

Certain Investigations on Bulk Viscous String Models of the Universe with BVDP

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Abstract. The present communication deals with the tilted Bianchi type- VI_0 bulk viscous string cosmological models in Sáez-Ballester theory of gravitation. To obtain an exact solution of the field equations, we consider three feasible physical relations, *viz.*, expansion scalar θ as proportional to one of the components of the shear scalar σ , bilinear varying deceleration parameter (BVDP) and bulk viscosity coefficient ξ as a simple power function of the energy density ρ . We also discuss the behaviour of physical and geometrical parameters of the obtained models. For the stability of the solution, the nature of energy conditions is explored graphically. The findings of the models are in good agreement with the observational data.

KEY WORDS: bilinear varying deceleration parameter, bulk viscous fluid, strings, Sáez-Ballester theory.

1 Introduction

Observational cosmology [1–7] suggests that the present universe is accelerating. Many cosmologists are trying to study the dynamic nature of the universe by adopting two different ways. Some of them are considering modified or alternate theories of gravity and others are using different types of fluid like quintessence, chaplygin gas, k-essence, tachyon, chameleon fields, phantom and quintom. All these alternatives have their own positive and negative aspects. Although Einstein's general theory of relativity (GTR) is the most acceptable theory to discuss all aspects of the universe but still it is unable to address few specific issues of the cosmos. Many attempts have been done to generalize the GTR such as Brans-Dicke theory [8], Sáez-Ballester theory [9], $f(R, T)$ theory of gravity [10], etc.

In this communication, we confine ourselves to investigate models of the universe in Sáez-Ballester theory of gravitation in which metric field is coupled with a dimensionless scalar field (ϕ) in a simple way. In this theory, the gravitational Lagrangian is considered as $L = -w\phi^n\phi_{,\beta}\phi^{,\beta} + R$, where w & n are the arbitrary constants and R is a scalar curvature. We also believe that this gravity theory provides an adequate description of the weak fields and a solution of missing matter cosmological problem in non-flat cosmology. Various authors including our own peer research group have constructed models of the universe in Sáez-Ballester theory and published fruitful results [11–17].

String cosmology plays a prominent role in the understanding of origin of galaxies, their evolution and large scale structures formation in the universe. It is believed that strings are the topological stable defects that might be generated during the symmetry breaking phase transition in the early cosmos [18]. The presence of the cosmic strings in the early cosmos can be justified through grand unified theories (GUT) [19–22]. These cosmic strings introduces density fluctuations which might lead to the formation of galaxies [23]. They possess stress-energy and are coupled to the gravitational field & hence produce gravitational effects. The formulation of the stress-energy tensor for a cloud of the strings with particles attached to them has been studied by Letelier [24]. Many authors have investigated string models of the universe to discuss the various aspects of the universe [25–27]. Mishra and Dua [28] have investigated Bianchi type-II bulk viscous string cosmological models in Saez-Ballester theory of gravity. Recently, Darabi et al. [29] have found the exact solution of FRW string cosmological model using the Hojman symmetry approach.

The concept of viscous cosmology was first given by Eckart [30]. Gron [31] investigated the Bianchi type-I viscous cosmological models. Research interest in studying bulk viscous properties in cosmic fluid has seen an increase over the past few years. It is believed that the matter behaved like a viscous fluid during the early cosmic evolution when neutrino decoupling occurred. During this era, the effect of viscosity was largest when the temperature was about 1 MeV (10^{10} K). An imbalance between the free paths of neutrinos and other particles gave rise to the viscosity in cosmic fluid.

Literature on the topic have discussed two types of viscous coefficients namely shear and bulk viscosity. However, cosmologists primarily focus on bulk viscosity to study the dynamics of the universe. The reasons include: First, the universe is spatially isotropic on a large scale. Second, cosmological models embedded with viscosity indicate a considerable contribution of the bulk viscosity during inflationary phase [32, 33].

Several studies have reported that bulk viscosity could possibly drive the early cosmic inflation [32, 34]. The inflation driven by bulk viscosity leads to a negative pressure, which in turn gives rise to repulsive gravity and causes the rapid cosmic expansion [35, 36].

As we all know that the present universe seems to be isotropic as well as homogeneous on a large scale, in such case, FRW models play a prominent role in describing various aspects of the universe. But many experimental evidences indicate the existence of anisotropy in the early universe. Therefore, studying Bianchi cosmological models is of considerable interest. These Bianchi universes are spatially homogeneous and anisotropic in nature. In the last ten years, many authors have investigated various Bianchi models of the universe to discuss the overall evolution of the universe. In the present paper, it is assumed that the geometry of the universe is described by tilted Bianchi type-VI₀ metric. In tilted Bianchi type models, fluid flow is not normal to the hyper-surfaces of homogeneity and fluid is not co-moving. The behaviour of the tilted Bianchi type cosmological models has been discussed by authors [37–39].

The Hubble parameter ($H = \dot{a}/a$) and deceleration parameter ($q = -\ddot{a}a/\dot{a}^2$) are two fundamental cosmological variables to study the expansion rate of the universe. Both these parameters originate from the Taylor series expansion of the scale factor $a(t)$ about the present time $t = t_0$. Hubble parameter describes the linear part of the time-dependence of $a(t)$. However, to study the whole dynamical system, it is necessary to examine the behaviour of deceleration parameter (DP) being the first nonlinear part in the series expansion of $a(t)$. The positive sign of the DP infers that the universe is decelerating while negative sign of the DP indicates that expansion is accelerated. For the universe which was decelerating in the past and is currently expanding with an accelerating rate [1–7], DP must indicate the flipping in its sign [40–42]. Therefore, DP is a time-variable quantity. Many authors have investigated cosmological models with time-dependent DP and discussed the dynamic nature of the universe [13, 17, 43–46]. Recently Mishra et al. [47] have constructed FRW models of the universe in Brans-Dicke theory with bilinear varying deceleration parameter (BVDP). Such DP provides an envelope for both constant as well as linearly varying deceleration parameter (LVDP). With this motivation, we investigate bulk viscous string cosmological model with BVDP in Sáez-Ballester theory of gravitation.

