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Phenomenological Description of Features of Even-Even $^{230\text{-}238}$U isotopes in Framework of Modified Sood Model

K.A. Gado[1,2]

[1]Department of Physics, Faculty of Science and Arts, Al-Mikhwah, Al-Baha University, Saudi Arabia
[2]Basic Sciences Department, Bilbeis Higher Institute for Engineering (BHIE), Bilbeis 44713, Sharqia, Egypt
Corresponding author E-mail: qjado76@gmail.com

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Abstract. A modified Sood model was used to investigate the effects of rotation - vibration. As a consequence, it is indicated that the energy expression as a rational polynomial (RP) in total angular momentum I provides an excellent approximation to the ground state band excitation energies in U ($A = 230$–238). The shape of the nucleus in uranium even-even isotopes is controlled by a softness parameter, and the softness is believed to prevent the nucleus from becoming a rigid entity. The reason is discussed on the basis of the centrifugal stretching as well as the shrinkage.

KEY WORDS: Ground state band, Sood model, nuclear structure.

1 Introduction

The goal of scientists since the discovery of nuclear physics till now has been to understand the reaction properties, structure, and properties of nuclei. Theoretical researchers are still working to establish a unified model that can predict the shape and properties of nuclei with the same precision. Rotating bands in the spectra of deformed nuclei, those characterized by a non-spherical spatial distribution of nuclear density [1], microscopic description of the rotational motion requires coherent contributions from many nucleons, and it is thus referred to as a collective motion. As a result, the nucleus as a whole rotates around an axis except for the nuclear symmetry axis. A prolate-nucleus rotating along an axis perpendicular to the nuclear symmetry axis is a schematic example of a collective rotation. From experimental spectra, it was discovered that the relationship between excitation energy E and total angular momentum I is typically

smooth, and that it may be approximated by $E \sim I(I+1)$ for spins that are not excessively high. A rotational band is the corresponding series of states with consecutively increasing angular momentum. Band head refers to the lowest level of a band. Many states with varied intrinsic structures can theoretically become band heads; the band built on the nucleus's ground state is known as the ground state band (GRB). All other bands are referred as side bands or Exciting bands. The yrast state [2] is the lowest energy state of a given angular momentum.

In Section 2, mathematical model will be discussed. Section 3 is devoted to results and discussion. The conclusion is reported in last section.

2 Mathematical Model

The formula $E(I) = A[I(I+1)]$ gives the lowest rotational energy levels for nuclei, according to Bohr-Mottelson [3]; where $A = 1/(2\vartheta)$, ϑ is the nuclear moment of inertia, and I is the angular momentum, the sequence 0^+, 2^+, 4^+, ... is followed. The nucleus was treated as a rigid rotator in the above equation, but the shape of the nucleus is really determined by the angular momentum due to centrifugal stretching or the rotation-vibration interaction. According to Holmberg et al. [4] and Huang et al. [5] the rotational energy may be expressed as an infinite power series

$$E(I) = A[I(I+1)] - B[I(I+1)]^2 + C[I(I+1)]^3 - D[I(I+1)]^4 + \cdots, \quad (1)$$

A, B, C, and D are fitting parameters.

$$E(I) = AJ - BJ^2 + CJ^3 - DJ^4 + \cdots \quad (2)$$

$$E(I) = AJ\Big\{1 - \Big(\frac{B}{A}\Big)J\Big[1 - \Big(\frac{C}{B}\Big)J + \Big(\frac{D}{B}\Big)J^2 \cdots\Big]\Big\}. \quad (3)$$

According to Sood [6], we may write $C/B = N(B/A)$, where $N \cong 2-3$. From molecular spectra theory Dunham [7], and we know that, $D/B \cong (C/B)^2$ according to Huang et al. [5]. Substitute the following for equation (3):

$$E(I) = AJ\Big\{1 - \Big(\frac{B}{A}\Big)J\Big[1 - N\Big(\frac{B}{A}\Big)J + \Big(N\Big(\frac{B}{A}\Big)J\Big)^2 - \cdots\Big]\Big\}. \quad (4)$$

