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Exponential and Logarithmic Expansion of Bianchi Type-III Cosmological Models in $f(T)$ Gravity

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Abstract. The main motive of this paper is to study the Bianchi Type-III cosmological model in the context of $f(T)$ gravity. We construct some $f(T)$ models for expalning the evolutionary behavior of equation of state parameter $\omega_{\rm DE}$ and energy density $\rho^{(*)}_{\rm DE}$. These parameters are calculated in terms of $|T/T_0|$ and redshift z. To solve the field equations of the theory we have used hybrid expansion law which shows a transition from decelerated phase to an accelerated phase in $f(T)$ theory gravity. We discuss these cosmological parameters graphically by taking different values of redshift and EoS parameter. The crossing of phantom divide line can be seen in exponential and combined $f(T)$ models whereas it can not be seen in logarithmic models.

KEY WORDS: Bianchi type III Universe, $f(T)$ theory of gravity, Equation of state parameter, Hybrid expansion law.

1 Introduction

The present Universe is dominated by two dark components containing dark matter (DM) and dark energy (DE) suggests the Type Ia Supernovae observational data. A matter without pressure, which is mainly used to explain galactic curves and large scale structure formation, is know as dark matter, whereas an exotic energy with negative pressure, is used to explain the present cosmic accelerating expansion is known as a dark energy. One of the greatest cosmological and astronomical problem of the 21st century is to understand the origin and nature of dark matter and dark energy. Recently, there are two main ways for the discussion of this accelerated expansion. One way is to introduce scalar fields models in Einstein gravity like phantom [1, 2], quintessence [3–5] and anisotropic fluids [6, 7] etc.

Another way is modifications of Einstein Hilbert action to obtain alternate theories of gravity like $f(R)$ gravity [8] and $f(T)$ gravity [9–12]. The $f(R)$ modified theory produces both cosmic inflation and mimic behavior of dark energy including present cosmic acceleration. $f(R)$ gravity tries to explain early inflation, late-time acceleration and even the flat galactic rotation curves through modification of the Einstein Hilbert action.

A new modified theory, named $f(T)$ theory of gravity, which instead of the curvature, is essentially based on the torsion of the space-time through the Weitzenbock connection. An important advantage of this theory is that its field equations are of second order and hence becomes easy to handle as compared to $f(R)$ theory. Within this theory, various interesting results have been found [13–16]. The relationship between $f(T)$ gravity and k-essence is studied by Myrzakulov [17]. Also, three new $f(T)$ models are investigated and described by Yang [18]. Bamba *et al*. [14] discussed different $f(T)$ models to investigate the cosmological evolutions of EoS parameter for DE.

Spatially homogeneous and anisotropic cosmological models plays a significant role in the description of large scale behavior of Universe and such models have been widely studied in the framework of general relativity in search of a realistic picture of the universe in its early stages. From last forty years, Bianchi type-III models in the presence of dark energy have been studied in general relativity. Adhav *et al.* [19] have investigated Bianchi type-III magnetized wet dark fluid cosmological model in general relativity.

In this paper we have analysed Bianchi type-III universe for some $f(T)$ models. The outline of the paper is: In Section (2) a brief review of $f(T)$ gravity is exposed. In Section (3) we formulate the field equations of $f(T)$ gravity. Section (4) gives the detailed construction of $f(T)$ models. Some concluding remarks are made in Section (5).

2 Some Basics of $f(T)$ gravity

Let us consider the action of $f(T)$ gravity given by [14]

$$I = \frac{1}{16\pi G} \int d^4xe(T + f(T) + L_m) \,, \tag{1}$$

where T is the torsion scalar, $f(T)$ is a general differentiable function of the torsion, L_m corresponds to the matter Lagrangian and $e = \sqrt{-g}$.

