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Proof of the Orthogonal–Pin Duality

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Abstract. This article contains the proof of a theorem on orthogonal–Pin duality that was cited without proof in a previous article in this journal (*Bulg. J. Phys.* **48** (2021) 390-399).

KEY WORDS: Orthogonal Lie algebra, orthogonal group, Spin and Pin groups, duality.

1 Introduction

In a previous article in this journal [1], I cited a theorem of dual Fock space representations of an orthogonal Lie algebra and a Pin group, but the space limits of that article as a conference contribution did not allow to include the proof. The present paper serves to render the proof, which is found so far only in a preprint [2], accessible in a journal. For the context of the theorem, which is Theorem 2 below, see [1, 3]. The proof is given in Section 6. Sections 2–5 provide necessary preliminaries. I thus specify my terminology in Section 2 and define the fermion Fock space in Section 3. In Section 4, I review the constructions on this space of a number conserving and a commuting number non-conserving representation of orthogonal Lie algebras [4], and in Section 5, I extend the number non-conserving representation to a representation of a Pin group as defined by Atiyah, Bott and Shapiro [5]. Section 7 extends the discussion in [1] by relating my result to contemporary work on Fock space dualities, and Section 8 provides a summary.

2 Terminology and Notation

While the theorem to be proved is one of linear algebra, I use the terminology and notation of quantum mechanics. Throughout, the base field is that of the complex numbers. Members of a finite-dimensional vector space V are written as Dirac kets $|i\rangle$ [6] and sometimes called states. Members of the dual space

V^* of linear forms on V are written as Dirac bras $\langle i|$, and $\langle i|(|j\rangle)$ as $\langle i|j\rangle$. When $(|i\rangle, i = 1,\dots,n)$ is a basis for a vector space V of finite dimension n, its dual basis $(\langle i|, i = 1,\dots,n)$ for V^* is defined by $\langle i|j\rangle = \delta_{ij}$ and vice versa, and when a ket and a bra have identical labels, they are understood to be corresponding members of dual bases. In particular, the identity on V may thus be written $\sum_i |i\rangle\langle i|$, where $|i\rangle$ scans an arbitrary basis for V. Bilinear forms on V are written as bras $\langle b|$, and $\langle b|(|i\rangle, |j\rangle)$ as $\langle b|ij\rangle$. Bilinear forms on V^* are written as kets $|b\rangle$, and $|b\rangle(\langle i|, \langle j|)$ as $\langle ij|b\rangle$. When $\langle b|$ is non-degenerate, a dual bilinear form $|b\rangle$ is defined by $\sum_k \langle b|jk\rangle\langle ik|b\rangle = \langle i|j\rangle$, which renders $|b\rangle$ non-degenerate, and vice versa. The bilinear form $|b\rangle$ is symmetric if and only if $\langle b|$ is symmetric.

3 Fock Space

The theorem to be proved concerns group representations on the *Fock space* of multiple kinds of fermions sharing a single-kind state space. I therefore introduce this concept. Consider k *kinds* of fermions inhabiting a common *single-kind state space* $\mathcal{S}$ of dimension d. Examples are spin up and down electrons in an atomic shell with orbital angular momentum l ($d = 2l+1, k = 2$), spin up and down nucleons in a nuclear shell with orbital angular momentum l ($d = 2l + 1$, $k = 4$), nucleons in a nuclear shell with orbital and spin angular momenta coupled to total angular momentum j ($d = 2j + 1, k = 2$). In terms of a state space $\mathcal{K}$ with a basic state $|\tau\rangle$ for each fermion kind τ, one can define, corresponding to every state $|p\rangle \otimes |\tau\rangle$ in $\mathcal{S} \otimes \mathcal{K}$, a *creation operator* $a^\dagger_{p\tau}$, and corresponding to every member $\langle p| \otimes \langle\tau|$ of $\mathcal{S}^* \otimes \mathcal{K}^*$, an *annihilation operator* $a_{p\tau}$. By definition, these operators obey

$$\{a_{p\tau}, a^\dagger_{q\upsilon}\} = \langle p|q\rangle\langle\tau|\upsilon\rangle, \quad \{a_{p\tau}, a_{q\upsilon}\} = \{a^\dagger_{p\tau}, a^\dagger_{q\upsilon}\} = 0 \tag{1}$$

in terms of the anticommutator $\{\cdot,\cdot\}$. Formally, $\mathcal{S} \otimes \mathcal{K}$ is just a dk-dimensional vector space, and the relations (1) are those of exterior and interior multiplication by members of $\mathcal{S} \otimes \mathcal{K}$ and $\mathcal{S}^* \otimes \mathcal{K}^*$ on the *exterior algebra* [7] on $\mathcal{S} \otimes \mathcal{K}$.