In this study, we use gravitational units ($G = c = 1$). The rest of the paper is organised as follows: Section 2 deals with the basic equations related to the model while Section 3 is devoted to finding the solution of field equations with the help of some suitable assumptions. In Section 4, we make use of statefinder diagnostic to discuss different DE models. In Section 5, we explore the time-variation of energy conditions. The main features of the proposed models are discussed in last section *i.e.* Section 6.

2 Basic Equations

Here, we assume that the space-time geometry is described by the spatially homogeneous, anisotropic and tilted Bianchi type-VI₀ metric, being expressed as

$$ds^2 = -dt^2 + a_1^2 dx^2 + a_2^2 e^{-2\alpha x} dy^2 + a_3^2 e^{2\alpha x} dz^2, \quad (1)$$

where, $a_1(t)$, $a_2(t)$ and $a_3(t)$ are the scale factors in x , y and z axes respectively. α is an arbitrary constant having the dimension as inverse of length.

The modified Einstein field equations (EFEs) in Sáez-Ballester theory (Sáez and Ballester, 1986) are given by

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R - w\phi^n \left(\phi_{,\mu}\phi_{,\nu} - \frac{1}{2}g_{\mu\nu}\phi_{,\beta}\phi^{,\beta} \right) = 8\pi T_{\mu\nu}, \quad (2)$$

where, ϕ is a scalar field satisfying the following relation

$$2\phi^n \phi_{;\mu}^{\mu} + n\phi^{n-1}\phi_{,\beta}\phi^{,\beta} = 0, \quad (3)$$

$R_{\mu\nu}$ is the Ricci tensor which is defined by

$$R_{\mu\nu} = \frac{\partial^2 \log \sqrt{-g}}{\partial x^\mu \partial x^\nu} - \frac{\partial \Gamma_{\mu\nu}^k}{\partial x^k} + \Gamma_{\mu m}^l \Gamma_{\nu l}^m - \Gamma_{\mu\nu}^k \frac{\partial \log \sqrt{-g}}{\partial x^k}, \quad (4)$$

$R = g^{\mu\nu} R_{\mu\nu}$ is the Ricci scalar, $g_{\mu\nu}$ is the metric tensor, g is its determinant, $\Gamma_{\mu\nu}^k$ are Christoffel symbols related to $g_{\mu\nu}$ and n & w are arbitrary dimensionless constants. Comma represents the partial derivative and semicolon denotes the covariant derivative.

For perfect fluid distribution, the stress-energy tensor $T_{\mu\nu}$ is given by

$$T_{\mu\nu} = pg_{\mu\nu} + (p + \rho)u_\mu u_\nu, \quad (5)$$

where, ρ is the energy density and p is pressure of the fluid. $u^\mu = (1, 0, 0, 0)$ is the four-velocity vector satisfying $u^\mu u_\mu = -1$. In case of dust, it will reduce to $T_{\mu\nu} = \rho u_\mu u_\nu$. This tensor $T_{\mu\nu}$ is symmetric w.r.t interchange of μ and ν and is divergence-less for energy and momentum to be conserved *i.e.* $T_{;\nu}^{\mu\nu} = 0$.

We investigate the behavior of this model in presence of bulk viscous fluid having one-dimensional cosmic strings. The reason behind assuming the matter fluid of specific interest is due to the fact that history of evolution of the universe shows the dominance of matter fluids during early cosmic evolution whereas, late-time cosmic evolution (present accelerated expansion phase) is filled and driven due to DE only.

Reddy et al. [48] have investigated Kaluza-Klein bulk viscous string model of the universe in Saez-Ballester theory of gravity, whereas, Naidu et al. [49] have investigated same model in Brans-Dicke gravity. Recently, Vinutha et al. [50] have investigated Kantowski-Sachs viscous string cosmological model in Saez-Ballester theory of gravity and reported that the model shows phantom behavior.

Now, for a bulk viscous fluid containing cosmic strings, the stress-energy tensor $T_{\mu\nu}$ is given by

$$T_{\mu\nu} = \bar{p}g_{\mu\nu} + (\bar{p} + \rho)u_\mu u_\nu - \lambda x_\mu x_\nu, \quad (6)$$

where,

$$\bar{p} = p - 3\xi H. \quad (7)$$

Here ρ is the total energy density for a cloud of the strings with particles attached to them, p is the pressure, \bar{p} is the effective pressure, λ is the string tension density, ξ is the coefficient of bulk viscosity and H is the Hubble parameter.

Also, the direction of string is described by a unit space-like vector x^μ . These vectors u^μ and x^μ satisfy the following conditions:

$$u^\mu u_\mu = -x^\mu x_\mu = -1, \quad u^\mu x_\mu = 0. \quad (8)$$

Without any loss of generality, the direction of strings is assumed to be parallel to x-direction so that $x^\mu = \left(0, \frac{1}{a_1}, 0, 0\right)$.

