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A Characterization of Nontrivial Zeros of Riemann Zeta Function

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Abstract. In this article, we give a characterization of nontrivial zeros of the Riemann zeta function using two real integrals. Using this characterization we can provide simple proof of the fact that the Riemann zeta function has no nontrivial real zeros. We also establish that this function takes negative values on the real axis within the critical region.

KEY WORDS: Riemann zeta function, Riemann hypothesis, nontrivial zeros, analytic continuation.

1 Introduction

Riemann's zeta function has been ruling the imagination of mathematicians ever since Riemann presented his monumental seminal paper "On the Number of Primes Less Than a Given Magnitude" [1]. Several equivalent forms of the Riemann hypothesis are available in the literature [2–4] which state that:

> *all the nontrivial zeros of zeta function $\zeta(s)$ lie on the critical line* $\left\{\frac{1}{2} + i\,t : t \in R\right\}$.

So the central part of the discussion of the Riemann hypothesis is locating the zeros (value of s for which $\zeta(s) = 0$) of the Riemann zeta function.

The first major step in any attempt to discuss the Riemann hypothesis is defining the Riemann zeta function to the whole of the complex plane. That is, analytically continuing the Dirichlet series

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}, \tag{1}$$

which is defined for all $s = \sigma + it$ with $\Re(s) > 1$ to the whole of the complex plane with the possible exception at the simple pole $s = 1$. Here $\Re(s)$ represents the real part of the complex number s.

Observe that the Dirichlet series (1) diverges in the half plane $\Re(s) \leq 1$ and using the integral test, we can see that it converges for all s with $\Re(s) > 1$ (in fact, it converges absolutely) and so analytic in the half plane $\Re(s) > 1$.

The Dirichlet series defined in $D = \{s : \Re(s) > 1\}$ is taken to define the Riemann zeta function in that region D. Since it is an analytic function there, we can make use of analytic continuation to the get zeta function to a bigger domain containing D. The following theorem [5] is the basis for the construction of the unique analytic continuation of the zeta function.

Theorem 1.1. *If f and g are two functions analytic in the domains D_1 and D_2 respectively such that $f = g$ on $D_1 \cap D_2 \neq \phi$, then g is called analytic continuation of f to D_2 and vice versa. More over, if it exists the analytic continuation is unique.*

The uniqueness of analytic continuation is extremely useful and it says in effect that knowing the value of a complex function in some finite complex domain uniquely determines the value of the function at every other point. With the help of analytic continuation, whenever we have a representation of a function by any one power series, any number of other power series can be found which together define the value of the function at all points of the domain [6].

The above theorem shows that if we can construct an analytic function $g(z)$ on $\mathbb{C} - \{1\}$, the complex plane excluding the point $z = 1$, which agrees with the Dirichlet series in D, then $g(z)$ is the required analytic continuation and it is unique.

In Section 2, we illustrate the explicit construction of the analytic continuation of the zeta function developed by Riemann to the whole of the complex plane excluding the point $z = 1$, which is a simple pole of the zeta function. In the third section, we give a characterization of nontrivial zeros of the zeta function and as a consequence, we establish that the zeta function assumes negative values on the real axis within the critical region. The gamma function for complex numbers with positive real part is defined by the convergent improper integral

$$\Gamma(s) = \int_0^\infty e^{-x} x^{s-1} dx, \quad \Re(s) > 0 \,. \tag{2}$$

This can be analytically continued to a meromorphic function which is analytic in the complex plane except at the non-positive integers $s = 0, -1, -2, -3, \ldots$ which are simple poles of gamma function [7]. The gamma function never vanishes (that is, it has no zeros) and so its reciprocal, $1/\Gamma(s)$ is analytic in the whole complex domain with simple zeros at the non-positive integers.