I call linear combinations of the creation and annihilation operators *field operators* and denote by $\mathcal{F}$ the space of such operators. They generate the *Clifford algebra* $\mathrm{Cl}(2dk)$ [8] of the anticommutator product in $\mathcal{F}$. The members of $\mathrm{Cl}(2dk)$ act on a space Φ that is isomorphic as a vector space to the subalgebra of $\mathrm{Cl}(2dk)$ generated by the creation operators, the *spinor space* of $\mathrm{Cl}(2dk)$ [9]. *This* is the Fock space. One state $|\rangle$ in Φ corresponds to the unit member of the subalgebra, and the action of $\mathrm{Cl}(2dk)$ on Φ is given by the convention that every annihilation operator kills $|\rangle$, which may thus be seen as a *vacuum state*. This renders $\mathrm{Cl}(2dk)$ identical to the algebra of linear transformations of Φ. One may alternatively identify Φ with the exterior algebra on $\mathcal{S} \otimes \mathcal{K}$, which renders the creation and annihilation operators identical to the operators of exterior and interior multiplication by members of $\mathcal{S} \otimes \mathcal{K}$ and $\mathcal{S}^* \otimes \mathcal{K}^*$. Note that I *do not* introduce any Hermitian inner product on Φ, so despite the notation, $a_{p\tau}$ and $a^\dagger_{p\tau}$ should not be seen as a Hermitian conjugates.

4 Fock Space Representations of Orthogonal Lie Algebras

I refer to [10] for the basic theory of orthogonal Lie algebras. The *orthogonal Lie algebra* $\mathfrak{o}(n)$ is the Lie algebra of infinitesimal linear transformations x of a vector space V of finite dimension n that preserves a non-degenerate, symmetric bilinear form $\langle b|$ on V in the sense that

$$\langle b|(x|i\rangle \otimes |j\rangle + |i\rangle \otimes x|j\rangle) = 0 \tag{2}$$

for every $|i\rangle, |j\rangle \in V$. Different $\langle b|$ make isomorphic Lie algebras. A convenient choice is

$$\langle b|ij\rangle = \delta_{i+j,0} \tag{3}$$

relative to a basis $(|i\rangle, i = -\Omega, \dots, \Omega)$ with $|0\rangle$ omitted when n is even, where $\Omega = \lfloor n/2 \rfloor$. In terms of basic elements

$$e_{ij} = |i\rangle\langle j| \tag{4}$$

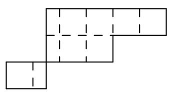
of the space of linear transformations of V, where $|i\rangle$ and $\langle j|$ belong to dual bases for V and V^*, the Lie algebra $\mathfrak{o}(n)$ defined by (3) is spanned by the transformations

$$\bar{e}_{ij} = e_{ij} - e_{-j,-i} \tag{5}$$

with $i + j > 0$. Its finite-dimensional irreducible representations are characterised by *highest weights* $\Lambda = (\Lambda_1, \dots, \Lambda_\Omega)$, where each Λ_i is the module eigenvalue of $\bar{e}_{ii}$ on a *highest weight vector* killed by the module action of every $\bar{e}_{ij}$ with $i > j$. (Isomorphic representations are considered identical unless otherwise specified.) The highest weight vector is unique within normalisation and the entire irreducible module is generated from it by polynomials in the module actions of the basic elements (5) of the Lie algebra. The highest weight components Λ_i are either integral or half-integral and obey

$$\begin{array}{ll} 0 \le \Lambda_1 \le \Lambda_2 \le \cdots \le \Lambda_\Omega & \text{when } n \text{ is odd}\,, \\ 0 \le |\Lambda_1| \le \Lambda_2 \le \cdots \le \Lambda_\Omega & \text{when } n \text{ is even}\,. \end{array} \tag{6}$$

Highest weights are conveniently visualized by generalized Young diagrams [3]. For example, the diagram



(7)

describes the highest weight $\Lambda = (-3/2, 5/2, 9/2)$. Rows of zero length are omitted. In particular the diagram is empty if every Λ_i is zero. In the case of $\mathfrak{o}(2)$, the edge whence the single row extends must be specified. By convention, $-\Lambda$ shall denote for even n the highest weight obtained from Λ by changing the sign of Λ_1.