If ρ_p denotes the energy density of the particles then the total energy density of the fluid is given by

$$\rho = \rho_p + \lambda. \quad (9)$$

The gravitational field equations (2) and (3) for the the metric (1) and stress-energy tensor (6) reduce to the following set of nonlinear differential equations:

$$\frac{\ddot{a}_2}{a_2} + \frac{\ddot{a}_3}{a_3} + \frac{\dot{a}_2 \dot{a}_3}{a_2 a_3} + \frac{\alpha^2}{a_1^2} - \frac{w}{2} \phi^n \dot{\phi}^2 = -8\pi(\bar{p} - \lambda), \quad (10)$$

$$\frac{\ddot{a}_1}{a_1} + \frac{\ddot{a}_3}{a_3} + \frac{\dot{a}_1 \dot{a}_3}{a_1 a_3} - \frac{\alpha^2}{a_1^2} - \frac{w}{2} \phi^n \dot{\phi}^2 = -8\pi\bar{p}, \quad (11)$$

$$\frac{\ddot{a}_1}{a_1} + \frac{\ddot{a}_2}{a_2} + \frac{\dot{a}_1 \dot{a}_2}{a_1 a_2} - \frac{\alpha^2}{a_1^2} - \frac{w}{2} \phi^n \dot{\phi}^2 = -8\pi\bar{p}, \quad (12)$$

$$\frac{\dot{a}_1 \dot{a}_2}{a_1 a_2} + \frac{\dot{a}_2 \dot{a}_3}{a_2 a_3} + \frac{\dot{a}_1 \dot{a}_3}{a_1 a_3} - \frac{\alpha^2}{a_1^2} + \frac{w}{2} \phi^n \dot{\phi}^2 = 8\pi\rho, \quad (13)$$

$$\frac{\dot{a}_2}{a_2} - \frac{\dot{a}_3}{a_3} = 0, \quad (14)$$

$$\frac{\ddot{\phi}}{\dot{\phi}} + \frac{n}{2} \frac{\dot{\phi}}{\phi} + \left(\frac{\dot{a}_1}{a_1} + \frac{\dot{a}_2}{a_2} + \frac{\dot{a}_3}{a_3} \right) = 0. \quad (15)$$

The energy conservation equation ($T_{;\nu}^{\mu\nu} = 0$) leads to the following expression:

$$\frac{\dot{\rho}}{\rho} - \frac{\lambda}{\rho} \frac{\dot{a}_1}{a_1} + (\eta + 1) \left(\frac{\dot{a}_1}{a_1} + \frac{\dot{a}_2}{a_2} + \frac{\dot{a}_3}{a_3} \right) = 0, \quad (16)$$

where, $\eta = \frac{\bar{p}}{\rho}$ is the EoS parameter of the cosmic fluid.

The various physical parameters such as spatial volume V , Hubble parameter H , expansion scalar θ , shear scalar σ and anisotropic parameter A_m are defined as

$$V = a_1 a_2 a_3, \quad (17)$$

$$H = \frac{1}{3} \left(\frac{\dot{a}_1}{a_1} + \frac{\dot{a}_2}{a_2} + \frac{\dot{a}_3}{a_3} \right) = \frac{1}{3} (H_1 + H_2 + H_3), \quad (18)$$

where, $H_1 = \frac{\dot{a}_1}{a_1}$, $H_2 = \frac{\dot{a}_2}{a_2}$ and $H_3 = \frac{\dot{a}_3}{a_3}$ are the directional Hubble factors in the directions of x , y and z axes, respectively

$$\theta = 3H, \quad (19)$$

$$\sigma^2 = \frac{1}{2} \left[\left(\frac{\dot{a}_1}{a_1} \right)^2 + \left(\frac{\dot{a}_2}{a_2} \right)^2 + \left(\frac{\dot{a}_3}{a_3} \right)^2 \right] - \frac{1}{6} \theta^2, \quad (20)$$

$$A_m = \frac{1}{3} \sum_{\mu=1}^3 \left(\frac{H_\mu - H}{H} \right)^2. \quad (21)$$

3 Assumptions and Solution of Field Equations

Integration of Eq. (14) gives

$$a_2 = k_1 a_3, \quad (22)$$

where, k_1 is an integrating constant. We choose k_1 as unity for simplicity, so that we have

$$a_2 = a_3. \quad (23)$$

Using the above relation, the gravitational field equations (10)-(14) and the scalar field equation (15) can be further simplified as

$$2 \frac{\ddot{a}_2}{a_2} + \left(\frac{\dot{a}_2}{a_2} \right)^2 + \frac{\alpha^2}{a_1^2} - \frac{w}{2} \phi^n \dot{\phi}^2 = -8\pi(\bar{p} - \lambda), \quad (24)$$

$$\frac{\ddot{a}_1}{a_1} + \frac{\ddot{a}_2}{a_2} + \frac{\dot{a}_1 \dot{a}_2}{a_1 a_2} - \frac{\alpha^2}{a_1^2} - \frac{w}{2} \phi^n \dot{\phi}^2 = -8\pi\bar{p}, \quad (25)$$

$$2 \frac{\dot{a}_1 \dot{a}_2}{a_1 a_2} + \left(\frac{\dot{a}_2}{a_2} \right)^2 - \frac{\alpha^2}{a_1^2} + \frac{w}{2} \phi^n \dot{\phi}^2 = 8\pi\rho, \quad (26)$$

$$\frac{\ddot{\phi}}{\phi} + \frac{n}{2} \frac{\dot{\phi}}{\phi} + \left(\frac{\dot{a}_1}{a_1} + 2 \frac{\dot{a}_2}{a_2} \right) = 0. \quad (27)$$

Eqs. (24)-(27) are four nonlinear differential independent equations involving seven unknown parameters, viz., a_1 , a_2 , p , ϕ , ρ , ξ and λ . Therefore, to solve the field equations explicitly, three additional feasible physical constraints relating these unknowns are needed.

For the above said purpose, we wish to assume the following physical constraints:

i) Expansion scalar θ is proportional to one of the components of shear scalar σ , *i.e.*

$$a_1 = a_2^l, \quad (28)$$

where, $l > 0$ is a constant [51, 52].

ii) We consider the bilinear form of deceleration parameter *i.e.*

$$q(t) = \frac{1 + \alpha_1 t}{1 + \alpha_2 t}, \quad (29)$$

where, $\alpha_1 < 0$ and $\alpha_2 > 0$ are the constants. Both α_1 and α_2 are having dimensions as that of the inverse of time. The above form of DP has been motivated from the authors [47] where $q(t) = \frac{\gamma(1-t)}{1+t}$, $\gamma > 0$ has been considered.

