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An Extension of the Explicit Calculation of the Principle of Least Action to a Power Series Potential

Yuji Kajiyama

Gifu Shotoku Gakuen University,
1-1, Takakuwa-Nishi, Yanaizu, Gifu, 501-6194, Japan

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Abstract. We present the general formula of $x(t)$ of a particle moving under any potential expressed by a power series of x without solving the equations of motion. If $x(t)$ is assumed to be expanded by a power series of t with unknown coefficients, one can explicitly perform the time integration of the action. By imposing the boundary conditions and the extremum conditions to the action, all the expansion coefficients and the function of $x(t)$ are determined. This approach will be an alternative way to analyze the motion in classical mechanics based not on the variational principle and the Euler-Lagrange equation.

KEY WORDS: analytical mechanics, inverted pendulum, Morse potential, power series solution, principle of least action.

1 Introduction

In analytical mechanics, the Euler-Lagrange equation as an equation of motion is derived from the principle of least action based on the variational principle [1] (see also Refs. [2–6] for discussions in an educational point of view), and one can analyze the motion of a particle by solving the equation of motion. In our previous paper [7], we have presented an alternative way to analyze the motion of a particle which is not based on the variational principle. In our approach, the motion of a particle can be described by explicitly performing the time integration of the action and by solving the extremum conditions without solving the equation of motion. We assume that the position x can be expressed by a power series of the time t [7–9],

$$x(t) = \sum_{k=0}^{\infty} a_k t^k, \tag{1}$$

with unknown parameters a_k. By performing the time integration of the action

$$S = \int_0^T L dt, \tag{2}$$

in the time region $0 \leq t \leq T$, one obtains the action as a function of $S(T, a_0, a_1, a_2 \cdots)$. Since a different set of the parameters $\{a_k\}$ corresponds to a different path of the particle, there exists the set $\{a_k\}$ such that the action takes the extremum value corresponding to the actual motion of the particle. To find the desirable set of $\{a_k\}$, one imposes the boundary condition $x(0) = a_0$ and $x(T) = D$ instead of $\delta x(0) = \delta x(T) = 0$ in the variational principle method, and then eliminates two degrees of freedom a_0 and a_1. Next, by solving the extremum condition

$$\frac{\partial S}{\partial a_j} = 0, \quad \text{for } j \geq 2, \tag{3}$$

all the parameters $\{a_k\}$ can be determined.

In Ref. [7], we have shown that our method works properly in some specific cases. In the present paper, we extend the discussion to *any* potential which can be expressed by power series of x. We find the general formula for the desirable set $\{a_k\}$ without specifying the form of the potential, and then apply the formula to several nontrivial examples of the potential. Our present approach will be an alternative way of understanding the principle of least action based not on the variational principle and the Euler-Lagrange equation.

2 General Formula

First we derive the general formula to obtain the coefficients a_k which minimize the action. Let us consider the motion of a particle of mass m along the x-axis under the potential $U(x)$ in the time region $0 \leq t \leq T$, and the Lagrangian L is given by

$$L = \frac{1}{2} m \dot{x}^2 - U(x). \tag{4}$$

The action is the sum of the kinetic term S_K and the potential term S_U as $S = S_K + S_U$. We assume that $x(t)$ is expanded by the power series of t as in Eq.(1) which satisfies the boundary conditions

$$x(0) = 0, \;\; x(T) = D \;\; \rightarrow \;\; a_0 = 0, \;\; a_1 T = D - \sum_{k=2}^{\infty} a_k T^k. \tag{5}$$

Thus, a_0 vanishes and a_1 is a function of T, D, and a_k $(k \geq 2)$ [1]. Notice that a_1 corresponds to the initial velocity v_0.

[1] For the case of $a_0 \neq 0$, the same discussions hold for the new variable $y = x - a_0$. Therefore we simply assume $a_0 = 0$. See the example discussed in subsection 3.1.

The kinetic term of the action S_K is given by [7]

$$S_K = \int_0^T \frac{1}{2} m\dot{x}^2 dt = \frac{1}{2} m\Big(\sum_{k,\ell=2}^{\infty} K_{k\ell} a_k a_\ell T^{k+\ell-1} + \frac{D^2}{T} \Big), \tag{6}$$

where

$$K_{k\ell} = \frac{(k-1)(\ell-1)}{k+\ell-1}, \tag{7}$$

and its partial derivative by a_j is

$$\frac{\partial S_K}{\partial a_j} = m \sum_{k=2}^{\infty} K_{kj} a_k T^{k+j-1} = m \sum_{N=0}^{\infty} K_{N+2,j} a_{N+2} T^{N+j+1}, \tag{8}$$

where $N = k - 2$.