Here h^i_μ are the components of the vierbein vector field h_i in a coordinate basis, that is $h_i = h^\mu_i \partial_\mu$. It is noted that in teleparallel gravity, the dynamical variable is the vierbein field $h_i x^\mu$. The variation of action with respect to this vierbein field leads to the following gravitational equations of motion:

$$\left[e^{-1}\partial_\mu\left(eS_i^{\mu\nu}\right) - h_i^\lambda T^\rho_{\mu\lambda}S_\rho^{\nu\mu}\right](1+f(T)) \\ + S_i^{\mu\nu}\partial_\mu(T)\, f_{TT} + \frac{1}{4}h_i^\nu\left[T+f(T)\right] = \frac{1}{2}k^2 h_i^\rho T_\rho^\nu, \quad (2)$$

where $S_i^{\mu\nu} = h_i^\rho S_\rho^{\mu\nu}$, $k^2 = 8\pi G$, $f_T \equiv df/dT$, and the energy momentum tensor is given by

$$T_\rho^\nu = \text{diag}(\rho_\text{M}, -P_\text{M}, -P_\text{M}, -P_\text{M}), \quad (3)$$

where ρ_M is the energy density and P_M is the pressure of matter inside the Universe.

The torsion scalar is given as

$$T = S_\rho^{\mu\nu}T^\rho_{\mu\nu}. \quad (4)$$

Here the torsion, contorsion and antisymmetric tensors are defined as

$$T^\rho_{\mu\nu} = \Gamma^\rho_{\nu\mu} - \Gamma^\rho_{\mu\nu} = h_i^\rho(\partial_\mu h_\nu^i - \partial_\nu h_\mu^i), \quad (5)$$

$$K_\rho^{\mu\nu} = -\frac{1}{2}\left(T_\rho^{\mu\nu} - T_\rho^{\nu\mu} - T_\rho^{\mu\nu}\right), \quad (6)$$

$$S_\rho^{\mu\nu} = \frac{1}{2}\left(K_\rho^{\mu\nu} + \delta_\rho^\mu T_\theta^{\theta\nu} - \delta_\rho^\nu T_\theta^{\theta\mu}\right). \quad (7)$$

The metric tensor is obtained from the dual vierbein as

$$g_{\mu\nu} = \eta_{ij}h_\mu^i h_\nu^j, \quad (8)$$

where $h_i h_j = \eta_{ij}$ and $\eta_{ij} = \text{diag}\,(1,-1,-1,-1)$ is the Minkowski metric for the tangent space satisfying the following properties [9, 20]

$$h_\mu^i h_j^\mu = \delta_j^i, \qquad h_\mu^i h_i^\nu = \delta_\mu^\nu. \quad (9)$$

3 Metric and Field Squations

The spatially homogeneous and anisotropic Bianchi type-III line element is given by [19]

$$ds^2 = dt^2 - A^2dx^2 - B^2e^{-2\alpha x}dy^2 - C^2dz^2, \quad (10)$$

where α is a non-zero constant and A, B, C are functions of cosmic time t.

Using equations (8) and (10), the tetrad components are obtained as

$$h_\mu^i = \text{diag}(1, A, Be^{-\alpha x}, C), \qquad h_i^\mu = \text{diag}(1, A^{-1}, B^{-1}e^{\alpha x}, C^{-1}). \quad (11)$$

After substituting Equations (5) and (7) in equation (4) the torsion T for B-III becomes

$$T = -2\left(\frac{\dot{A}\dot{B}}{AB} + \frac{\dot{B}\dot{C}}{BC} + \frac{\dot{A}\dot{C}}{AC}\right). \tag{12}$$

The average scale factor R, the mean Hubble parameter H and the anisotropy parameter Δ are given by

$$R = (ABC)^{\frac{1}{3}}, \quad H = \frac{1}{3}\Big(\frac{\dot{A}}{A} + \frac{\dot{B}}{B} + \frac{\dot{C}}{C}\Big), \quad \Delta = \frac{1}{3}\sum_{i=1}^{3}\Big(\frac{H_i - H}{H}\Big)^2, \tag{13}$$

whereH_i are the directional Hubble paramerters in x, y and z direction respectively given as,

$$H_1 = \frac{\dot{A}}{A}, \qquad H_2 = \frac{\dot{B}}{B}, \qquad H_3 = \frac{\dot{C}}{C}.$$