Further, we assume $\alpha_1 = -\alpha_3$, $\alpha_3 > 0$, so that the Eq. (29) takes the form

$$q(t) = \frac{1 - \alpha_3 t}{1 + \alpha_2 t} \quad (\alpha_2, \alpha_3 > 0). \quad (30)$$

From Eq. (30), it can be easily verified that q is positive for $\alpha_3 t < 1$ and q is negative for $\alpha_3 t > 1$. This means DP shows transition in the phase of expansion of the universe *i.e.* from early decelerated phase to present accelerated phase. Also, $q \rightarrow -\alpha_3/\alpha_2$ as $t \rightarrow \infty$.

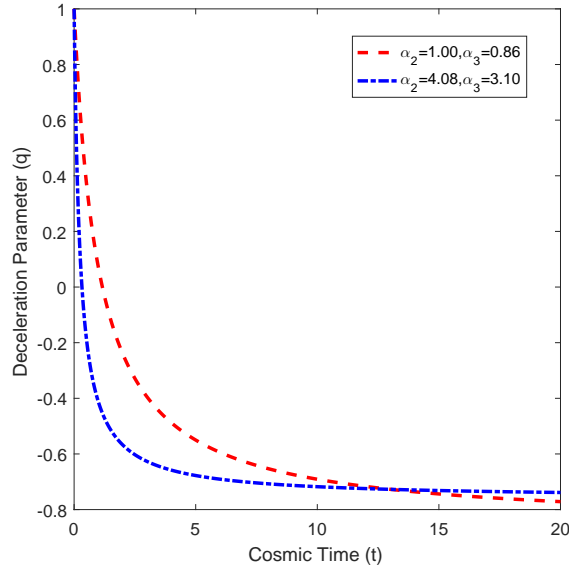


Figure 1. Variation of deceleration parameter q versus cosmic time t .

The observed value of DP at the present epoch ($t_0 = 13.8$ Gyr) is -0.73 [53]. Using these values, the following relationship between the constants α_2 and α_3 can be obtained:

$$13.8\alpha_3 - 10.07\alpha_2 = 1.73. \quad (31)$$

In this paper, we plot all the graphs of the physical parameters for different values of (α_2, α_3) satisfying the above relation. Also, we take the units of time as in billion years (Gyr).

We know that

$$\frac{d}{dt} \left(\frac{1}{H} \right) = q(t) + 1. \quad (32)$$

Integration of Eq. (32) yields

$$\frac{1}{H} = t + \int q(t)dt + c_0, \quad (33)$$

where, c_0 is a constant of integration.

For assumed bilinear form of deceleration parameter, Eq. (33) becomes

$$H = \frac{1}{\alpha_4 t + \alpha_5 \log(1 + \alpha_2 t) + c_1}, \quad (34)$$

where, $\alpha_4 = \left(1 - \frac{\alpha_3}{\alpha_2}\right)$ and $\alpha_5 = \left\{ \frac{1}{\alpha_2} + \frac{\alpha_3}{(\alpha_2)^2} \right\}$.

Here c_1 is a constant. Since expansion rate is very large at time of inflation, therefore $c_1 = 0$. Eq. (34) now becomes

$$H = \frac{1}{\alpha_4 t + \alpha_5 \log(1 + \alpha_2 t)}. \quad (35)$$

Further, H can be expanded as

$$H = \frac{1}{\alpha_6 t} \left[1 - \alpha_7 \left\{ \frac{t}{2} - \frac{\alpha_2 t^2}{3} + \frac{(\alpha_2)^2 t^3}{4} - \frac{(\alpha_2)^3 t^4}{5} + \dots \right\} \right]^{-1}, \quad (36)$$

where, $\alpha_6 = \alpha_4 + \alpha_5 \alpha_2$ and $\alpha_7 = \frac{\alpha_5 (\alpha_2)^2}{\alpha_6}$.

$$\begin{aligned} H = & \frac{1}{\alpha_6 t} + \frac{\alpha_7}{\alpha_6 t} \left\{ \frac{t}{2} - \frac{\alpha_2 t^2}{3} + \frac{(\alpha_2)^2 t^3}{4} - \frac{(\alpha_2)^3 t^4}{5} + \dots \right\} \\ & + \frac{(\alpha_7)^2}{\alpha_6 t} \left\{ \frac{t}{2} - \frac{\alpha_2 t^2}{3} + \frac{(\alpha_2)^2 t^3}{4} - \frac{(\alpha_2)^3 t^4}{5} + \dots \right\}^2 \\ & + \frac{(\alpha_7)^3}{\alpha_6 t} \left\{ \frac{t}{2} - \frac{\alpha_2 t^2}{3} + \frac{(\alpha_2)^2 t^3}{4} - \frac{(\alpha_2)^3 t^4}{5} + \dots \right\}^3 + \dots, \end{aligned} \quad (37)$$

$$H = k_0 \frac{1}{t} + k_1 + k_2 t + k_3 t^2 + k_4 t^3 + O(t^4), \quad (38)$$

where,

$$\begin{aligned} k_0 &= \frac{1}{2}, & k_1 &= \frac{\alpha_2 + \alpha_3}{8}, \\ k_2 &= \frac{(\alpha_2 + \alpha_3)(3\alpha_3 - 5\alpha_2)}{96}, \\ k_3 &= \frac{\alpha_2 + \alpha_3}{4} \left\{ \frac{(\alpha_2 + \alpha_3)^2}{32} - \frac{\alpha_2(\alpha_2 + \alpha_3)}{6} + \frac{(\alpha_2)^2}{4} \right\}, \\ k_4 &= \frac{\alpha_2 + \alpha_3}{4} \left\{ \frac{(\alpha_2 + \alpha_3)^3}{128} - \frac{\alpha_2(\alpha_2 + \alpha_3)^2}{16} + \frac{13(\alpha_2)^2(\alpha_2 + \alpha_3)}{72} - \frac{(\alpha_2)^3}{5} \right\}. \end{aligned}$$

