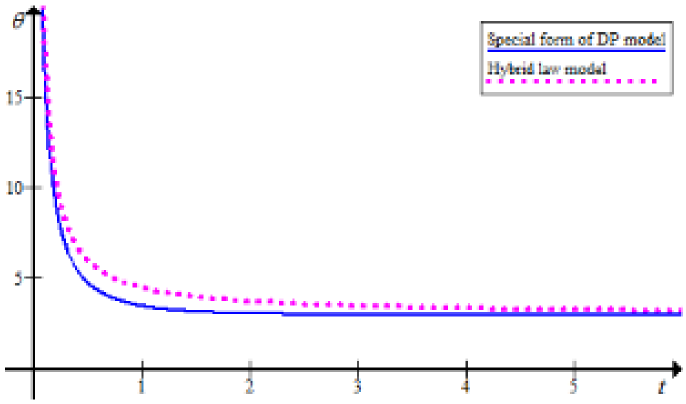

Figure 2. Plot of the scalar expansion θ versus time t for $k = 2$, $n = \alpha_2 = 1$, $\alpha_1 = 0.5$.

4.3 The deceleration parameter q

From the observations of the researchers it is found that the deceleration parameter $q$shows a transition of the universe from initial decelerating phase to present accelerating phase [27–34]. For the acceleration of the universe it is required that the parameter q lies between $-1 < q < 0$ and for decelerating universe condition is $q > 0$. Figure 4 shows the evolution of deceleration parameter q versus time t. It is observed that for different values of k (in special form of deceleration parameter model) and α_1(in hybrid law model) the deceleration parameter q represent two types of models: accelerating model (for $-1 < k < 0$ and $\alpha_1 < 1$) and the model with transition from the early decelerating phase to

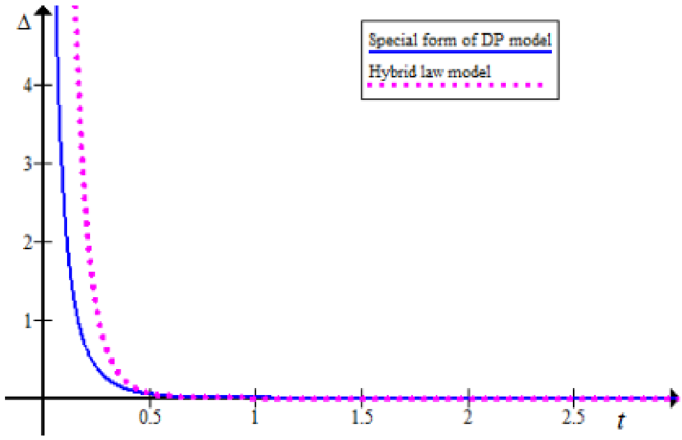

Figure 3. Plot of the anisotropy parameter Δ versus time t for $k = 2$, $L = n = \alpha_2 = 1$, $\alpha_1 = 0.5$ and $\beta = 0.132124$.

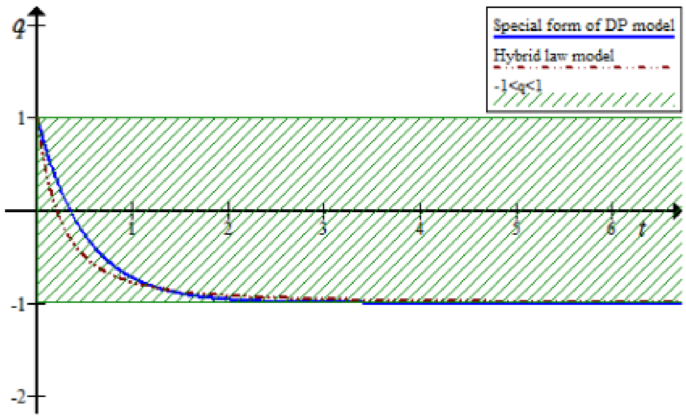

Figure 4. Plot of the deceleration parameter q versus time t for $k = 2$, $\alpha_2 = 1$, $\alpha_1 = 0.5$.

present accelerating phase (for $k > 0$ and $\alpha_1 \geq 1$) which is in good agreement with the recent observations of supernovae.