Using the transformation $x = n^2\pi u$, and changing s to $\frac{s}{2}$ in the gamma function, we get

$$\frac{1}{n^s}\Gamma\left(\frac{s}{2}\right) = \pi^{\frac{s}{2}} \int_0^\infty e^{-n^2\pi u} u^{\frac{s}{2}-1} du \tag{3}$$

Taking the summation from 1 to infinity and then interchanging summation and integration (interchange of summation and integration is possible as the convergence is absolute), we get

$$\Gamma\left(\frac{s}{2}\right)\sum_1^\infty \frac{1}{n^s} = \pi^{\frac{s}{2}}\sum_1^\infty \int_0^\infty e^{-n^2\pi u} u^{\frac{s}{2}-1} du = \pi^{\frac{s}{2}} \int_0^\infty \left(\sum_1^\infty e^{-n^2\pi u}\right) u^{\frac{s}{2}-1} du$$

and this can be expressed in the form

$$\Gamma\left(\frac{s}{2}\right)\zeta(s) = \pi^{\frac{s}{2}} \int_0^\infty \psi(u)\, u^{\frac{s}{2}-1} du \tag{4}$$

where $\psi(x) = \sum\limits_1^\infty e^{-n^2\pi x}$.

The function $\psi(x)$ and the Jacobi theta function

$$\vartheta(x) = \sum_{-\infty}^\infty e^{-n^2\pi x}$$

are connected by the relation

$$1 + 2\psi(x) = \vartheta(x)$$

Using the Poisson summation, the following important functional equation of the Jacobi theta function can be proved [8,9].

$$x^{\frac{1}{2}}\vartheta(x) = \vartheta(x^{-1}), \quad x > 0 \tag{5}$$

In terms of $\psi(x)$, this takes the form

$$2\psi\left(\frac{1}{x}\right) = x^{\frac{1}{2}} - 1 + 2x^{\frac{1}{2}}\psi(x). \tag{6}$$

Splitting the integral $\int\limits_0^\infty \psi(u)\, u^{\frac{s}{2}-1} du$ in equation (4) into two parts

$$\int_0^\infty \psi(u)\, u^{\frac{s}{2}-1} du = \int_0^1 \psi(u)\, u^{\frac{s}{2}-1} du + \int_1^\infty \psi(u)\, u^{\frac{s}{2}-1} du \tag{7}$$

and modifying the first integral in the right-hand side of equation (7) using the relation (6) and by the transformation $u = \frac{1}{x}$, we can express the first integral as

$$\int_0^1 \psi(u)\, u^{\frac{s}{2}-1} du = \frac{1}{2} \int_1^\infty \left(x^{\frac{1}{2}} - 1 + 2x^{\frac{1}{2}}\psi(x)\right) \left(x^{-\frac{s}{2}-1}\right) dx$$
$$= \frac{1}{s(s-1)} + \int_1^\infty \psi(x)\, x^{-\frac{s}{2}-\frac{1}{2}} dx$$

Putting these values in (4), we get

$$\Gamma\left(\frac{s}{2}\right)\zeta(s) = \pi^{\frac{s}{2}} \left[\frac{1}{s(s-1)} + \int_1^\infty \left(x^{\frac{s}{2}-1} + x^{-\frac{s}{2}-\frac{1}{2}}\right) \psi(x) dx \right]. \tag{8}$$

Due to the exponential decay of the function

$$\psi(x) = \sum_1^\infty e^{-n^2 \pi x},$$

the integral within the bracket in (8) converges for all complex numbers s and hence defines an entire function throughout the complex plane [10].

The gamma function $\Gamma(\frac{s}{2})$ has simple poles at $s = 0, -2, -4, -6, \ldots$ and hence the zeta function has zeros at $s = -2, -4, -6, \ldots$. These zeros are called *trivial zeros* of the zeta function and any other zeros of the zeta function are called *nontrivial zeros*. Note that the pole at $s = 0$ of the gamma function cancels with the term $\dfrac{1}{s(s-1)}$ of the zeta function and it has only one simple pole due to the term $\dfrac{1}{s(s-1)}$ at $z = 1$. So, Riemann zeta function is a meromorphic function having a simple pole at $z = 1$. Eventually, we are in a position to define the Riemann zeta function:

Definition 1.1. *The Riemann zeta function is defined as the analytic continuation of the Dirichlet series (1) to the meromorphic function*

$$\zeta(s) = \frac{\pi^{\frac{s}{2}}}{\Gamma\left(\frac{s}{2}\right)} \left[\frac{1}{s(s-1)} + \int_1^\infty \left(x^{\frac{s}{2}} + x^{\frac{1}{2}(1-s)}\right) \psi(x) \frac{dx}{x} \right] \tag{9}$$

defined in the whole of the complex plane with the exception at the simple pole at $s = 1$ having residue 1.