As for the potential term S_U defined as

$$S_U = -\int_0^T U(x) dt, \tag{9}$$

we assume that the potential $U(x)$ can be expanded as the power series of x, as

$$U(x) = \sum_{n=0}^{\infty} \frac{1}{n!} U_0^{(n)} x^n = \sum_{n=0}^{\infty} \frac{1}{n!} U_0^{(n)} \sum_{k_1 \cdots k_n = 0}^{\infty} a_{k_1} \cdots a_{k_n} t^{k_1 + \cdots + k_n}, \tag{10}$$

where $U_0^{(n)} = d^n U(x)/dx^n|_{x=0}$. By substituting Eq. (10) into Eq. (9), and after a long calculation, we find

$$\frac{\partial S_U}{\partial a_j} = -\sum_{N=0}^{\infty} \sum_{n=1}^{N+1} \sum_{\delta=1}^{n} \sum_{k_1 \cdots k_{\delta-1}=2}^{N} U_0^{(n)} \frac{1}{(n-\delta)!} \frac{1}{(\delta-1)!} \times \left[\frac{1}{N+j+1} - \frac{1}{N+2} \right] a_1^{n-\delta} a_{k_1} \cdots a_{k_{\delta-1}} \times T^{N+j+1} \delta_{N, k_1 + \cdots + k_{\delta-1} + n - \delta}. \tag{11}$$

See Appendix for the detail of its derivation.

Now we require the coefficients of each order of T^{N+j+1} to be zero as the minimum condition $\partial(S_K + S_U)/\partial a_j = 0$ from Eqs.(8) and (11), and then we obtain

$$a_{N+2} = -\frac{1}{(N+1)(N+2)} \sum_{n=1}^{N+1} \sum_{\delta=1}^{n} \sum_{k_1 \cdots k_{\delta-1}=2}^{N} \frac{1}{m} U_0^{(n)} \frac{1}{(n-\delta)!} \frac{1}{(\delta-1)!} \times a_1^{n-\delta} a_{k_1} \cdots a_{k_{\delta-1}} \delta_{N, k_1 + \cdots + k_{\delta-1} + n - \delta}, \tag{12}$$

N	a_k	Eq. (12)
$N=0$	a_2	$-\frac{1}{2}U_0^{(1)}$
$N=1$	a_3	$-\frac{1}{6}U_0^{(2)}a_1$
$N=2$	a_4	$-\frac{1}{12}\left[U_0^{(2)}a_2+\frac{1}{2}U_0^{(3)}a_1^2\right]$
$N=3$	a_5	$-\frac{1}{20}\left[U_0^{(2)}a_3+U_0^{(3)}a_1a_2+\frac{1}{3!}U_0^{(4)}a_1^3\right]$
$N=4$	a_6	$-\frac{1}{30}\left[U_0^{(2)}a_4+U_0^{(3)}\left(a_1a_3+\frac{1}{2}a_2^2\right)+\frac{1}{2}U_0^{(4)}a_1^2a_2+\frac{1}{4!}U_0^{(5)}a_1^4\right]$
$N=5$	a_7	$-\frac{1}{42}\left[U_0^{(2)}a_5+U_0^{(3)}\left(a_1a_4+a_2a_3\right)+\frac{1}{2}U_0^{(4)}\left(a_1^2a_3+a_1a_2^2\right)\right.$ $\left.+\frac{1}{3!}U_0^{(5)}a_1^3a_2+\frac{1}{5!}U_0^{(6)}a_1^5\right]$
$N=6$	a_8	$-\frac{1}{56}\left[U_0^{(2)}a_6+U_0^{(3)}\left(a_1a_5+a_2a_4+\frac{1}{2}a_3^2\right)\right.$ $+U_0^{(4)}\left(a_1a_2a_3+\frac{1}{2}a_1^2a_4+\frac{1}{3!}a_2^3\right)$ $\left.+U_0^{(5)}\left(\frac{1}{4}a_1^2a_2^2+\frac{1}{3!}a_1^3a_3\right)+\frac{1}{4!}U_0^{(6)}a_1^4a_2+\frac{1}{6!}U_0^{(7)}a_1^6\right]$

Table 1. List of coefficients a_k for $a_0=0$ and $N\leq 6$. The overall factor $1/m$ should be multiplied.

which is the main result of our study. It allows us to find all the parameters a_k sequentially. Notice that $a_1=0$ corresponds to the initial condition $\dot{x}(0)=0$, and $n=\delta$ in this case. Table 1 shows a list of a_k for $a_0=0$ and $N\leq 6$ without the factor $1/m$ calculated from Eq. (12).