For $i = 0 = \nu$ and $i = 1 = \nu$, we get the field equations as follows:

$$T + f(T) - 4\left(\frac{\dot{A}\dot{B}}{AB} + \frac{\dot{B}\dot{C}}{BC} + \frac{\dot{C}\dot{A}}{CA}\right)(1 + f(T)) = 2k^2\rho_{\rm M}, \tag{14}$$

$$\begin{aligned} &2\Big(\frac{\dot{A}\dot{B}}{AB} + 2\frac{\dot{B}\dot{C}}{BC} + \frac{\dot{C}\dot{A}}{CA} + \frac{\ddot{B}}{B} + \frac{\ddot{C}}{C}\Big)(1 + f(T)) \\ &\quad - 4\Big(\frac{\dot{B}}{B} + \frac{\dot{C}}{C}\Big)\Big[\Big(\frac{\ddot{A}}{A} - \frac{\dot{A}^2}{A^2}\Big)\Big(\frac{\dot{B}}{B} + \frac{\dot{C}}{C}\Big) + \Big(\frac{\ddot{B}}{B} - \frac{\dot{B}^2}{B^2}\Big)\Big(\frac{\dot{C}}{C} + \frac{\dot{A}}{A}\Big) \\ &\quad + \Big(\frac{\ddot{C}}{C} - \frac{\dot{C}^2}{C^2}\Big)\Big(\frac{\dot{A}}{A} + \frac{\dot{B}}{B}\Big)\Big] f_{TT} - (T + f(T)) = 2k^2 P_{\rm M}. \end{aligned} \tag{15}$$

By using the Hybrid expansion law

$$V = \left(t^a e^{bt}\right)^{\frac{3}{c}}, \tag{16}$$

where a, b and c are non-negative constants.

The transition of the universe from the initial decelerating phase to the present accelerating phase is shown by the above relation which shows that the obtained deceleration parameter exhibits time dependence. Physical acceptance of such type of parameters is considered or exist. Pradhan and Amirhashchi [21], Pradhan *et al*. [22], Yadav [23], Das and Sultana [24] have also used such type of considerations.

Using this condition, we have found

$$\Delta = 0. \tag{17}$$

Here we have obtained the isotropic behavior of the expanding Universe for $\Delta = 0$. Using this condition the above field equation becomes

$$H^2 = \frac{-8\pi G\rho_{\rm M}}{9} + \frac{f}{18} + \frac{Tf_T}{9}\,, \tag{18}$$

$$(H^2)' = \frac{4\pi G P_{\rm M}}{3(1+f_T)} + \frac{T+f(T)}{12(1+f_T)} + \frac{a}{3ct^2}\,. \tag{19}$$

For $T + f(T) = T$ the above equations reduces to field equations as

$$H^2 = \frac{-8\pi G}{9}(\rho_{\rm M} + \rho_{\rm DE})\,, \tag{20}$$

$$(H^2)' = \frac{4\pi G}{3}(P_{\rm M} + P_{\rm DE}) + \frac{T}{12} + \frac{a}{3ct^2}\,. \tag{21}$$

Here we consider only non-relativistic matter with pressure zero, i.e. $P_{\rm M} = 0$.

The energy density and pressure of the effective dark energy can be described by comparing Eq.(18) with Eq.(20) and Eq.(19) with Eq.(21) as

$$\rho_{\rm DE} = \frac{-1}{16\pi G}(f + 2Tf_T), \tag{22}$$

$$P_{\rm DE} = \frac{3}{4\pi G}\Big(\frac{f(T) - Tf_T}{12(1+f_T)}\Big). \tag{23}$$

The energy conservation equations corresponding to dark matter and dark energy reads as

$$\dot\rho_{\rm M} + 3H\rho_{\rm M} = 0,$$
$$\dot\rho_{\rm DE} + 3H\rho_{\rm DE}(1+\omega_{\rm DE}) = 0.$$

Here, $\omega_{\rm DE} = \dfrac{P_{\rm DE}}{\rho_{\rm DE}}$ is the effective EoS parameter for $f(T)$ gravity.

The equation of state for dark energy is

$$\omega_{\rm DE} = -\Big[\frac{f(T) - Tf_T}{(1+f_T)(f+2Tf_T)}\Big]. \tag{24}$$

4 Cosmological Solutions for Exponential, Logarithmic and Combined Exponential & Logarithmic Models

4.1 Exponential $f(T)$ model

We assume the exponential $f(T)$ model [14,25], given by

$$f(T) = \alpha T\left(1 - e^{PT_0/T}\right) \tag{25}$$

with $\alpha = -\dfrac{1-\Omega_{\rm M}^{(0)}}{1-(1-2p)e^{p}}$ and p is a constant.