Riemann observed that the expression

$$\frac{1}{s(s-1)} + \int_1^\infty \left(x^{\frac{s}{2}} + x^{\frac{1}{2}(1-s)}\right)\psi(x)\frac{dx}{x} = K(s) \tag{10}$$

in (9) is invariant under the transformations $s \to 1-s$ so that the value of the above expression is symmetric about the critical line $s = \frac{1}{2}$.

We recall some observations about the zeta function defined in the half plane $\Re(s) > 1$. Treating s as a real variable, Euler observed that

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} = \Pi_p \left(1 + \frac{1}{p^s} + \frac{1}{p^{2s}} + \frac{1}{p^{3s}} + \cdots\right) \tag{11}$$

where the infinite product is taken for all possible primes p. Note that the series

$$1 + \frac{1}{p^s} + \frac{1}{p^{2s}} + \frac{1}{p^{3s}} + \cdots$$

is a geometric series with $\frac{1}{p^s} < 1$ for all $s > 1$ and so it is convergent for all $s > 1$. Observing that the infinite product is also convergent, we get the Euler's product representation of the zeta function in the half plane $\Re(s) > 1$

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} = \Pi_p \left(1 - \frac{1}{p^s}\right)^{-1} \tag{12}$$

where the infinite product is taken for all possible primes p.

From the above product representation of zeta function, it is easily concluded that this function never vanishes ($\zeta(s) \neq 0$) in the half plane $\Re(s) > 1$. In 1899, the famous Belgian mathematician de La Vallée Poussin proved that it does not vanish on the line $s = 1$. That is, $\zeta(1 + it) \neq 0$ for any real number t [1]. Hence the zeta function has no zeros in the half plane $\Re(s) \geq 1$. Since the function is symmetric in the critical region about the line $s = \frac{1}{2}$, we get that $\zeta(0 + it) \neq 0, \quad \forall t$. Thus, it has no zeros in the half-plane $\Re(s) \leq 0$ other than the trivial zeros. The Riemann hypothesis is concerned with the zeros of the zeta function in the infinite strip $0 \leq \Re(s) \leq 1$. This infinite strip is called a *critical strip* and the zeros of the zeta function in this strip are called *nontrivial zeros* of the zeta function. With the above observation, we will be mostly concentrating on the infinite strip $0 < \Re(s) < 1$. In the following section, we present a characterization of nontrivial zeros of the zeta function in this region in terms of integrals.

2 Results and Discussion

Lemma 2.1. *A necessary and sufficient condition for a complex number* $s = \sigma + it$ *with* $0 < \Re(s) < 1$ *to be a nontrivial zero of the Riemann zeta function* $\zeta(s)$ *is that*

$$\int_1^\infty \left(x^{\frac{s}{2}} + x^{\frac{1}{2}(1-s)}\right)\psi(x)\frac{dx}{x} = \frac{1}{s(1-s)}, \quad 0 < \Re(s) < 1 \tag{13}$$

Proof. When we restrict our attention to the infinite strip $0 < \Re(s) < 1$ and the fact that for any s the functions $\pi^{\frac{s}{2}}$ and $\Gamma\left(\frac{s}{2}\right)$ takes finite non-zero values in this strip, we get the following:

$$\begin{aligned}\zeta(s) = 0 \iff & \frac{\pi^{\frac{s}{2}}}{\Gamma\left(\frac{s}{2}\right)}\left[\frac{1}{s(s-1)} + \int_1^\infty \left(x^{\frac{s}{2}} + x^{\frac{1}{2}(1-s)}\right)\psi(x)\frac{dx}{x}\right] = 0 \\ \iff & \int_1^\infty \left(x^{\frac{s}{2}} + x^{\frac{1}{2}(1-s)}\right)\psi(x)\frac{dx}{x} = \frac{1}{s(1-s)}, \quad 0 < \Re(s) < 1\end{aligned}$$

Hence we get a simple formulation (13) for nontrivial zeros of the Riemann zeta function in $0 < \Re(s) < 1$. □

Observe that the above result is a direct implication from the definition of the zeta function, but this observation possibly gives a different approach for extending the zero-free region of the zeta function in the critical region. The above equivalent formulation can be expressed in terms of two real integrals as given below. In some situations, this formulation may be more convenient to prove some properties of the zeta function.