4.4 The effective equation of state parameter of DE ω_{eff}^{de}

The effective EoS parameter of viscous and non-viscous DE for Model-I: The behavior of the effective EoS parameter of viscous and non-viscous DE versus time t is as shown in Figure 5. From figure it is observed that the EoS parameter of non-viscous DE starts from phantom region $\omega_{\text{eff}}^{de} < -1$ and enter into quintessence region, then after some finite time it approaches to cosmolog-

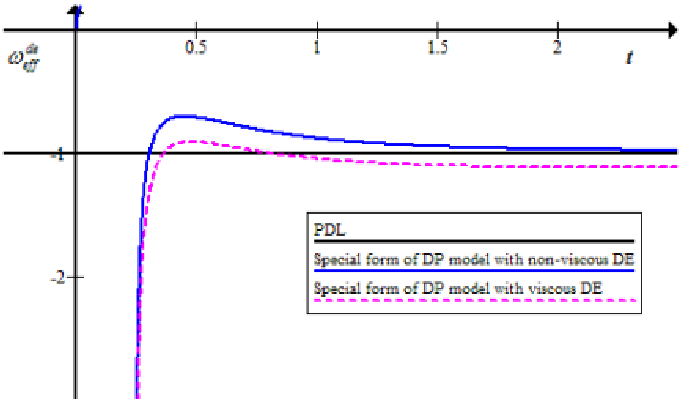

Figure 5. Plot of EoS parameter versus time t for $\beta = 0.132124$, $k = 2$, $\bar{\omega} = L = \varphi_0 = \alpha = 1$, $\eta = 0.5$, $\Omega_0^m = 0.3$.

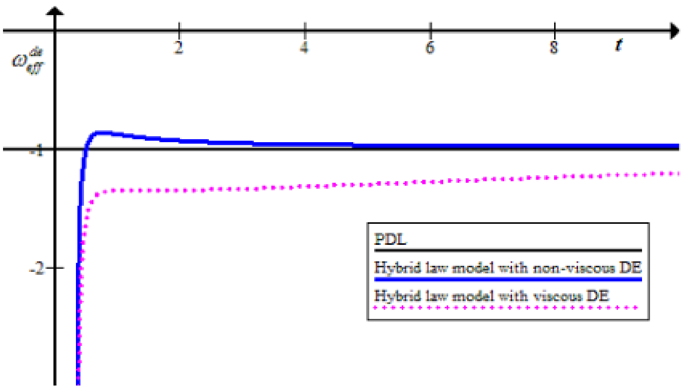

Figure 6. Plot of EoS parameter versus time t for $\beta = 0.132124$, $\bar{\omega} = L = \varphi_0 = \alpha = n = \alpha_2 = 1$, $\alpha_1 = \eta = 0.5$, $\Omega_0^m = 0.3$.

ical constant $\omega_{\text{eff}}^{de} = -1$, whereas the EoS parameter of viscous DE starts from phantom region, after some finite time it crosses PDL and finally reaches in the phantom region.

The effective EoS parameter of viscous and non-viscous DE for Model-II: The behavior of the effective EoS parameter of viscous and non-viscous DE versus time t is as shown in Figure 6. From figure it is observed that, the EoS parameter of non-viscous DE starts from phantom region, crosses PDL and then after some finite time it behaves like a quintessence DE, whereas the EoS parameter of viscous DE starts from phantom region and after some finite time it behaves like a phantom DE. This behavior of EoS parameter is due to the bulk viscosity.

5 The Jerk and Snap Parameters

Among all the cosmological parameters the most fundamental cosmological parameters are Hubble parameter $H = \frac{\dot{a}}{a}$ (where a is average scale factor and $\dot{a}$ is its first derivative) and deceleration parameter $q = -\frac{\ddot{a}a}{\dot{a}}$ (related to second order derivative of a). The other geometrical parameters can be determined by expanding a in the neighborhood of t_0 by Taylor theorem as

$$a(t) = a(t_0 + t - t_0) = a_0 + \frac{(t-t_0)}{1!}\dot{a}_0 + \frac{(t-t_0)^2}{2!}\ddot{a}_0 + \frac{(t-t_0)^3}{3!}\dddot{a}_0 + \frac{(t-t_0)^4}{4!}\ddddot{a}_0 + \cdots, \quad (76)$$

where a_0 is the value of a at present time t_0. The other cosmological parameters like jerk, snap, lerk etc. are defined from Eq. (76). The jerk parameter j located in the third order term of the Taylor series expansion and snap parameter s is related to the fourth order term of the Taylor series expansion. These parameters are defined in [35–37] as