In the next section, we will give several nontrivial examples to confirm that the formula Eq. (12) can indeed describe the motion.

3 Examples

3.1 Inverse-square potential

We first present an example of a nonlinear equation of motion of a unit mass $m=1$ in an inverse-square potential given by

$$U(x)=\frac{1}{2x^2}, \tag{13}$$

as an example that has an exact solution. The potential Eq. (13) corresponds to the centrifugal potential $U(r)=\vec{L}^2/(2\mu r^2)$, with the angular momentum $\vec{L}$ and the reduced mass μ in the radial direction.

First we will find the analytic solution of the equation of motion. The equation of motion of a particle of unit mass $m=1$ is $\ddot{x}=x^{-3}$, and its solution under the initial conditions $x(0) = 1,\ \dot{x}(0) = 1$ is given by

$$x(t) = \sqrt{1 + 2t + 2t^2}. \tag{14}$$

Now we define the new parameter $y = x - 1$, and rewrite the potential with respect to y as

$$U(y) = \frac{1}{2(y+1)^2} = \frac{1}{2}\sum_{n=0}^{\infty}(-1)^n(n+1)y^n. \tag{15}$$

The equation of motion for y is $\ddot{y} = (y+1)^{-3}$ and its initial conditions are $y(0) = 0,\ \dot{y}(0) = 1$. Therefore the exact solution is

$$y(t) = \sqrt{1 + 2t + 2t^2} - 1 = t + \frac{1}{2}t^2 - \frac{1}{2}t^3 + \frac{3}{8}t^4 - \frac{1}{8}t^5 + \cdots . \tag{16}$$

The solution Eq. (16) can be obtained from Eq. (12) (or Table 1) by applying the results in the previous section for y. In fact, since $U_0^{(n)} = \frac{1}{2}(-1)^n(n+1)!$ from Eq. (15) and $a_0 = 0,\ a_1 = 1$ corresponding to the initial conditions for y, we obtain the parameter a_k as

$$a_2 = \frac{1}{2}, \quad a_3 = -\frac{1}{2}, \quad a_4 = \frac{3}{8}, \quad a_5 = -\frac{1}{8}, \ldots \tag{17}$$

which reproduces Eq. (16). In this way, by directly applying the principle of least action to the Lagrangian, one can correctly describe the motion without solving the equation of motion.

3.2 Inverted pendulum

As the second example, we consider an inverted pendulum that a mass m attached at the edge of a massless rod of the length ℓ. Under the gravitational acceleration g, the Lagrangian

$$L = \frac{1}{2}m\ell^2\dot{\theta}^2 - mg\ell\cos\theta, \tag{18}$$

gives the equation of motion

$$\ddot{\theta} - \omega^2 \sin\theta = 0, \tag{19}$$

where $\omega = \sqrt{g/\ell}$ and the angle θ of the rod is measured from the vertical. We first confirm that the power series solution Eq. (12) coincides with the analytic solution under a small-angle approximation $|\theta| \ll 1$. Next, we will find that the power series solution agrees well the numerical solution of Eq. (19) in the whole range of θ.

For the small-angle approximation $|\theta| \ll 1$, Eq. (19) reduces to $\ddot{\theta} - \omega^2\theta = 0$. Under the initial condition $\theta(0) = 0$ and $\dot{\theta}(0) = \omega_0$ (which is not related to ω), it can be solved as

$$\begin{aligned} \theta(t) &= \frac{\omega_0}{\omega}\sinh(\omega t) \\ &= \frac{\omega_0}{\omega}\left[\omega t + \frac{1}{3!}(\omega t)^3 + \frac{1}{5!}(\omega t)^5 + \cdots\right], \end{aligned} \tag{20}$$

On the other hand, since the potential is given by

$$U = m\ell^2\omega^2\left(1 - \frac{1}{2}\theta^2\right), \tag{21}$$

$U_0^{(n)} = 0$ except $U_0^{(2)} = -m\omega^2$, where $U_0^{(n)} = d^nU/d(\ell\theta)^n|_{\theta=0}$.