Remark. The Lie algebras $\mathfrak{o}(1)$, $\mathfrak{o}(2)$ and $\mathfrak{o}(4)$ are special. $\mathfrak{o}(1)$ is 0-dimensional, and its only irreducible representation consist of the operator 0 on a 1-dimensional vector space. It concurs with the systematics to assign to this representation $\Lambda = ()$ (the 0-tuple) and an empty diagram. $\mathfrak{o}(2)$ is 1-dimensional. Its irreducible representations are described by an arbitrary complex number Λ_1, but only those with an integral or half-integral Λ_1 occur in the sequel. $\mathfrak{o}(4)$ is isomorphic to $\mathfrak{o}(3) \oplus \mathfrak{o}(3)$, and its irreducible modules with highest weights (Λ_1, Λ_2) are products of irreducible $\mathfrak{o}(3)$ modules with single highest weight components $(\Lambda_2 \pm \Lambda_1)/2$ [4].

Every $\mathfrak{o}(n)$ is the Lie algebra of the *orthogonal group* $\mathrm{O}(n)$ of linear transformation g of V that preserve $\langle b|$ in the sense that

$$\langle b|(g|i\rangle \otimes g|j\rangle) = \langle b|(|i\rangle \otimes |j\rangle) \tag{8}$$

for every $|i\rangle, |j\rangle \in V$. This implies $\det g = \pm 1$. The elements $g \in \mathrm{O}(n)$ with $\det g = 1$ form the subgroup $\mathrm{SO}(n)$ of *proper* orthogonal transformations.

The Fock space Φ carries two commuting faithful representations of orthogonal Lie algebras [4]. The first one is a representation of $\mathfrak{o}(d)$ defined by

$$x \mapsto \sum_{pq\tau} a^\dagger_{p\tau} \langle p|x|q\rangle a_{q\tau} \tag{9}$$

for every $x \in \mathfrak{o}(d)$. The second one is a representation of $\mathfrak{o}(2k)$ given by

$$\bar{e}_{\tau\upsilon} \mapsto \sum_p a_{p\tau} a^\dagger_{p\upsilon} - \delta_{\tau\upsilon} \frac{d}{2} \,,$$

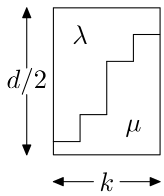
$$\bar{e}_{\tau,-\upsilon} \mapsto \sum_{pq} \langle b|pq\rangle a_{p\tau} a_{q\upsilon} \,, \qquad \bar{e}_{-\tau,\upsilon} \mapsto \sum_{pq} a^\dagger_{p\tau} a^\dagger_{q\upsilon} \langle pq|b\rangle \,. \tag{10}$$

in terms of the bilinear form $\langle b|$ that defines $\mathfrak{o}(d)$. For $\mathfrak{o}(2k)$, the bilinear form (3) is implicit. In (9) and (10), summation indices p and q scan the labels of a basis for $\mathcal{S}$ and the indices τ and υ scan the set of fermion kinds. In [4], these representations are called number conserving and number non-conserving, respectively. By a calculation of characters, I prove in [4] the following.

Theorem 1 ($\mathfrak{o}(d)$–$\mathfrak{o}(2k)$ duality)**.** *The fermion Fock space Φ has the decomposition*

$$\Phi = \bigoplus \mathrm{X}_\lambda \otimes \Psi_\mu, \tag{11}$$

where X_λ and Ψ_μ carry representations of $\mathfrak{o}(d)$ and $\mathfrak{o}(2k)$. The summation in (11) runs over all pairs (λ, μ) of highest weights such that the λ Young diagram and a reflected and rotated copy of the μ Young diagram fill a $d/2 \times k$ frame without overlap as in the following example, where $(d, k) = (11, 4)$, $\lambda = (1, 2, 2, 3, 4)$ and $\mu = (1/2, 3/2, 7/2, 9/2)$.