We take the current torsion $T_0 = T(z=0)$ and the redshift z is as follows

$$\frac{1}{c}\Big(\frac{a}{t}+b\Big) = \frac{-\dot{Z}}{(1+Z)} \tag{26}$$

$$\Omega_{\rm M}^{(0)} \equiv \rho_{\rm M}^{(0)}/\rho_{\rm crit}^{(0)} = 0.26\,,$$

where $\rho_{\rm M}^{(0)}$ is the present value of energy density of non-relativistic matter which is defined by Komatsu *et al.* [26]; $\rho_{\rm crit}^{(0)} = \dfrac{3H_0^2}{8\pi G}$ is the critical density [27] with H_0 as the present Hubble parameter. Remember that equation (25) has the only parameter p if value of $\Omega_{\rm M}^{(0)}$ is known.

The EoS parameter in terms of $|T/T_0|$ is

$$\omega_{\rm DE} = \frac{PT_0 e^{\frac{PT_0}{T}}}{TE} \tag{27}$$

where

$$E = \Big[1+\alpha(1-e^{PT_0/T}) + \frac{\alpha PT_0}{T}e^{PT_0/T}\Big]\Big[3(1-e^{PT_0/T}) + \frac{2PT_0 e^{PT_0/T}}{T}\Big].$$

Figure 1 shows the graphical behavior of $\omega_{\rm DE}$ as a function of $|T/T_0|$. It can be noted from the left graph that, only for $p=0.1$, $\omega_{\rm DE}$ crosses the phantom divide line but for rest of the values of p, $\omega_{\rm DE}$ approaches to -1 but does not cross the phantom divide line ($\omega_{\rm DE}=-1$). In the right graph it can be seen that for $p=-0.1$, $\omega_{\rm DE}$ crosses the phantom divide line but stays in phantom phase for very little time, then crosses the phantom divide line and stays in non phantom phase permanently and for rest of the values of p, $\omega_{\rm DE}$ approaches to -1 but does

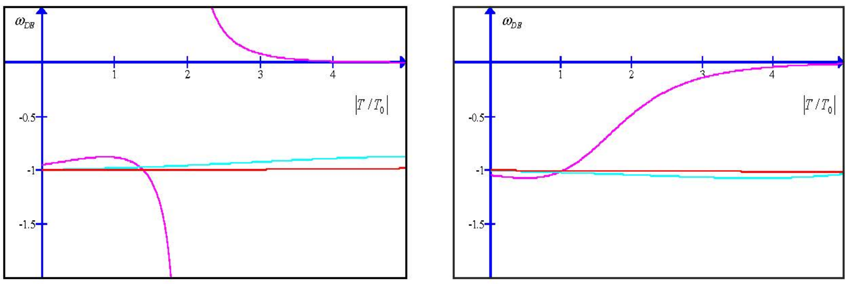


Figure 1. Variation of $\omega_{\rm DE}$ versus $|T/T_0|$ for $p>0$ and $p<0$ in the left and the right graph, respectively.

not cross the phantom divide line ($\omega_{\rm DE} = -1$). So, the universe persists in the Dark Energy era as $|T/T_0|$ tends to infinity.

Equating Eq. (16) in Eq. (12), we get a torsion scalar which is a function of redshift z.

$$T = \frac{-6\dot{Z}^2}{(1+Z)^2}, \qquad T_0 = -6\dot{Z}^2. \tag{28}$$

On using Eq. (28), Eq.(27) becomes

$$\omega_{\rm DE} = \frac{Pe^{P(1+Z)^{-2}}}{D(1+Z)^2}, \tag{29}$$

where

$$D = \left[1 + \alpha(1 - e^{P(1+Z)^{-2}}) + \alpha P(1+Z)^{-2} e^{P(1+Z)^{-2}}\right] \times \left[3(1 - e^{P(1+Z)^{-2}}) + 2P(1+Z)^{-2} e^{P(1+Z)^{-2}}\right].$$

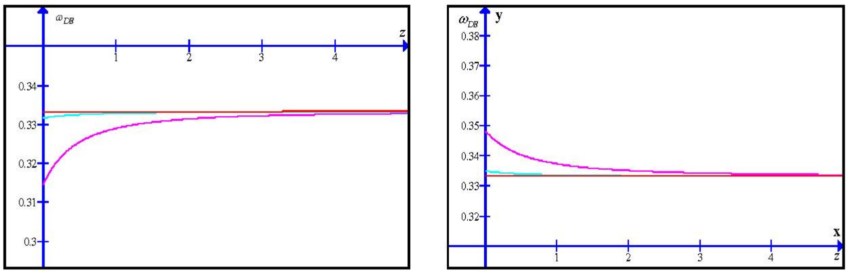


Figure 2. Variation of $\omega_{\rm DE}$ versus z for $p > 0$ and $p < 0$ in the left and the right graph, respectively.