Since $H = \frac{1}{3} \left(\frac{\dot{a}_1}{a_1} + \frac{\dot{a}_2}{a_2} + \frac{\dot{a}_3}{a_3} \right)$, so on integration, we get the following expression:

$$a_1 a_2 a_3 = (c_2 \sqrt{t})^3 \exp\{3M_1(t)\}, \quad (39)$$

where, c_2 is a constant of integration and

$$M_1(t) = k_1 t + \frac{k_2 t^2}{2} + \frac{k_3 t^3}{3} + \frac{k_4 t^4}{4} + O(t^5).$$

With the help of Eqs. (23), (28) and (39), we obtain the following values of metric potentials:

$$a_1 = [c_2 \sqrt{t} \exp\{M_1(t)\}]^{\frac{3l}{l+2}}, \quad (40)$$

$$a_2 = a_3 = [c_2 \sqrt{t} \exp\{M_1(t)\}]^{\frac{3}{l+2}}. \quad (41)$$

Using Eqs. (27) and (39), we obtain the following value of the scalar field ϕ :

$$\phi = \left\{ \frac{n+2}{2} \left(\int \frac{\phi_0}{(c_2 \sqrt{t})^3 \exp\{3M_1(t)\}} dt + \phi_1 \right) \right\}^{\frac{2}{n+2}}, \quad (42)$$

where, ϕ_0 and ϕ_1 are the constants of integration.

Using Eqs. (40)-(41) in (24)-(26), we get the following expressions for total energy density ρ , effective pressure \bar{p} and string tension density λ :

$$\begin{aligned} 8\pi\rho &= \frac{9(2l+1)}{(l+2)^2} H^2 - \alpha^2 [c_2 \sqrt{t} \exp\{M_1(t)\}]^{\frac{-6l}{l+2}} \\ &\quad + \frac{w}{2} \phi_0^2 [c_2 \sqrt{t} \exp\{M_1(t)\}]^{-6}, \end{aligned} \quad (43)$$

$$\begin{aligned} -8\pi\bar{p} &= \frac{3(l+1)}{l+2} \dot{H} + \frac{9(l^2+l+1)}{(l+2)^2} H^2 - \alpha^2 [c_2 \sqrt{t} \exp\{M_1(t)\}]^{\frac{-6l}{l+2}} \\ &\quad - \frac{w}{2} \phi_0^2 [c_2 \sqrt{t} \exp\{M_1(t)\}]^{-6}, \end{aligned} \quad (44)$$

$$8\pi\lambda = \frac{3(1-l)}{l+2} \dot{H} + \frac{9(1-l)}{l+2} H^2 + 2\alpha^2 [c_2 \sqrt{t} \exp\{M_1(t)\}]^{\frac{-6l}{l+2}} \quad (45)$$

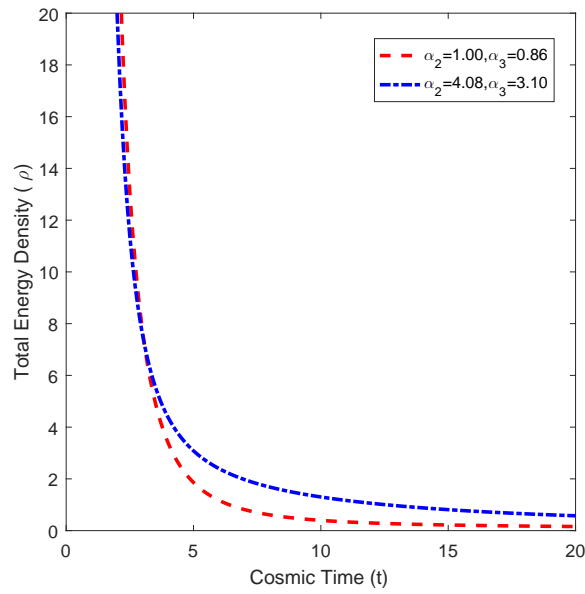


Figure 2. Variation of energy density ρ versus cosmic time t for $l = 1.326$ and $w = \phi_0 = \alpha = c_2 = 1$

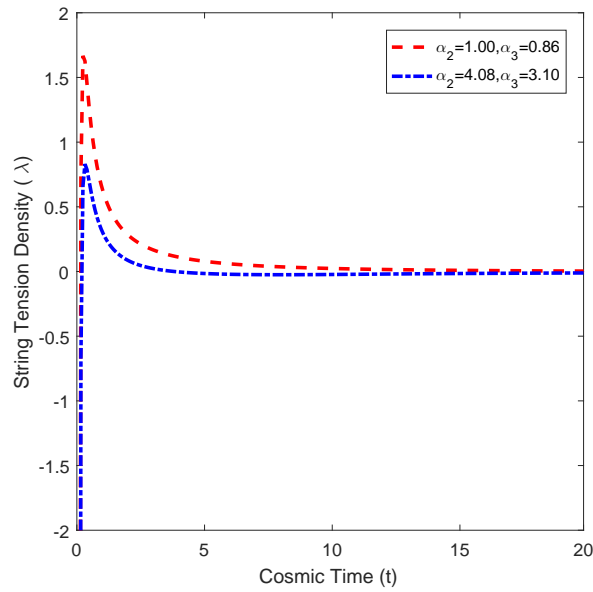


Figure 3. Variation of string tension density λ versus cosmic time t for $l = 1.326$ and $\alpha = c_2 = 1$