Theorem 2.1. *A necessary and sufficient condition for a complex number* $s = \sigma + it$ *with* $0 < \Re(s) < 1$ *to be a nontrivial zero of the zeta function* $\zeta(s)$ *is that the two integrals*

$$\int_1^\infty \left[x^{\frac{\sigma}{2}} + x^{\frac{1-\sigma}{2}}\right]\cos\left(\frac{t\log x}{2}\right)\psi(x)\frac{dx}{x} = \frac{\sigma - \sigma^2 + t^2}{[\sigma^2 + t^2]\left[(1-\sigma)^2 + t^2\right]} \tag{14}$$

and

$$\int_1^\infty \left[x^{\frac{\sigma}{2}} - x^{\frac{1-\sigma}{2}}\right]\sin\left(\frac{t\log x}{2}\right)\psi(x)\frac{dx}{x} = \frac{t(2\sigma - 1)}{[\sigma^2 + t^2]\left[(1-\sigma)^2 + t^2\right]} \tag{15}$$

are simultaneously true for some real number t.

Proof. Note that $x^{\mu \pm i\eta} = x^{\mu} e^{\pm i\eta \ln x} = x^{\mu} \left[\cos(\eta \log x) \pm i \sin(\eta \log x)\right]$. Therefore,

$$\begin{aligned} x^{\frac{s}{2}} + x^{\frac{1}{2}(1-s)} &= x^{\frac{\sigma+it}{2}} + x^{\frac{1-\sigma-it}{2}} \\ &= x^{\frac{\sigma}{2}} e^{i\frac{t\log x}{2}} + x^{\frac{1-\sigma}{2}} e^{-i\frac{t\log x}{2}} \\ &= \left[x^{\frac{\sigma}{2}} + x^{\frac{1-\sigma}{2}}\right] \cos\left(\frac{t\log x}{2}\right) + i\left[x^{\frac{\sigma}{2}} - x^{\frac{1-\sigma}{2}}\right] \sin\left(\frac{t\log x}{2}\right) \end{aligned}$$

Also,

$$\frac{1}{s(1-s)} = \frac{\sigma - \sigma^2 + t^2 + it(2\sigma - 1)}{[\sigma^2 + t^2]\,[(1-\sigma)^2 + t^2]}$$

Putting these values in (13) and equating real and imaginary parts on both sides, we get

$$\int_1^{\infty} \left[x^{\frac{\sigma}{2}} + x^{\frac{1-\sigma}{2}}\right] \cos\left(\frac{t\log x}{2}\right) \psi(x)\frac{dx}{x} = \frac{\sigma - \sigma^2 + t^2}{[\sigma^2 + t^2]\,[(1-\sigma)^2 + t^2]}$$

and

$$\int_1^{\infty} \left[x^{\frac{\sigma}{2}} - x^{\frac{1-\sigma}{2}}\right] \sin\left(\frac{t\log x}{2}\right) \psi(x)\frac{dx}{x} = \frac{t(2\sigma - 1)}{[\sigma^2 + t^2]\,[(1-\sigma)^2 + t^2]}$$

□

Using theorem (2.1) we give a simple proof to the following theorem.

Theorem 2.2. *The zeta function $\zeta(s)$ has no nontrivial real zeros.*

Proof. We shall prove that at least one of the two integrals (14) and (15) is not true when $t = 0$. Bear in mind that when $t = 0$ the integral (15) is true for any $0 < \sigma < 1$. So we compute both sides of integral (14) for $t = 0$.

For $x \geq 1$,

$$x^{\frac{\sigma-2}{2}} + x^{-(\frac{\sigma+1}{2})} = \frac{1}{x^{\frac{2-\sigma}{2}}} + \frac{1}{x^{\frac{\sigma+1}{2}}} \leq 2, \quad 0 < \sigma < 1$$

So when we give $t = 0$ in the left hand side of (14), we get

$$\begin{aligned}\int_1^\infty \left[x^{\frac{\sigma}{2}} + x^{\frac{1-\sigma}{2}}\right] \cos\left(\frac{t\log x}{2}\right) \psi(x)\frac{dx}{x} &= \int_1^\infty \left[x^{\frac{\sigma-2}{2}} + x^{-(\frac{\sigma+1}{2})}\right] \psi(x)dx \\ &\leq 2\int_1^\infty \psi(x)dx = 2\int_1^\infty \sum_1^\infty e^{-n^2\pi x}dx \\ &= 2\sum_1^\infty \int_1^\infty e^{-n^2\pi x}dx = 2\sum_1^\infty \left[\frac{e^{-n^2\pi x}}{-n^2\pi}\right]_1^\infty \\ &= 2\sum_1^\infty \frac{e^{-n^2\pi}}{n^2\pi} < 1\end{aligned}$$

Absolute convergence of the series guarantees the interchange of summation and integration.