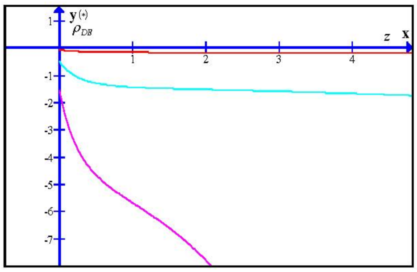
Figure 2 shows the graphical behavior of $\omega_{\rm DE}$ as a function of z. Here the universe remains in the non-phantom phase(quintessence) for both $p > 0$ and $p < 0$.

$$\rho_{\rm DE}^{(*)} = \frac{\dot{Z}^2(1+Z)^{-2}\alpha}{8\pi G \rho_{\rm DE}^{(0)}}\left[3(1 - e^{P(1+Z)^2}) + \frac{2Pe^{P(1+Z)^2}}{(1+Z)^2}\right], \tag{30}$$

where$\rho_{\rm DE}^{(0)} = 0.74{\rho_{\rm crit}}^{(0)}$ [14].

Figure 3 shows the variation of $\rho_{\rm DE}^{(*)}$ in terms of redshift z for $p > 0$ and $p < 0$. It can be seen that for $p > 0$ and smaller values of z, $\rho_{\rm DE}^{(*)}$ expand slightly whereas it takes a constant value for greater values of z. For $p < 0$, at first $\rho_{\rm DE}^{(*)}$ decline with increasing z and reach a constant value as $z \to \infty$. Notice that for both cases of p, $\rho_{\rm DE}^{(*)}$ gain different values at $z = 0$.

By using an approximate method we verify here the relevancy of the exponential $f(T)$ model for phantom and non-phantom phases.



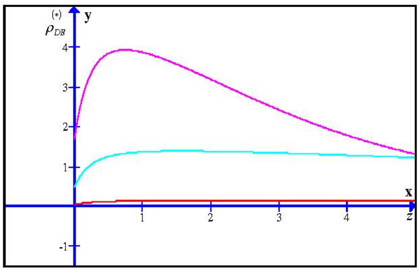


Figure 3. Variation of $\rho_{\rm DE}^{(*)}$ versus redshift z for $p > 0$ and $p < 0$ in the left and the right graph, respectively.

From Eq. (25), we derive

$$f_T = \alpha\Big(1 - e^{PT_0/T} + \frac{PT_0}{T}e^{PT_0/T}\Big). \tag{31}$$

By considering $X = pT_0/T, T_0/T \leq 1$ in Eqs. (25) and (31), it follows that [28] Considering $X = PT_0/T, T_0/T \leq 1$

$$\frac{f}{T} \approx -\alpha\Big(X + \frac{X^2}{2}\Big), \qquad f_T \approx \frac{\alpha X^2}{2}\,. \tag{32}$$

Substituting these values in Eq. (24), $\omega_{\rm DE}$ takes the following form:

$$\omega_{\rm DE} \approx -\frac{(1+X)}{\Big(1 + \dfrac{\alpha X^2}{2}\Big)\Big(1 - \dfrac{X}{2}\Big)}. \tag{33}$$

By taking $\alpha \sim O(1)$, with this assumption the nature of EoS parameter depends on the sign of p as well on X. The non-phantom phase of the universe is followed by $X > 0$ hence $\omega_{\rm DE} > -1$. On the other hand the phantom phase of the universe is followed by $X < 0$ hence $\omega_{\rm DE} < -1$.

4.2 Logarithmic $f(T)$ model

Here we examine the logarithmic $f(T)$ model [14] as given by

$$f(T) = \beta T_0\Big(\frac{qT_0}{T}\Big)^{-1/2} In\Big(\frac{qT_0}{T}\Big), \tag{34}$$

where $\beta \equiv \dfrac{1-\Omega_{\rm M}^{(0)}}{2q^{-1/2}}$ and q is a positive constant.