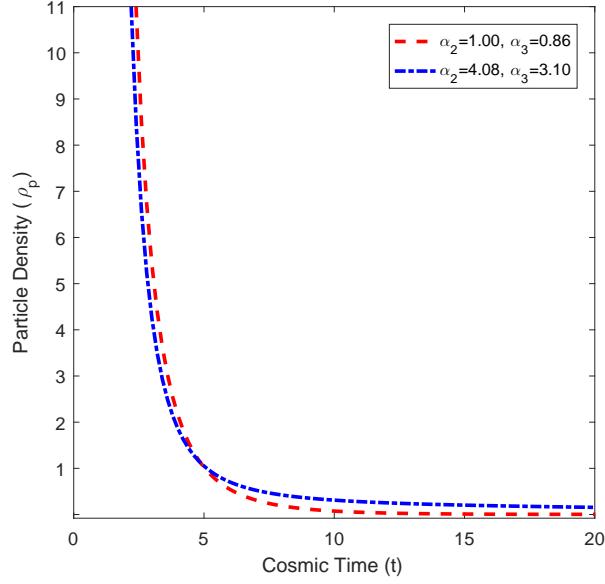


Figure 4. Variation of particle energy density ρ_p versus cosmic time t for $l = 1.326$ and $w = \phi_0 = \alpha = c_2 = 1$

In order to find the other parameters, we have to assume one more physical relation which we shall consider in next two subsections.

3.1 Model with bulk viscosity coefficient $\xi = \xi_0 \rho^k$, $k > 0$

In this model of the universe, we assume coefficient of bulk viscosity as a power function of density [54–56] *i.e.*

$$\xi = \xi_0 \rho^k, \quad (46)$$

where, ξ_0 and k (> 0) are real constants. $k = 1$ in Eq. (46) (considered by Murphy [57] for small density) corresponds to a radiative fluid [58]. However, near the big bang, k ($0 \leq k \leq \frac{1}{2}$) gives more realistic cosmological models [59].

Using the above equation, we get the following expressions for ξ and p :

$$8\pi\xi = 8\pi\xi_0 \left\{ \frac{9(2l+1)}{8\pi(l+2)^2} H^2 - \frac{\alpha^2}{8\pi} T^{-\frac{6l}{l+2}} + \frac{w}{16\pi} \phi_0^2 T^{-6} \right\}^k, \quad (47)$$

where, $T = c_2 \sqrt{t} \exp\{M_1(t)\}$

$$\begin{aligned} 8\pi p = & -\frac{3(l+1)}{l+2} \dot{H} - \frac{9(l^2+l+1)}{(l+2)^2} H^2 + \alpha^2 T^{-\frac{6l}{l+2}} + \frac{w}{2} \phi_0^2 T^{-6} \\ & + 24\pi\xi_0 H \left\{ \frac{9(2l+1)}{8\pi(l+2)^2} H^2 - \frac{\alpha^2}{8\pi} T^{-\frac{6l}{l+2}} + \frac{w}{16\pi} \phi_0^2 T^{-6} \right\}^k. \end{aligned} \quad (48)$$

3.2 Model with bulk viscosity coefficient $\xi = \xi_0 = \text{constant}$

In this model of the universe, the coefficient of bulk viscosity is assumed to be a constant *i.e.*

$$\xi = \xi_0. \quad (49)$$

Using the above equation, we get the following expression for p :

$$8\pi p = -\frac{3(l+1)}{l+2} \dot{H} - \frac{9(l^2+l+1)}{(l+2)^2} H^2 + \alpha^2 T^{\frac{-6l}{l+2}} + \frac{w}{2} \phi_0^2 T^{-6} + 24\pi\xi_0 H. \quad (50)$$

For the above two models, the other cosmological parameters such as expansion scalar θ , spatial volume V , directional Hubble parameters H_μ , shear scalar σ are obtained as

$$\theta = \frac{3}{\alpha_4 t + \alpha_5 \log(1 + \alpha_2 t)}, \quad (51)$$

$$V = (c_2 \sqrt{t})^3 \exp\{3M_1(t)\}, \quad (52)$$

$$H_1 = \frac{3l}{l+2} \left\{ \frac{1}{\alpha_4 t + \alpha_5 \log(1 + \alpha_2 t)} \right\}, \quad (53)$$

$$H_2 = H_3 = \frac{3}{l+2} \left\{ \frac{1}{\alpha_4 t + \alpha_5 \log(1 + \alpha_2 t)} \right\}, \quad (54)$$

$$\sigma^2 = 3 \left(\frac{l-1}{l+2} \right)^2 \left\{ \frac{1}{\alpha_4 t + \alpha_5 \log(1 + \alpha_2 t)} \right\}^2. \quad (55)$$

The isotropic condition $\frac{\sigma^2}{\theta^2}$ can be obtained as

$$\frac{\sigma^2}{\theta^2} = \frac{1}{3} \left(\frac{l-1}{l+2} \right)^2. \quad (56)$$

Anisotropic parameter (A_m) can be obtained as

$$A_m = 2 \left(\frac{l-1}{l+2} \right)^2. \quad (57)$$

From above set of the equations, it is observed that $V = 0$ at $t = 0$. Moreover, it is increasing exponentially with the passage of cosmic time. The physical parameters such as H , θ , H_μ and σ diverge in the beginning and $\rightarrow 0$ as $t \rightarrow \infty$. The isotropic condition $\sigma^2/\theta^2 = 0$ for $l = 1$. Hence, our model remains isotropic throughout the evolution for $l = 1$. However for $l \neq 1$, σ^2/θ^2 remains a non zero constant. Thus for $l \neq 1$, anisotropy is maintained throughout the evolution of the universe.