Here we assume that θ is expanded by ωt as

$$\theta(t) = \sum_{k=0}^{\infty} a_k t^k \equiv \sum_{k=0}^{\infty} b_k\,(\omega t)^k, \tag{22}$$

where $a_k = b_k\omega^k$. For instance in the case of $N = 1$, $a_3 = -(1/6)(U_0^{(2)}/m)a_1 = (1/6)\omega^2 a_1$ from Table 1, and thus one obtains $b_3 = (1/3!)b_1$. In a similar fashion, one obtains

$$\begin{aligned} &b_3 = \frac{1}{3!}b_1, \quad b_5 = \frac{1}{5!}b_1, \quad b_7 = \frac{1}{7!}b_1, \ldots, \\ &b_2 = b_4 = b_6 = \cdots = 0, \end{aligned} \tag{23}$$

which is equivalent to Eq. (20) with $b_1 = \omega_0/\omega(a_1 = \omega_0)$.

Next, without any approximation, the potential is given by

$$U = m\ell^2\omega^2\cos\theta, \tag{24}$$

and $U_0^{(n)} = m\omega^2\cos(n\pi/2)$ for all $n \geq 1$. Therefore the coefficients b_k in Eq. (22)

$$\begin{aligned} &b_3 = \frac{1}{3!}b_1, \quad b_5 = \frac{1}{5!}(b_1 - b_1^3), \quad b_7 = \frac{1}{7!}(b_1 - 11b_1^3 + b_1^5), \ldots, \\ &b_2 = b_4 = b_6 = \cdots = 0, \end{aligned} \tag{25}$$

are obtained. Notice that b_k has the higher orders of b_1 corresponding to the deviations from $\sinh(\omega t)$.

Figure1 shows the relation of the angle θ and ωt for $b_1 = 0.02$. Thick black and red solid curve represents the power series solution up to b_7 and the numerical solution of the equation of motion Eq. (19), respectively. Blue dashed curve represents the analytical solution Eq. (20) under the small-angle approximation, which is apart from the numerical solution for $\theta \simeq \pi/2$. One sees that the power series solution agrees with the numerical solution in the whole range of θ.

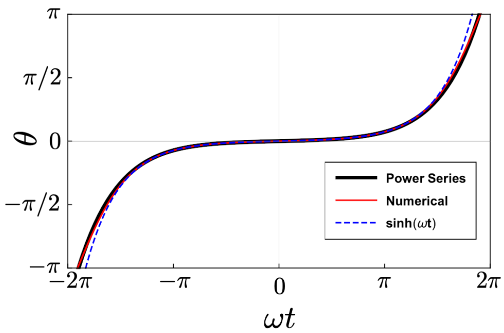

Figure 1. The relation of the angle θ and ωt in the inverted pendulum. The power series solution is drawn by the coefficients up to b_7.

3.3 Morse potential

Finally let us consider the motion under the Morse potential [10], which is an interatomic interaction model of a diatomic molecule, as an example that an analytic solution can not be written by elementary functions. The Lagrangian is given by

$$L = \frac{1}{2} m\dot{x}^2 - U_0 \left(1 - e^{-\alpha x}\right)^2 , \tag{26}$$

where U_0 and α are positive constants. One can readily find $U_0^{(n)} = (2^{n-1} - 1)(-\alpha)^{n-2} m\omega^2$ in this case, where $\omega = \alpha\sqrt{2U_0/m}$. The motion under the Morse potential can be approximately treated as the harmonic oscillation of the angular velocity ω for $|\alpha x| \ll 1$.

Here we assume that x is expanded by ωt as

$$x(t) = \sum_{k=0}^{\infty} a_k t^k \equiv \frac{1}{\alpha} \sum_{k=0}^{\infty} b_k \left(\omega t\right)^k \tag{27}$$

where $a_k = b_k \omega^k/\alpha$. For instance in the case of $N = 1$, $a_3 = -(1/6)(U_0^{(2)}/m)a_1 = -(1/6)\omega^2 a_1$ from Table 1, and thus one obtains $b_3 = -(1/6)b_1$. Therefore, one finds that the parameters b_k up to b_7 are given by

$$\begin{aligned}
&b_2 = 0, \quad b_3 = -\frac{1}{6} b_1, \quad b_4 = \frac{1}{8} b_1^2, \\
&b_5 = -\frac{1}{20}\left(\frac{7}{60} b_1^3 + b_3\right), \\
&b_6 = \frac{1}{30}\left(\frac{5}{8} b_1^4 + 3 b_1 b_3 - b_4\right), \\
&b_7 = -\frac{1}{42}\left(\frac{31}{120} b_1^5 + \frac{7}{2} b_1^2 b_3 - 3 b_1 b_4 + b_5\right), \ldots .
\end{aligned} \tag{28}$$