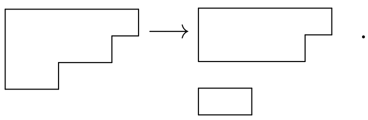


If the border between the diagrams hits the bottom of the frame (which is possible only when d is even), $\mathrm{X}_\lambda = \bar{\mathrm{X}}_\lambda \oplus \bar{\mathrm{X}}_{-\lambda}$*, where* $\bar{\mathrm{X}}_{\pm\lambda}$ *are irreducible with highest weights* $\pm\lambda$*, and* Ψ_μ *is irreducible with highest weight* μ*. If the border hits the left edge (as in the example),* X_λ *is irreducible with highest weight* λ*, and* $\Psi_\mu = \bar\Psi_\mu \oplus \bar\Psi_{-\mu}$*, where* $\bar\Psi_{\pm\mu}$ *are irreducible with highest weights* $\pm\mu$*.*

The direct product of an $\mathfrak{o}(d)$ highest weight vector of $\bar{\mathrm{X}}_{\pm\lambda}$ or X_λ and an $\mathfrak{o}(2k)$ highest weight vector of Ψ_μ or $\bar\Psi_{\pm\mu}$ is an $\mathfrak{o}(d) \oplus \mathfrak{o}(2k)$ highest weight state of the product module. In [3], I construct these states. To this end, I define modified $\mathfrak{o}(d)$ Young diagrams of maximal depth d whose rows are labelled $p = \lfloor d/2 \rfloor, \lfloor d/2 \rfloor - 1, \ldots$ from the top with $p = 0$ omitted when d is even. The columns are labeled $\tau = 1, 2, \ldots$ from the left. For each such diagram $D,$ I set

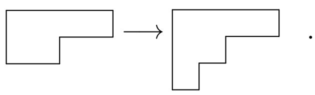
$$\phi_D = \left(\prod_{p\tau \in D} a^\dagger_{p\tau} \right) |\rangle, \tag{12}$$

where the product runs over the cells of D with each cell labelled by its row p and column τ. The order of the factors $a^\dagger_{p\tau}$ is immaterial. When d is even and $\lambda_1 > 0$, I consider besides the λ diagram the diagram obtained by moving its bottom row one step down. For example, for $d = 6$ and $\lambda = (2, 4, 5)$, the change is



(13)

I then set $D = \lambda$ for the original diagram and $D = -\lambda$ for the changed diagram. Then $\phi_{\pm\lambda}$ are $\mathfrak{o}(d) \oplus \mathfrak{o}(2k)$ highest weight states of $\bar{\mathrm{X}}_{\pm\lambda} \otimes \Psi_\mu$. Otherwise if $\tilde\lambda_1$ is the depth of the first column, I extend this column to the depth $d - \tilde\lambda_1$. For example, for $d = 5$ and $\lambda = (2, 4)$, the change is



(14)

I set $D = \lambda$ for the original diagram and $D = \lambda'$ for the changed diagram. Then ϕ_λ and $\phi_{\lambda'}$ are $\mathfrak{o}(d) \oplus \mathfrak{o}(2k)$ highest weight states of $\mathrm{X}_\lambda \otimes \bar\Psi_{\pm\mu}$.

5 Fock Space Representations of $\mathrm{Pin}(2k)$

Definitions of a groups $\mathrm{Pin}(n)$ vary in the literature, but every definition gives within isomorphism the same subgroups $\mathrm{Spin}(n)$ (see below), isomorphic for $n \geq 3$ to the universal covering groups [11] of $\mathrm{SO}(n)$. I adopt the definition due to Atiyah, Bott and Shapiro [5] as naturally generalized from the real to the complex case by Goodman and Wallach [12]. Since I am concerned only with the groups $\mathrm{Pin}(2k)$, I shall not consider the case when n is odd.

Atiyah, Bott and Shapiro construct $\mathrm{Pin}(2k)$ within the Clifford algebra $\mathrm{Cl}(2k)$, which is obtained by setting $d = 1$ in Section 3. I denote in this case the space $\mathcal{F}$ of field operators by $\mathcal{F}_1$ and omit the index 1 in $a_{1\tau}$ and $a^\dagger_{1\tau}$. Then $\mathrm{Pin}(2k)$ is the group of all products of elements $\alpha \in \mathcal{F}_1$ such that $\alpha^2 = -1$. An operator ι on $\mathrm{Cl}(2k)$ is given by $\alpha \mapsto -\alpha$ for $\alpha \in \mathcal{F}_1$, and an operator θ on $\mathrm{Cl}(2k)$ inverts the order of the factors in any product of members of $\mathcal{F}_1$. The bilinear form $\{\alpha, \beta\}$ defines a realization of $\mathrm{O}(2k)$ on $\mathcal{F}_1$, which I identify with $\mathrm{O}(2k)$. It can be shown that for every $g \in \mathrm{Pin}(2k)$ such that