If the terms between the square brackets in equation (4) are an infinite geometric series, the sum rule of the infinite geometric series may be used to obtain

$$E(I) = A\left\{\frac{J + M\Big(\frac{B}{A}\Big)J^2}{1 + N\Big(\frac{B}{A}\Big)J}\right\}, \quad (5)$$

where $M = N - 1$. Due to the general large difference between B/A and N: $(B/A \gg N)$, the two are multiplied by each other, simplifying the explanation

considerably. Assume $MB = b$ and $N(B/A) = d$, where b and d are new parameters

$$E\left(I\right) = \frac{AJ + bJ^2}{1 + dJ}\,, \tag{6}$$

where the usage of a rational function, as seen in equation (6), presents a non-linearity problem and necessitates an iterative rather than a direct technique, as in linear least-squares problems, i.e., computing a rational least-squares approximation. The authors utilized the formula $N = 2.85 - 0.05I$ to analyze the energy levels of nuclei based on earlier work [6–8]. However, in this work, we will introduce three parameters-model as a new version of the traditional Sood model, dubbed "MSM3" in order to enhance theoretical conclusions and make them more consistent with experimental data. Choose to express the excitation energies $E(I)$ of the ground-state band in terms of a plot between the nuclear moments of inertia and the squared rotational frequencies, as suggested in the current study, by utilizing the relations

$$\vartheta_1 = \frac{2\vartheta}{\hbar^2} = \frac{\left(1 + dJ\right)^2}{A + bJ(2 + dJ)}\,, \tag{7}$$

$$\hbar\omega = \frac{2AJ^{\frac{1}{2}} + 2bJ^{\frac{3}{2}}(2 + dJ)}{\left(1 + dJ\right)^2}\,. \tag{8}$$

Softness is defined as the fractional change in moment of inertia between the first excited state and the ground state [9]:

$$\sigma = \frac{\vartheta_1|_{J=6}}{2\vartheta_0} - \frac{1}{2}\,. \tag{9}$$

3 Results and Discussion

Many two-parameter and, more recently, three-parameter mathematical formulations based on semiempirical, phenomenological, or semi-classical considerations for the description of nucleus energy levels have been offered in the lack of a unified model of nucleus [9]. A nucleus (diatomic molecule) can be imagined as a virtually rigid dumbbell that can rotate along a fixed axis in space and so has angular momentum, which must be quantized according to Born [10]. On Bohr's theory, the first term in Eq. (1), where A is the dumbbell's moment of inertia about the axis, gives the energy of the rotating dumbbell. We expect that the spectrum by first term in Eq. (1) be purely rotational since we disregard any possible oscillation of the nucleus relative to one another. The classification of the two components as rotational and potential energy, respectively, is basically true, according to discussions of the possible theoretical origin of the first term in Eq. (1). Nevertheless, have been disregard potential energy to modify this formula in order to extend its range of validity have focused on the first

Table 1. Experimental and calculated energies using four parameter model (FPM4), traditional Sood model (TSM2) and modified Sood model (MSM3) for even-even $^{230-238}$U actinide nuclei

Nucleus	I^{π}	Experimental energy E_{Levels} (MeV)	Calculated energy FPM4 E_{Levels} (MeV)	TSM2 E_{Levels} (MeV)	SMS3 E_{Levels} (MeV)
^{230}U	2^+	0.05172	0.05151	0.05209	0.05148
	4^+	0.16950	0.16908	0.16987	0.16899
	6^+	0.34710	0.34713	0.34823	0.34696
	8^+	0.57820	0.57854	0.58125	0.57844
	10^+	0.85640	0.85642	0.86196	0.85659
	12^+	1.17570	1.17540	1.18217	1.17566
	14^+	1.53160	1.53178	1.53217	1.53144
	16^+	1.92120	1.92117	1.90034	1.92126
		rmsd	**0.00024**	**0.00805**	**0.00023**
^{232}U	2^+	0.04757	0.04736	0.04821	0.04727
	4^+	0.15657	0.15614	0.15721	0.15587
	6^+	0.32260	0.32247	0.32228	0.32204
	8^+	0.54100	0.54106	0.53793	0.54061
	10^+	0.80580	0.80600	0.79771	0.80582
	12^+	1.11150	1.11169	1.09405	1.11201
	14^+	1.45370	1.45351	1.41795	1.45415
	16^+	1.82810	1.82792	1.75868	1.82812
	18^+	2.23150	2.23169	2.10328	2.23079
	20^+	2.65970	2.65966	2.43601	2.65999
		rmsd	**0.00020**	**0.08541**	**0.00045**
^{234}U	2^+	0.04350	0.04303	0.04416	0.04307
	4^+	0.14335	0.14216	0.14401	0.14221
	6^+	0.29607	0.29444	0.29523	0.29431
	8^+	0.49704	0.49564	0.49277	0.49509
	10^+	0.74120	0.74072	0.73074	0.73965
	12^+	1.02380	1.02443	1.00221	1.02306
	14^+	1.34080	1.34187	1.29892	1.34080
	16^+	1.68780	1.68908	1.61105	1.68901
	18^+	2.06300	2.06337	1.92672	2.06466
	20^+	2.46420	2.46333	2.23152	2.46549
	22^+	2.88970	2.88851	2.50771	2.88998
	24^+	3.33900	3.33828	2.73328	3.33718
	26^+	3.80800	3.81004	2.88053	3.80663
	28^+	4.29700	4.29629	2.91411	4.29820
		rmsd	**0.00110**	**0.49035**	**0.00130**