Now we consider the right-hand side of (14). Putting $t = 0$ in the expression $\frac{\sigma - \sigma^2 + t^2}{[\sigma^2 + t^2][(1-\sigma)^2 + t^2]}$, we get

$$\frac{\sigma - \sigma^2 + t^2}{[\sigma^2 + t^2][(1-\sigma)^2 + t^2]} = \frac{\sigma - \sigma^2}{[\sigma^2][(1-\sigma)^2]} = \frac{1}{\sigma(1-\sigma)}$$

Let $g(\sigma) = \frac{1}{\sigma(1-\sigma)} = \frac{1}{(1-\sigma)} + \frac{1}{\sigma}$. The first and second derivatives of the function $g(\sigma)$ are given by

$$g'(\sigma) = \frac{1}{(1-\sigma)^2} - \frac{1}{\sigma^2}, \quad g''(\sigma) = \frac{2}{(1-\sigma)^3} + \frac{2}{\sigma^3}$$

The point of maximum or minimum is given by solving $g'(\sigma) = 0$ and we get $\sigma = \frac{1}{2}$ and for this value $g''(\frac{1}{2}) = 32 > 0$.

So that the function $g(\sigma)$ attains its minimum value at $\sigma = \frac{1}{2}$ and the minimum value is $g(\frac{1}{2}) = 4$.

That is,

$$g(\sigma) \geq 4, \forall\sigma \in (0, 1).$$

This shows that

$$\int_1^\infty \left[x^{\frac{\sigma}{2}} + x^{\frac{1-\sigma}{2}}\right] \psi(x)\frac{dx}{x} \neq \frac{1}{\sigma(1-\sigma)}, \quad 0 < \sigma < 1$$

Hence Riemann zeta function has no nontrivial real zeros. □

In fact, we can prove the following theorem.

Theorem 2.3. *On the real axis where* $0 < \sigma < 1$, $\zeta(\sigma) < 0$

Proof. On the real axis when $0 < \sigma < 1$, note that the function $\frac{1}{s(1-s)} = \frac{1}{\sigma(1-\sigma)} \geq 4$, so that we get $\frac{1}{\sigma(\sigma-1} \leq -4$ and the value of the integral

$$\int_1^\infty \left[x^{\frac{\sigma}{2}} + x^{\frac{1-\sigma}{2}}\right] \cos\left(\frac{t\log x}{2}\right) \psi(x)\frac{dx}{x} = \int_1^\infty \left[x^{\frac{\sigma}{2}} + x^{\frac{1-\sigma}{2}}\right] \psi(x)\frac{dx}{x} < 1$$

Since the integral in (15) vanishes on the real line, we get

$$\begin{aligned} \zeta(\sigma) &= \frac{\pi^{\frac{\sigma}{2}}}{\Gamma\left(\frac{\sigma}{2}\right)} \left[\frac{1}{\sigma(\sigma-1)} + \int_1^\infty \left(x^{\frac{\sigma}{2}} + x^{\frac{1}{2}(1-\sigma)}\right) \psi(x)\frac{dx}{x}\right] \\ &< \frac{\pi^{\frac{\sigma}{2}}}{\Gamma\left(\frac{\sigma}{2}\right)} \left[-4+1\right] < 0 \end{aligned}$$

as other factors are positive. Hence we get $\zeta(\sigma) < 0$, when $0 < \sigma < 1$. □

3 Conclusions

In summary, we provide a necessary and sufficient condition for a complex number to be a nontrivial zero of the Riemann zeta function. A simple proof of the fact that the zeta function has no nontrivial real zeros is given, and as a consequence, we proved that the Riemann zeta function takes negative values on the real axis within the critical region. The characterization of nontrivial zeros of the zeta function will be helpful to widen the zero-free region of the Riemann zeta function.

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