$$g(\iota\theta g) = 1, \tag{15}$$

the transformation

$$u: \quad \alpha \mapsto (\iota g)\alpha(\iota\theta g), \quad \forall \alpha \in \mathcal{F}_1, \tag{16}$$

belongs to this $\mathrm{O}(2k)$ and that the map $\mathrm{Pin}(2k) \to \mathrm{O}(2k) : g \mapsto u$ is surjective [5, 12]. Evidently, elements $\pm g \in \mathrm{Pin}(2k)$ map to the same u, so $g \mapsto u$ provides a *double covering* of $\mathrm{O}(2k)$ by $\mathrm{Pin}(2k)$. The Lie algebra of $\mathrm{Pin}(2k)$ is then isomorphic to $\mathfrak{o}(2k)$, and I identify it with $\mathfrak{o}(2k)$

The products of an even number of factors $\alpha \in \mathcal{F}_1$ with $\alpha^2 = -1$ form a subgroup $\mathrm{Spin}(2k)$ of $\mathrm{Pin}(2k)$ which double covers $\mathrm{SO}(2k)$ and is the maximal connected subset of $\mathrm{Pin}(2k)$. The Lie algebra of $\mathrm{Spin}(2k)$ is that of $\mathrm{Pin}(2k)$, that is, $\mathfrak{o}(2k)$. The relation (16) leads to a defining relation

$$(\iota x)\alpha + \alpha(\iota\theta x) = 0, \quad \forall \alpha \in \mathcal{F}_1, \tag{17}$$

for the members x of $\mathfrak{o}(2k)$. It is easily verified that (17) holds when x is any commutator $[\beta, \gamma]$ of field operators $\beta, \gamma \in \mathcal{F}_1$. The span of the set of these commutators is exactly the span of the set of operators on the right in (10) for $d = 1$, so it exhausts $\mathfrak{o}(2k)$. Since the representation (10) is faithful, one therefore gets a representation ρ of $\mathfrak{o}(2k)$ on the Fock space Φ by composing the map (10) with the inverse map for $d = 1$.

I now choose basic vectors $|f\rangle, |g\rangle, \dots$ in $\mathcal{S}$ such that $\langle b|fg\rangle = \delta_{fg}$ and set $\alpha_f = \sum_\tau (u_\tau a_{f\tau} + v_\tau a^\dagger_{f\tau})$ when $\alpha = \sum_\tau (u_\tau a_\tau + v_\tau a^\dagger_\tau)$ with numeric coefficients u_τ and v_τ. The map ρ is then given by

$$\rho: \quad [\alpha, \beta] \mapsto \sum_f [\alpha_f, \beta_f], \quad \forall \alpha, \beta \in \mathcal{F}_1. \tag{18}$$

This evidently expands to a representation of Spin$(2k)$ given by

$$\rho: \quad g \mapsto \prod_f g_f, \tag{19}$$

where g_f is obtained by substituting every $\alpha \in \mathcal{F}_1$ by α_f in the expression for g as a member of Cl$(2k)$. Indeed, since g is the product of an even number of factors α, the factors in (19) commute. To get a representation of Pin$(2k)$, it then suffices to construct $\rho(\alpha)$ for one $\alpha \in \mathcal{F}_1$ with $\alpha^2 = -1$ such that (i) for every $g, g' \in$ Spin$(2k)$ such that $\alpha g = g'\alpha$ the identity $\rho(\alpha)\rho(g) = \rho(g')\rho(\alpha)$ holds, and (ii) $\rho(\alpha)^2 = \rho(-1)$. Because Spin$(2k)$ is connected, the condition (i) is equivalent to (i') $\alpha x = x'\alpha \Rightarrow \rho(\alpha)\rho(x) = \rho(x')\rho(\alpha)$ for every $x, x' \in \mathfrak{o}(2k)$.