Table 1. (continued)

Nucleus	I^{π}	Experimental energy E_{Levels} (MeV)	Calculated energy FPM4 E_{Levels} (MeV)	TSM2 E_{Levels} (MeV)	SMS3 E_{Levels} (MeV)
^{236}U	2^+	0.04524	0.04487	0.04611	0.04522
	4^+	0.14948	0.14852	0.15035	0.14953
	6^+	0.30978	0.30850	0.30821	0.31015
	8^+	0.52224	0.52117	0.51444	0.52311
	10^+	0.78230	0.78198	0.76288	0.78365
	12^+	1.08530	1.08583	1.04627	1.08667
	14^+	1.42630	1.42741	1.35603	1.42715
	16^+	1.80091	1.80165	1.68188	1.80043
	18^+	2.20390	2.20407	2.01143	2.20246
	20^+	2.63170	2.63109	2.32963	2.62987
	22^+	3.08120	3.08020	2.61797	3.07990
	24^+	3.55000	3.54995	2.85346	3.55081
	26^+	4.03900	4.03971	3.00718	4.04096
	28^+	4.54900	4.54907	3.04224	4.54952
	30^+	5.07700	5.07684	2.91080	5.07600
		rmsd	**0.00072**	**0.76926**	**0.00110**
^{238}U	2^+	0.04491	0.04435	0.04578	0.04451
	4^+	0.14841	0.14683	0.14928	0.14732
	6^+	0.30721	0.30516	0.30602	0.30594
	8^+	0.51830	0.51591	0.51079	0.51678
	10^+	0.77570	0.77476	0.75746	0.77543
	12^+	1.07650	1.07682	1.03885	1.07705
	14^+	1.41530	1.41696	1.34641	1.41665
	16^+	1.78820	1.79015	1.66994	1.78945
	18^+	2.19070	2.19181	1.99716	2.19106
	20^+	2.61870	2.61803	2.31309	2.61765
	22^+	3.06720	3.06578	2.59938	3.06598
	24^+	3.53450	3.53285	2.83320	3.53346
	26^+	4.01730	4.01764	2.98583	4.01807
	28^+	4.51650	4.51860	3.02064	4.51833
	30^+	5.03430	5.03336	2.89013	5.03323
		rmsd	**0.00146**	**0.764241**	**0.00109**

term, has left its $I\,(I{+}1)$ dependence intact. It has long been recognized that first term in Eq. (1) is inadequate in most nuclei for the higher levels and corrections of the form $B\,[I\,(I{+}1)]^2$, as suggested by rotation-vibration coupling effects in molecules, have been used. Now that we're looking at the rotations of the molecule (nucleus) as a whole (rotations) and the vibrations of individual atoms on the molecule (vibrations), we can get a decent approximation to the findings by suggesting the extension first term in Eq. (1) owing to the centrifu-

gal stretching or rotation–vibration interaction [11], to an infinite power series in $I(I+1)$.