Figure 2 shows the time evolution of x for $U_0/m = 1$ and $b_1 = 0.1$. Thick black and red solid curve represent the power series solution up to b_{17} and the numerical solution of the equation of motion, respectively. Blue dashed curve represents the exact solution of the harmonic oscillator of the angular velocity ω defined above. The amplitude of the harmonic oscillation is b_1/α, which is depicted by the horizontal dashed lines. It can be seen that the power series solution agrees with the numerical solution for $|\omega t| \lesssim \pi$. The approximation of the power series solution will be better if one takes b_k more.

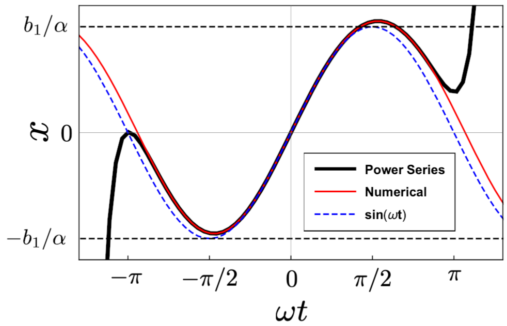

Figure 2. The relation of the position x and ωt in the Morse potential. The power series solution is drawn by the coefficients up to b_{17}.

4 Conclusions

We have found the general formula for the coefficients $\{a_k\}$ of the power series ansatz for $x = \sum a_k t^k$ by explicitly performing the time integration of the action and by solving its extremum conditions for any potential $U(x)$ expressed by a power series of the position x. We presented several nontrivial examples to show that our approach properly describe the motion of a particle. Unlike the standard method of analytical mechanics, our approach is not based on the variational principle and the Euler-Lagrange equation.

It may be simpler to assume another way of expansion of $x(t)$ and $U(x)$, such as a Fourier series expansion, depending on the form of the potential. If the most appropriate way of expansion is taken, it is expected that the convergence of $x(t)$ will be much better.

In this paper, we have restricted ourselves to the motion in one dimension. It will be possible to extend this method to the motion in d dimensions if $2d$ boundary conditions and extremum conditions are imposed. More general discussions are left to future work.

Appendix

Here we show the derivation of Eq. (11). The potential term S_U defined in Eq. (9) is explicitly integrated by assuming the power series expansion Eq. (10) as

$$S_U = -\sum_{n=0}^{\infty}\sum_{\alpha=0}^{n}\sum_{k_1\cdots k_\alpha=2}^{\infty}\frac{1}{n!}U_0^{(n)}\binom{n}{n-\alpha} \times \frac{1}{k_1+\cdots+k_\alpha+(n-\alpha)+1}(a_1T)^{n-\alpha}a_{k_1}\cdots a_{k_\alpha}T^{k_1+\cdots+k_\alpha+1}, \quad (29)$$

where $\binom{n}{n-\alpha}$ is the binomial coefficient, and

$$(a_1T)^{n-\alpha} = (D-\sum_{\ell=2}^{\infty}a_\ell T^\ell)^{n-\alpha} = \sum_{\beta=0}^{n-\alpha}\binom{n-\alpha}{\beta}(-1)^\beta D^{n-(\alpha+\beta)}\sum_{\ell_1\cdots\ell_\beta=2}^{\infty}a_{\ell_1}\cdots a_{\ell_\beta}T^{\ell_1+\cdots+\ell_\beta}, \quad (30)$$

because of Eq. (5).