This condition holds if there is a map $\beta \mapsto \beta'$ of $\mathcal{F}_1$ into itself such that $\alpha\beta = \beta'\alpha$ for $\beta \in \mathcal{F}_1$ and $\rho(\alpha)\beta_f = s\beta'_f\rho(\alpha)$ for every β and f with a fixed sign $s = +$ or $-$. The map $\beta \mapsto \beta'$ exists as $\beta \mapsto -\alpha\beta\alpha = -\beta - \{\alpha, \beta\}\alpha$. Further, $\alpha_f\beta_g = -\beta_g\alpha_f$ for $f \neq g$ and $\alpha_f\beta_f = s\beta'_f\alpha_f$, so $\rho(\alpha)\beta_f = s\beta'_f\rho(\alpha)$ holds with

$$\rho(\alpha) = c\prod_f \alpha_f \tag{20}$$

and $s = (-)^{d-1}$, where c is a numeric factor. Because the factors α_f in (20) anticommute, their order is immaterial. To be specific, I set $f = \Omega, \Omega-1, \dots, -\Omega$ with 0 omitted when d is even and assume this order of the factors in (20).

As to the condition (ii), first notice

$$\begin{aligned}\rho(-1) = \rho(\exp i\pi[a_1, a_1^\dagger]) &= \exp \rho(i\pi[a_1, a_1^\dagger]) \\ &= \exp i\sum_f [a_{f1}, a_{f1}^\dagger] = \prod_f \exp i\pi[a_{f1}, a_{f1}^\dagger] = (-1)^d.\end{aligned} \tag{21}$$

Further, by $\alpha_f\alpha_g = -\alpha_g\alpha_f$ for $f \neq g$ and $\alpha_f^2 = -1$, the expression (20) gives $\rho(\alpha)^2 = (-)^{d+\Omega}c^2$, so (ii) holds when c is a square root of $(-1)^\Omega$. I choose $c = i^\Omega$.

I now set

$$\alpha = a_1^\dagger - a_1, \tag{22}$$

whence follows

$$a'_1 = -a_1^\dagger, \quad (a^\dagger)'_1 = -a_1, \quad a'_\tau = -a_\tau, \quad (a_\tau^\dagger)' = -a_\tau^\dagger, \quad \tau > 1, \tag{23}$$

and with $\sigma = \rho(\alpha)$,

$$\begin{aligned}\sigma a_{f1} &= (-)^d a_{f1}^\dagger\sigma, & \sigma a_{f1}^\dagger &= (-)^d a_{f1}\sigma, \\ \sigma a_{f\tau} &= (-)^d a_{f\tau}\sigma, & \sigma a_{f\tau}^\dagger &= (-)^d a_{f\tau}^\dagger\sigma, \quad \tau > 1, \forall f\,.\end{aligned} \tag{24}$$

Further,

$$\sigma|\rangle = i^\Omega \left(\prod_f a^\dagger_{f1} \right) |\rangle \tag{25}$$

with the ordering of the indices f as in (20). The operator σ is similar but not identical to the operator denoted by this symbol in [3]. It shares, in particular, with the latter the property that it commutes with the module action of every $g \in \mathrm{SO}(d)$ by the representation of $\mathrm{O}(d)$ on Φ defined in [3] and anticommutes with the module action of every $g \in \mathrm{O}(d) \setminus \mathrm{SO}(d)$. The restriction to $\mathfrak{o}(d)$ of this representation is given by (9). The commutation or anticommutation property of the present σ is an easy consequence of the fact that the product in (20) acquires a factor $\det g$ by the change of basis $|f\rangle \mapsto g|f\rangle$.

Next I change the basis for $\mathcal{S}$ to the set of vectors $|p\rangle, |q\rangle, \dots$ given by

$$|\pm p\rangle = \sqrt{\tfrac{1}{2}}(|f\rangle \pm i|-f\rangle) \quad \text{for } p = f > 0, \quad |p\rangle = |f\rangle \quad \text{for } p = f = 0. \tag{26}$$