This indicates, the equivalence of the mathematical formulations of the FPM4 and MSM3 models in respect of their energy expressions. For each ground state band, the parameters of the model in question are determined by using a computer simulated search program to reproduce the observed energies, the observed energy levels produce by parameters of the model by the value of parameters of the model which leads to the minimum of root mean square deviation (rmsd), where n is the number of data points involved in the fitting procedure [12].

$$rmsd = \sqrt{\frac{1}{n}\sum_{i=1}^{n}\left[E_i^{\text{exp.}} - E_i^{\text{theor.}}\right]^2}. \tag{10}$$

The results obtained by Sood in [13] has indicated the inadequacy of using any truncated series in $I(I+1)$ to describe experimental rotation levels, this is not achieved in this mass number range as shown in the fourth column of the previous table, due to softness for these nuclei, and we suggested a phenomenological the energy expression as a rational polynomial (RP) in total angular momentum, equation (6) which led to improved agreement, as shown in the last column of Table 1. MSM3 model giving excellent fits to energy levels of these nuclei, also

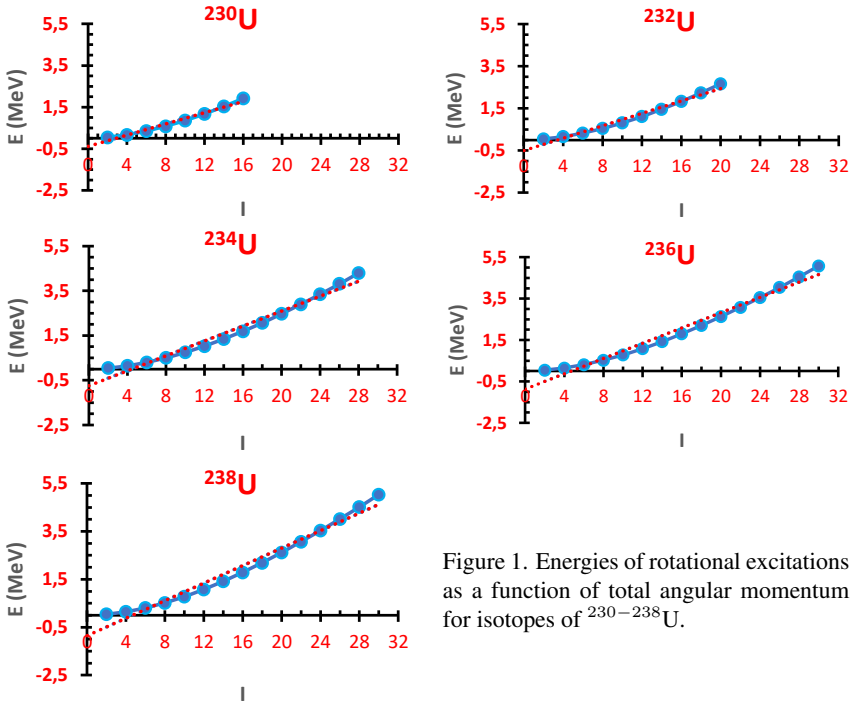

Figure 1. Energies of rotational excitations as a function of total angular momentum for isotopes of $^{230-238}$U.

describes features brought into evidence by experiments [14–18]. The quality of fits obtained by the RP formula is compared with that obtained in the FPM4 model in Table 1 for a set of Uranium nuclei. Although the RP formula have free parameters less than the FPM4 model, the improvement in the agreement is better than expected in this regard, particularly for the deformed nuclei. This is seen in Table 1 by comparison of RP formula results with traditional Sood model, referred to as the TSM2 model, obtained by mathematical treatment the FPM4 model expression.

Let us define the following dependence by the energies of rotational excitations are approximately proportional to the quantity $I(I+1)$, i.e., relation between energies of rotational excitations and the previous quantity is linearity. For visual representation of this dependence, the reader is referred to Figure 1. Therefore, moment of inertia of first excited state becomes appreciably violated, mean that its negative value. This explains why the relation between rotational excitation energies and the quantity $I(I+1)$ is not completely linear, and object in the first place is arrive at clear ideas with regard nuclear structure for these nuclei. We expect a change in the shape of the nucleus, i.e., centrifugal stretching since rotation causes departure of the distribution of particles (protons and neutrons) inside the nucleus, negative/positive values of softness correspond to shrinkage/stretching.

Figure 2, a plot of softness as a function of the mass number A, for the isotopes of $^{230-238}$U. Figure 2 shows stretching (positive softness values) evidenced that isotopes of $^{230-238}$U are soft rotors and does not hard rotors (negative softness values). We can compare with the rotational frequency and kinematic moment of inertia which extract by using the experimental transition energies as follows.

The results indicate that the back-bending does not appear, which means that the moment of inertia has no effect on the deformation of the nuclei in this mass range. From Figure 3, one notice that the difference between the experimental data and the results of the traditional Sood model decreases with increasing the mass number of nuclei, while the results of FPM4 and MSM3 models are close and close to the experimental results, which reinforces the idea of equivalence of mathematical formulas between them.