$$j = \frac{\dddot{a}}{aH^3}, \quad (77)$$

$$s = \frac{\ddddot{a}}{aH^4}. \quad (78)$$

For the special form of deceleration parameter model, we obtain

$$j = (1-k)(1-2k) + 3k(1-k)e^{-kt}\left(e^{kt}-1\right) + k^2 e^{-2kt}\left(e^{kt}-1\right)^2 \quad (79)$$

and

$$s = (1-k)(1-2k)(1-3k) + 6k(1-k)(1-2k)e^{-kt}(e^{kt}-1) + 7k^2(1-k)^2 e^{-2kt}(e^{kt}-1)^2 + k^3 e^{-3kt}(e^{kt}-1)^3. \quad (80)$$

For the hybrid law model, we obtain

$$j = \frac{\alpha_1(\alpha_1-1)(\alpha_1-2) + 3\alpha_1\alpha_2(\alpha_1-1)t + 3\alpha_1\alpha_2^2 t^2 + \alpha_2^3 t^3}{(\alpha_1+\alpha_2 t)^3} \quad (81)$$

and

$$s = \Big[\alpha_1(\alpha_1-1)(\alpha_1-2)(\alpha_1-3) + 4\alpha_1\alpha_2(\alpha_1-1)(\alpha_1-2)t + 6\alpha_1(\alpha_1-1)\alpha_2^2 t^2 + 4\alpha_1\alpha_2^3 t^3 + \alpha_2^4 t^4\Big]\Big[\alpha_1+\alpha_2 t\Big]^{-4}. \quad (82)$$

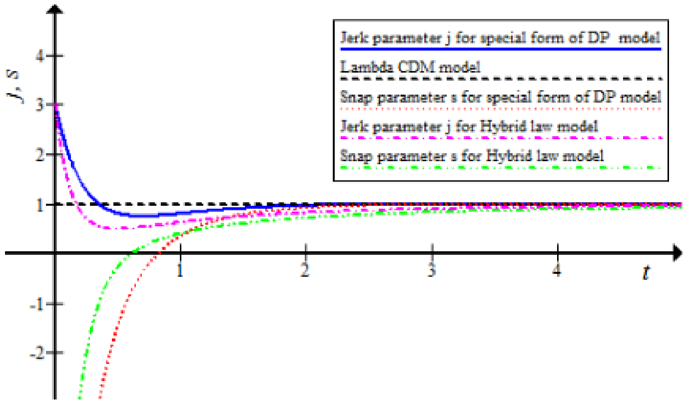

Figure 7. Plot of jerk and snap parameters versus time t for $k = 2$, $\alpha_1 = 0.5$, $\alpha_2 = 1$.

Figure 7 shows the evolution of jerk and snap parameters for both the models versus time t. We compare our dark energy models with standard ΛCDM model by studying the bahavior of the geometrical parameters j and s. At early stage of evolution the value of jerk parameter matches with the value $j = 2.16^{+0.81}_{-0.75}$ obtained from three kinematical data sets: the gold sample of type Ia supernovae [38], the SNIa data from the SNLS project [39] and the X-ray galaxy cluster distance measurements [40]. After some finite time $j \to 1$ this is consistent with the observations of standard ΛCDM [40, 41]. It is observed that for both the models the jerk parameter lies in the positive range throughout its evolution. For both the models, at early stage of evolution the snap parameter start from negative range and after some finite time $s \to 1$ (this is consistent with ΛCDM model) i.e in entire evolution of s there is a transition from negative to positive range (this result resemble with the result of Nagpal *et al.* [41]).

6 Conclusions

In this paper we have presented the Bianchi type-V universe filled with DM and viscous DE in the framework of Brans-Dicke theory of gravitation. Here we have discussed two models: special form of deceleration parameter model and hybrid law model. It is found that in both the models, the universe approaches to isotropy and it is conformally flat for large values of t, which matches with the recent cosmological observations which says that the universe is flat or isotropic on very large scale. The choice of special form of deceleration parameter and hybrid expansion law represents two types of models: i) accelerating model (for $-1 < k < 0$ and $\alpha_1 < 1$); and ii) the model with transition from the early decelerating phase to present accelerating phase (for $k > 0$and $\alpha_1 \geq 1$). In both the models it is observed that due to viscosity the effective equation of state parameter of DE behaves like phantom model of the universe. Also the estimated values of jerk and snap parameter are consistent with ΛCDM model.

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