Then $\langle b|pq\rangle = \delta_{p+q,0}$, so the description of highest weight vectors in Section 4 applies. The dual basis is given by

$$\langle \pm p| = \sqrt{\tfrac{1}{2}}(\langle f| \mp i\langle -f|) \quad \text{for } p = f > 0, \quad \langle p| = \langle f| \quad \text{for } p = f = 0, \tag{27}$$

so (24) becomes

$$\begin{aligned} \sigma a_{p1} &= (-)^d a^\dagger_{-p,1}\sigma, & \sigma a^\dagger_{p1} &= (-)^d a_{-p,1}\sigma, \\ \sigma a_{p\tau} &= (-)^d a_{p\tau}\sigma, & \sigma a^\dagger_{p\tau} &= (-)^d a^\dagger_{p\tau}\sigma, \quad \tau > 1, \forall f\,. \end{aligned} \tag{28}$$

Since the inverse of the transformation (26) has determinant i^Ω, equation (25) becomes

$$\sigma|\rangle = (-)^\Omega \left(\prod_p a^\dagger_{p1} \right) |\rangle \tag{29}$$

with the indices p ordered as the indices f in (20).

6 $\mathfrak{o}(d)$-$\mathrm{Pin}(2k)$ Duality

Now consider the action of σ on the highest weight states (12). With an appropriate ordering of the factors $a^\dagger_{p\tau}$ in (12), one can write

$$\phi_D = \left(\prod_{p \in R} a^\dagger_{p1} \right) \phi_{D'}, \tag{30}$$

where R is the set of labels p of the rows in D, taken in the product to be ordered from the top, and D' is D without the first column. By (28) and (29), one gets

$$\sigma\phi_{D'} = (-)^{\Omega}\left(\prod_p a_{p1}^\dagger\right)\phi_{D'}, \tag{31}$$

whence by (28),

$$\sigma\phi_D = (-)^{|R|d+\Omega}\left(\prod_{p\in R} a_{-p,1}\right)\left(\prod_p a_{p1}^\dagger\right)\phi_{D'}. \tag{32}$$

For every D in Section 4 except the one on the right in (13), this gives

$$\sigma\phi_D = (-)^{|R|+\Omega}\left(\prod_{p\in\complement R} a_{-p,1}^\dagger\right)\phi_{D'}, \tag{33}$$

where the complement $\complement R$ is relative to the set of possible indices p and the product is taken in the order of decreasing $-p$.

When d is even and $\lambda_1 > 0$, equation (33) becomes

$$\sigma\phi_\lambda = \phi_\lambda. \tag{34}$$

Let $\phi_\lambda = |\chi\rangle \otimes |\psi\rangle$ with $|\chi\rangle \in \mathrm{X}_\lambda$ and $|\psi\rangle \in \Psi_\mu$. Since σ commutes with the representation of $\mathfrak{o}(d)$ on Φ and it was shown that for every $x \in \mathfrak{o}(2k)$ there is an $x' \in \mathfrak{o}(2k)$ such that $\sigma\rho(x) = \rho(x')\sigma$, it follows from (34) that $|\chi\rangle \otimes \Psi_\mu$ is invariant to σ. The space $|\chi\rangle \otimes \Psi_\mu$ then carries a representation of $\mathrm{Pin}(2k)$, which is irreducible because $|\chi\rangle \otimes \Psi_\mu$ is irreducible as an $\mathfrak{o}(2k)$ module. By the natural identification of $|\chi\rangle \otimes \Psi_\mu$ with Ψ_μ, this representation becomes an irreducible representation of $\mathrm{Pin}(2k)$ on Ψ_μ and $|\psi\rangle$ a module eigenvector of α with eigenvalue 1.

A similar calculation gives

$$\sigma\phi_{-\lambda} = -\phi_{-\lambda}. \tag{35}$$

It follows again that Ψ_μ carries an irreducible representation of $\mathrm{Pin}(2k)$, but this is inequivalent to the previous one because the module eigenvalue of α on $|\psi\rangle$ is opposite. (The latter may also be deduced as in [3] from the anticommutation of σ with the module action of certain reflection in $\mathrm{O}(d)$.) It is convenient to assign highest weights and Young diagrams as follows to irreducible $\mathrm{Pin}(2k)$ representations whose restrictions to $\mathfrak{o}(2k)$ are irreducible with highest weights μ and such that the highest weight vectors of these $\mathfrak{o}(2k)$ representations defined by identifying the transformations on the left in (10) with those on the right for $d = 1$ are module eigenvectors of α with eigenvalues ± 1. If the eigenvalue is

1, the diagram is identical to the $\mathfrak{o}(2k)$ diagram. If the eigenvalue is -1, the depth $\tilde{\mu}_1$ of the first column is increased to $2k - \tilde{\mu}_1$. The highest weight is the set of row lengths of the diagram, where the rows are labelled from the top by $\tau = k, k-1, \dots, -k$ with $\tau = 0$ omitted and the length 0 is assigned to empty rows.