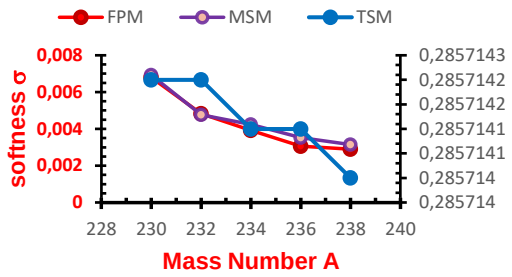

Figure 2. Variation of softness as a function of mass number for isotopes of $^{230-238}$U.

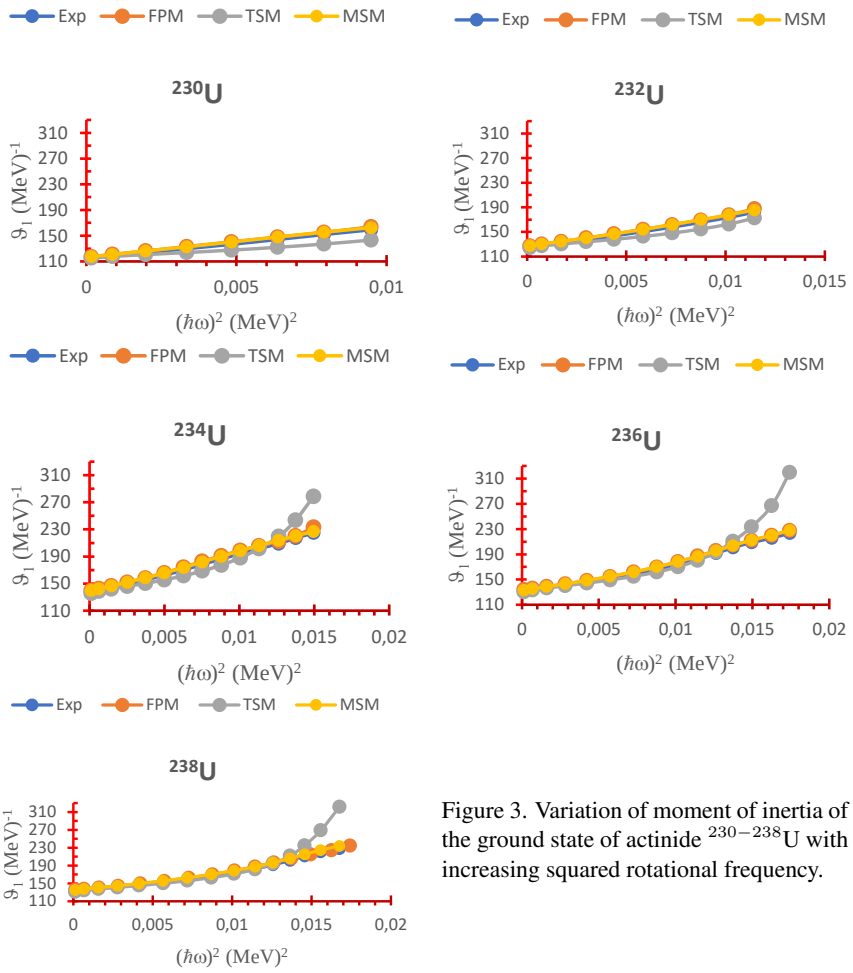

Figure 3. Variation of moment of inertia of the ground state of actinide $^{230-238}$U with increasing squared rotational frequency.

4 Conclusion

In this paper, we have given an overview on a specific topic which attracts much attention in contemporary nuclear structure research, namely the study of modified Sood model is a good theoretical approximation for describing the nuclear structure of neutron-rich deformed nuclei. In this work, ground-state properties of Uranium even-even $A = 230 - 238$ isotopes have been studied in the framework of the modified Sood model with another two models; four parameter model and traditional Sood model. The numerically calculated results of energy levels, kinematic moment of inertia, rotational frequency, softness is discussed and compared with the corresponding values obtained in the present work, based on the three models on this study. The energy levels, softness and moment of

inertia curves are plotted for some isotopes of the investigated nucleus, the transition in the shapes was clear with the change in the number of neutrons. The predictions obtained from this study by using modified Sood model (MSM3) show surprisingly good agreements in comparison with the experimentally observed energy levels and another the theoretical models results.

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