Since the extremum condition is the differentiation of the action by a_j $(j\geq 2)$, its a_k dependence is given by

$$\frac{\partial S_U}{\partial a_j} \sim \frac{\partial}{\partial a_j}(a_{k_1}\cdots a_{k_\alpha}a_{\ell_1}\cdots a_{\ell_\beta}) = \alpha\delta_{k_\alpha j}a_{k_1}\cdots a_{k_{\alpha-1}}a_{\ell_1}\cdots a_{\ell_\beta} + \beta\delta_{\ell_\beta j}a_{k_1}\cdots a_{k_\alpha}a_{\ell_1}\cdots a_{\ell_{\beta-1}}, \quad (31)$$

which is a sum of products of $\alpha+\beta-1$ coefficients. By renumbering the indices k and ℓ as $k_1\cdots k_{\alpha+\beta-1}$, we obtain

$$\frac{\partial S_U}{\partial a_j} = -\sum_{n=0}^{\infty}\sum_{\alpha=0}^{n}\sum_{\beta=0}^{n-\alpha}\sum_{k_1\cdots k_{\alpha+\beta-1}=2}^{\infty}\frac{1}{n!}U_0^{(n)} \times \binom{n}{n-\alpha}\binom{n-\alpha}{n-(\alpha+\beta)}(-1)^\beta D^{n-(\alpha+\beta)} \times \left[\frac{\alpha}{k_1+\cdots+k_{\alpha-1}+j+(n-\alpha)+1} + \frac{\beta}{k_1+\cdots+k_\alpha+(n-\alpha)+1}\right] \times a_{k_1}\cdots a_{k_{\alpha+\beta-1}}T^{k_1+\cdots+k_{\alpha+\beta-1}+j+1}. \quad (32)$$

Since the constant D can be rewritten by the parameter a_k as

$$D^{n-(\alpha+\beta)} = \Big(a_1T + \sum_{\ell=2}^{\infty} a_\ell T^\ell\Big)^{n-(\alpha+\beta)}$$
$$= \sum_{\gamma=0}^{n-(\alpha+\beta)} \binom{n-(\alpha+\beta)}{n-(\alpha+\beta+\gamma)} (a_1T)^{n-(\alpha+\beta+\gamma)}$$
$$\times \sum_{\ell_1\cdots\ell_\gamma=2}^{\infty} a_{\ell_1}\cdots a_{\ell_\gamma} T^{\ell_1+\cdots+\ell_\gamma}, \quad (33)$$

Equation (32) is as follows:

$$\frac{\partial S_U}{\partial a_j} = -\sum_{n=0}^{\infty}\sum_{\alpha=0}^{n}\sum_{\beta=0}^{n-\alpha}\sum_{\gamma=0}^{n-(\alpha+\beta)}\sum_{k_1\cdots k_{\alpha+\beta+\gamma-1}=2}^{\infty} U_0^{(n)} \frac{(-1)^\beta}{\alpha!\beta!\gamma![n-(\alpha+\beta+\gamma)]!}$$
$$\times \Big[\frac{\alpha}{k_1+\cdots+k_{\alpha-1}+j+(n-\alpha)+1} + \frac{\beta}{k_1\cdots k_\alpha+(n-\alpha)+1}\Big]$$
$$\times (a_1T)^{n-(\alpha+\beta+\gamma)} a_{k_1}\cdots a_{k_{\alpha+\beta+\gamma-1}} T^{k_1+\cdots+k_{\alpha+\beta+\gamma-1}+j+1}, \quad (34)$$

where the indices $\ell_1,\cdots,\ell_\gamma$ have been redefined as $k_{\alpha+\beta},\cdots,k_{\alpha+\beta+\gamma-1}$, and

$$\frac{1}{n!}\binom{n}{n-\alpha}\binom{n-\alpha}{n-(\alpha+\beta)}\binom{n-(\alpha+\beta)}{n-(\alpha+\beta+\gamma)}$$
$$= \frac{1}{\alpha!\beta!\gamma!\,[n-(\alpha+\beta+\gamma)]!}, \quad (35)$$

has been used.

In a certain value of δ defined by $\alpha+\beta+\gamma=\delta$, the terms of $\delta=0$ vanish and those of $\delta\geq 1$ can be non-zero in Eq. (34). In fact, the first and the second term in the square bracket $[\cdots]$ of Eq. (34) has a non-zero value when $\alpha=\delta,\ \beta=\gamma=0$ and $\alpha=\delta-1,\ \beta=1,\ \gamma=0$, respectively, and the other terms are cancelled out each other. Therefore the summations of $\alpha,\ \beta$, and γ are reduced to the summation of δ. Moreover, we define $N=k_1+\cdots+k_{\delta-1}+n-\delta$ and then the order of the each term of Eq. (34) about T is T^{N+j+1}. Therefore we find that Eq. (34) reduces to Eq. (11).

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