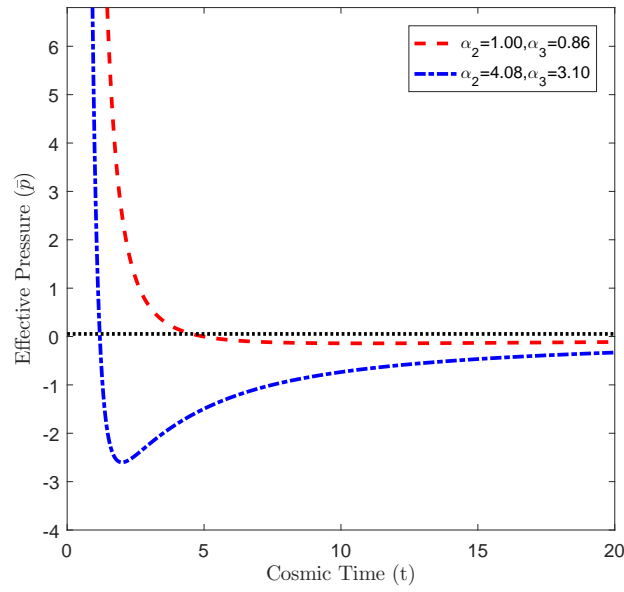


Figure 5. Variation of effective pressure \bar{p} versus cosmic time t for $l = 1.326$ and $w = \phi_0 = \alpha = c_2 = 1$.

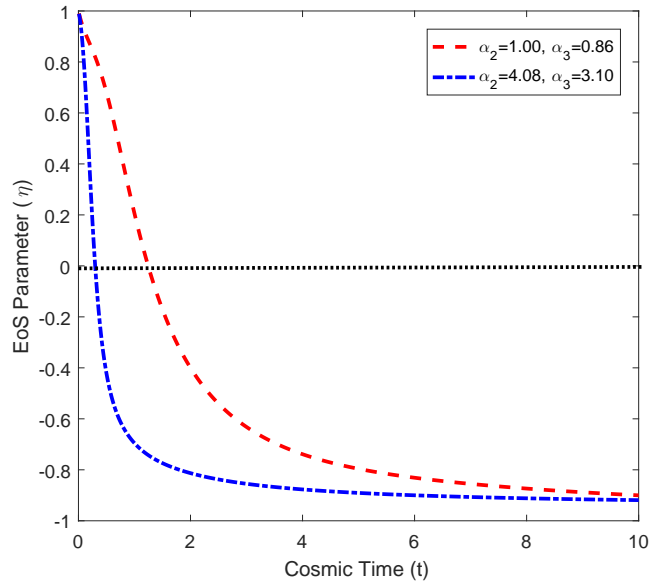


Figure 6. Variation of EoS parameter η versus cosmic time t for $l = 1.326$ and $w = \phi_0 = \alpha = c_2 = 1$.

4 Statefinder Diagnostic

Sahni et al. [60] have introduced two geometrical parameters r and s defined by

$$r = a^2 \frac{d^3 a}{dt^3} \left(\frac{da}{dt} \right)^{-3}, \quad s = \frac{2(1-r)}{3(1-2q)}, \quad (58)$$

where, a is an average scale factor and q is deceleration parameter.

Similar to H and q , these parameters involve scale factor and its higher order time derivatives. Both r and s are dimensionless quantities. In this technique, trajectories of the pair (s, r) are plotted in $s - r$ plane to examine the evolution of DE models. Since q , r and s depend only on scale factor and its higher order derivatives, this technique does not require any knowledge about theory of gravity. Thus, the approach is model-independent.

In $s - r$ plane, the fixed point $(s, r) = (0, 1)$ represents the Λ CDM model and $(s, r) = (1, 1)$ represents standard cold dark matter (SCDM) model. Other than Λ CDM and SCDM model, this method is widely used in literature to distinguish various DE models such as Chaplygin gas ($s < 0, r > 1$), quintessence ($s > 0, r < 1$), etc. by locating their respective regions on (s, r) trajectories.

For the presented model, the pair (s, r) is obtained as

$$s = \frac{2 [2T_1^2 - 3T_1 + T_2^2 \alpha_5 (\alpha_4 t + \alpha_5 \log(1 + \alpha_2 t))]}{3T_3}, \quad (59)$$

$$r = 1 + 2T_1^2 - 3T_1 + T_2^2 \alpha_5 (\alpha_4 t + \alpha_5 \log(1 + \alpha_2 t)), \quad (60)$$

where

$$T_1 = \alpha_4 + \frac{\alpha_2 \alpha_5}{1 + \alpha_2 t},$$

$$T_2 = \frac{\alpha_2}{1 + \alpha_2 t},$$

$$T_3 = \frac{1 - (\alpha_2 + 2\alpha_3)t}{1 + \alpha_2 t}.$$

The behaviour of cosmographic parameters at $t = 0$ and $t \rightarrow \infty$ is represented in Table 1.

Table 1. Behaviour of cosmographic parameters for $\alpha_4 = 0$ i.e. $\alpha_2 = \alpha_3$

Time (t) (Gyr)	$a(t)$	$H(t)$	$q(t)$	r	s
$t = 0$	0	∞	1	3	4/3
$t \rightarrow \infty$	∞	0	-1	1	0

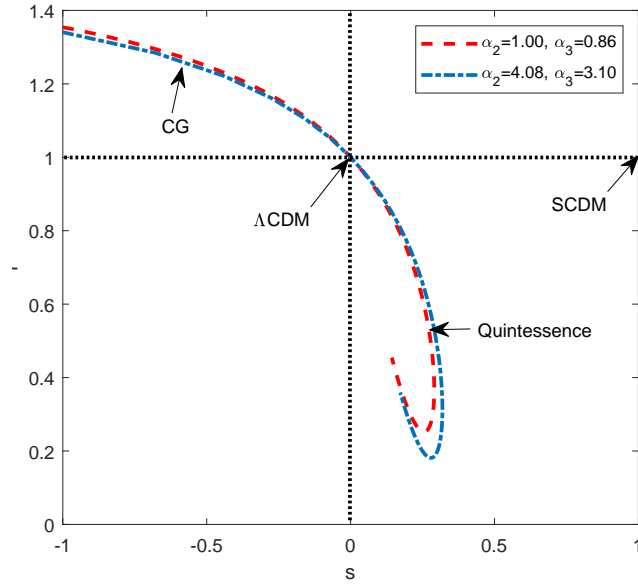


Figure 7. Plot of (s, r) trajectories.