Here Eq. (34) contains only one parameter q if value of $\Omega_{\rm M}^{(0)}$.

The EoS parameter independent of q is

$$\omega_{\rm DE} = -\Big[\frac{1}{2}\log\Big(\frac{T_0}{T}\Big)+1\Big]\Bigg[\Bigg(2+\Big(\frac{1-\Omega_{\rm M}^{(0)}}{2}\Big)\Big(\frac{T_0}{T}\Big)^{1/2}\log\Big(\frac{T_0}{T}\Big) - (1-\Omega_{\rm M}^{(0)})\Big(\frac{T_0}{T}\Big)^{1/2}\Bigg)\Big(\log\Big(\frac{T_0}{T}\Big)-1\Big)\Bigg]^{-1} \quad (35)$$

On using Eq. (28) in Eq. (35), $\omega_{\rm DE}$ takes the form

$$\omega_{\rm DE} = -\Big[1+\frac{1}{2}\log(1+Z)^2\Big]\Bigg[\Big[2+\Big(\frac{1-\Omega_{\rm M}^{(0)}}{2}\Big)(1+Z)\log(1+Z)^2 - (1-\Omega_{\rm M}^{(0)})(1+Z)^2\Big][\log(1+Z)^2-1]\Bigg]^{-1} \quad (36)$$

since $T=\dfrac{-6\dot{Z}^2}{(1+Z)^2}$ and $T_0=-6\dot{Z}^2$.

Figure 4 shows the graphical representation of $\omega_{\rm DE}$ versus $|T/T_0|$ and $\omega_{\rm DE}$ versus z for $q=1$ and $\Omega_{\rm M}^{(0)}=0.26$. From the left and the right graph we can see that initially the model displays the properties of both matter and radiation.Then the universe enters in DE era and hence we can say that the universe stays in non-phantom phase.

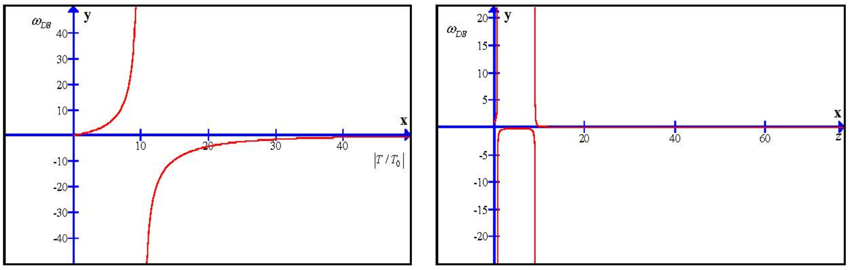


Figure 4. Variation of $\omega_{\rm DE}$ versus $|T/T_0|$ in the left side and $\omega_{\rm DE}$ versus z in right side for logarithmic $f(T)$ model.

4.3 Combined exponential and logarithmic $f(T)$ model

Here we assume the combined exponential and logarithmic $f(T)$ models [14] given by

$$f(T)=\gamma\Bigg[T_0\Big(\frac{uT_0}{T}\Big)^{-1/2}In\Big(\frac{uT_0}{T}\Big)-T(1-e^{uT_0/T})\Bigg]. \quad (37)$$

In this model also only one positive constant parameter u is used. The EoS parameter for DE in terms of $|T/T_0|$ is given as

$$\omega_{\rm DE} = -\frac{1}{I}\Big[\frac{1}{u}\sqrt{\frac{uT_0}{T}}\Big(\frac{1}{2}\log\Big(\frac{uT_0}{T}\Big)-1\Big)-2(1-e^{uT_0/T})-e^{uT_0/T}\frac{uT_0}{T}\Big], \quad (38)$$

where

$$I = \Big[1+\gamma\Big\{\frac{1}{u}\sqrt{\frac{uT_0}{T}}\Big(\frac{1}{2}\log\Big(\frac{uT_0}{T}\Big)-1\Big)-(1-e^{uT_0/T})-e^{uT_0/T}\frac{uT_0}{T}\Big\}\Big] \times\Big[\frac{2}{u}\sqrt{\frac{uT_0}{T}}\Big(\log\Big(\frac{uT_0}{T}\Big)-1\Big)-3(1-e^{uT_0/T})-2e^{uT_0/T}\frac{uT_0}{T}\Big]. \quad (39)$$

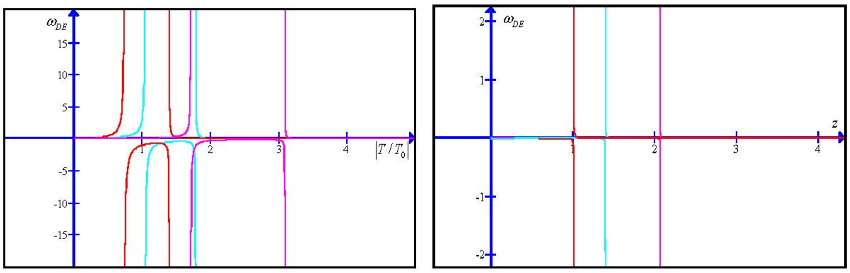


Figure 5. Variation of $\omega_{\rm DE}$ versus $|T/T_0|$ in the left side and $\omega_{\rm DE}$ versus z in right side for combined exponential $f(T)$ model and logarithmic $f(T)$ model.

As $\omega_{\rm DE}$ is a function of $|T/T_0|$, with the increase in $|T/T_0|$, $\omega_{\rm DE}$ decreases and intercepts the phantom divide line. Thereafter $\omega_{\rm DE}$ starts increasing and cuts the phantom divide line and consequently it reaches a constant value.

Using Eq. (28) in Eq. (1my39), we get $\omega_{\rm DE}$ in terms of z as follows:

$$\omega_{\rm DE} = -\frac{1}{F}\Bigg[\frac{1}{u}\sqrt{u(1+Z)^2}\Big(\frac{1}{2}\log(u(1+Z)^2)-1\Big) - 2(1-e^{u(1+Z)^2})-e^{u(1+Z)^2}u(1+Z)^2\Bigg], \quad (40)$$

where

$$F = \Bigg[1+\gamma\Bigg\{\frac{1}{u}\sqrt{u(1+Z)^2}\Big(\frac{1}{2}\log(u(1+Z)^2)-1\Big)-(1-e^{u(1+Z)^2}) - e^{u(1+Z)^2}u(1+Z)^2\Bigg\}\Bigg]\Bigg[\frac{2}{u}\sqrt{u(1+Z)^2}\big(\log(u(1+Z)^2)-1\big) - 3(1-e^{u(1+Z)^2})-2e^{u(1+Z)^2}u(1+Z)^2\Bigg]$$

Initially the universe is in phantom phase for different value of u. $\omega_{\rm DE}$ decreases with increase in z and enters in non-phantom phase by crossing the phantom divide line and attains the constant value.

5 Conclusion

This paper is devoted to the study of well known fact of the universe expansion in the context of $f(T)$ gravity where T denotes the torsion scalar. For Bianchy type-III universe model, we have investigated the cosmological evolution of EoS parameter $\omega_{\rm DE}$ and energy density $\rho_{\rm DE}^{(*)}$ for some renowned $f(T)$ models. To obtain a determinate solution of the field equations we have used hybrid expansion law for the average scale factor. Also we have discussed some geometrical and physical properties of the model with the following observations.

- In Model-I, for $\omega_{\rm DE}$ vs $|T/T_0|$ only for $p = 0.1$ crosses $\omega_{\rm DE}$ phantom divide line for both the cases $p > 0$ and $p < 0$ and for rest of the values $\omega_{\rm DE}$ approaches to $\omega_{\rm DE} = -1$ but does not cross the phantom divide line. The universe survives in the non-phantom phase for $\omega_{\rm DE}$ vs z. In the case of $\rho_{\rm DE}^{(*)}$ vs z, the universe begins in the phantom phantom phase, but when the value of p falls, it crosses the phantom divide line and enters the non-phantom phase, eventually reaching a constant value.
- In Model-II, the crossing of the phantom divide line is not visible.
- In Model-III, for $\omega_{\rm DE}$ vs $|T/T_0|$ reveals that the universe transitions from non-phantom to phantom phase at first. The cosmos then crosses the phantom split line once more and returns to its beginning phase. For $\omega_{\rm DE}$ and z, on the other hand, the phantom divide line is only crossed once from phantom to non-phantom phase.

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