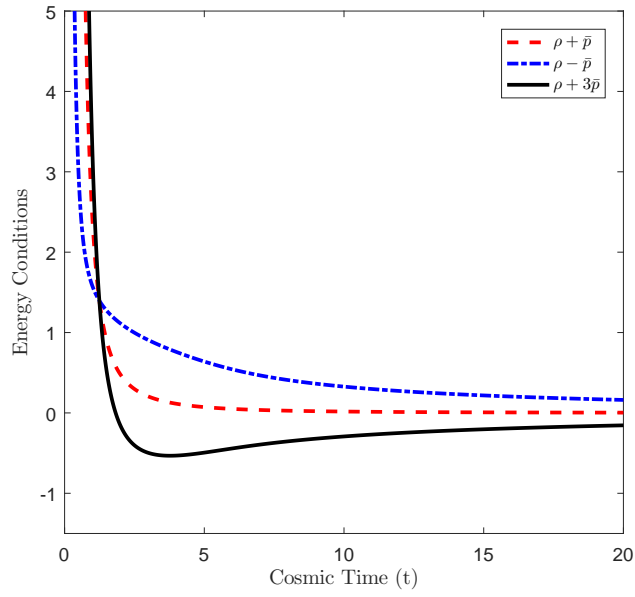


Figure 8. Variation of energy conditions *versus* cosmic time t for $l = 1.326$, $\alpha_2 = 1.00$, $\alpha_3 = 0.86$ and $w = \phi_0 = \alpha = c_2 = 1$.

5 Energy Conditions

In this section, the time-varying behaviour of the energy conditions is explored. Some of the important energy conditions are given as follows:

- Weak energy conditions (WEC) $\Rightarrow \rho \geq 0, \bar{p} + \rho \geq 0$.
- Dominant energy conditions (DEC) $\Rightarrow \rho \geq 0, -\bar{p} + \rho \geq 0$.
- Strong energy conditions (SEC) $\Rightarrow \rho + \bar{p} \geq 0, 3\bar{p} + \rho \geq 0$.

Using Eqs. (43) and (44), we analyse the behaviour of above mentioned energy conditions.

6 Concluding remarks

In this study, we have examined the behaviour of tilted Bianchi type-VI₀ bulk viscous string models of universe in Sáez-Ballester theory of gravitation. We have explored the time-varying behaviour of various cosmological parameters with the help of their pictorial representations. The detail observations of the constructed models are as follows:

- From Eq. (30), it is clear that DP shows signature flipping *i.e.* $q > 0$ for $\alpha_3 t < 1$ & $q < 0$ for $\alpha_3 t > 1$. The same behaviour of DP can be visualised through Figure 1. Thus, we have obtained the model of the universe that evolves from early decelerated expansion phase to the present accelerated expansion phase. Such feature of the model is compatible with the observational data.
- The constructed model of the universe originates from an initial Big-Bang singularity and evolves exponentially with the passage of cosmic time.
- The Hubble parameter H is positive and decreasing function of time. Also, $H \rightarrow 0$ as $t \rightarrow \infty$.
- Figures 2, 3 and 4 represent the time-varying behaviour of total energy density ρ , string tension density λ and the particle density ρ_p of the system respectively. Total energy density of the fluid is positive throughout the cosmic evolution and is decreasing function of the time. Also, $\rho \rightarrow 0$ as $t \rightarrow \infty$. However, string tension density is initially negative at $t = 0$, reaches to positive value, gradually decreases with the passage of time, and converges to zero at present epoch. This behaviour of λ indicates that we have obtained cosmological model in which the strings are present only in the early universe and they disappear at the present epoch. String tension density λ can be both negative or positive. The negative value

of the λ indicates that there are no strings in the universe *i.e.* we have an anisotropic fluid of particles [61]. The density of the particles is also positive decreasing function of time t .

- Figure 5 represents the behaviour of effective pressure \bar{p} *w.r.t.* cosmic time t . \bar{p} changes its behaviour from positive to negative value at present epoch. The negative value of \bar{p} is due to the large negative bulk viscous pressure. This negative effective pressure at present epoch and in late time could be responsible for gearing up the cosmic expansion .
- Figure 6 represents the time varying behaviour of EoS parameter. η is initially positive and becomes negative at present. Thus the constructed model of the universe evolves from early non-dark region to present dark region. The model behaves like quintessence dark energy model ($-1 < \eta < 0$) at present.
- Figure 7 represents the evolution of (s, r) trajectories. It is observed that the trajectories start evolving in the Chaplygin gas (CG) region, cross the Λ CDM fixed point and thereon stay in the quintessence region. Thus, the obtained model behaves like quintessence DE model at present and in the distant future. However, for $\alpha_2 = \alpha_3$, $(s, r) \rightarrow (0, 1)$ as $t \rightarrow \infty$ *i.e.* the model behaves similar to Λ CDM in the future.
- Figure 8 represents the time varying behaviour of WEC, DEC and SEC. It shows that both WEC and DEC are well satisfied throughout the cosmic evolution. However, SEC is satisfied only during the early stages of the evolution of the universe and is violated at the present epoch. For quintessence dark energy model, SEC must be violated and thus ensures the viability of the obtained models.

All the above findings of the obtained models are in good agreement with the observational data. Therefore, we conclude that this study could be helpful to the other authors for further studying dark energy string cosmological model in Sáez-Ballester theory of gravitation.

Declaration of interests

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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