When d is even and $\lambda_1 = 0$, equation (33) gives

$$\sigma\phi_\lambda = (-)^{\mu_1}\phi_{\lambda'}, \quad \sigma\phi_{\lambda'} = (-)^{\mu_1}\phi_\lambda, \tag{36}$$

where μ_1 is the length of the lowest row in the diagram complementary to λ in Theorem 1. When d is odd, one gets

$$\sigma\phi_\lambda = (-)^{\mu_1 - 1/2}\phi_{\lambda'}, \quad \sigma\phi_{\lambda'} = (-)^{\mu_1 + 1/2}\phi_\lambda. \tag{37}$$

When the factor $(-)^{\mu_1}$ or $(-)^{\mu_1 - 1/2}$ is included in $\phi_{\lambda'}$, these maps become

$$\sigma\phi_\lambda = \phi_{\lambda'}, \quad \sigma\phi_{\lambda'} = \phi_\lambda \tag{38}$$

and

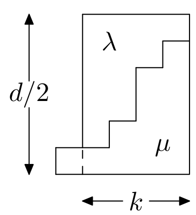
$$\sigma\phi_\lambda = \phi_{\lambda'}, \quad \sigma\phi_{\lambda'} = -\phi_\lambda, \tag{39}$$

respective. By arguments similar to those spelled out above, it hence follows that the $\mathfrak{o}(2k)$ module Ψ_μ of Theorem 1 carries an *irreducible* representation of $\mathrm{Pin}(2k)$ with $\mathfrak{o}(2k)$ highest weight vectors $|\pm\mu\rangle \in \bar{\Psi}_{\pm\mu}$ such that α has the module action $\alpha|\mu\rangle = |-\mu\rangle$, $\alpha|-\mu\rangle = \pm|\mu\rangle$ according to whether μ has integral or half-integral components. I assign to a $\mathrm{Pin}(2k)$ representation composed of irreducible $\mathfrak{o}(2k)$ representations with highest weights $\pm\mu$, where $\mu_1 > 0$, and with this module action of α, the Young diagram of μ and the highest weight obtained by appending k elements 0 at the front of μ. Thus I arrive at the following implication of Theorem 1.

Theorem 2 ($\mathfrak{o}(d)$–$\mathrm{Pin}(2k)$ duality)**.** *The fermion Fock space Φ has the decomposition*

$$\Phi = \bigoplus \mathrm{X}_\lambda \otimes \Psi_\mu, \tag{40}$$

where X_λ and Ψ_μ carry irreducible representations of $\mathfrak{o}(d)$ and $\mathrm{Pin}(2k)$ with highest weights λ and μ. The sum in (40) runs over all pairs of (λ, μ) such that the λ Young diagram and a reflected and rotated copy of the μ Young diagram fill a $d/2 \times k$ frame without overlap when a negative λ_1 cancels a part of the μ Young diagram extruding the frame as in the following example, where $(d, k) = (12, 4)$, $\lambda = (-1, 1, 2, 2, 3, 4)$ and $\mu = (0, 0, 0, 1, 1, 2, 4, 5)$.



(41)

Remark. It is seen from (21) that $\rho(-1) = 1$ for even d. The representation ρ then factors through $\mathrm{O}(2k)$. In this case the $\mathrm{Pin}(2k)$ Young diagrams are the usual $\mathrm{O}(2k)$ Young diagrams [11]. For even d, Theorem 2 is in fact obtained quickly from the theorem of $\mathrm{O}(d)$-$\mathfrak{o}(2k)$ duality [1, 3] by writing $a_{p\tau}$ and $a^\dagger_{p\tau}$ as $a^\dagger_{-p,-\tau}$ and $a_{-p,-\tau}$ for $p < 0$ so that the Fock space representation of $\mathfrak{o}(2k)$ becomes number conserving and that of $\mathfrak{o}(d)$ number non-conserving.

7 Discussion

Several authors discuss dual actions of finite- and infinite-dimensional Lie algebras and, more generally, Lie superalgebras on fermion or boson Fock spaces or combinations. A part of this work is reviewed in [1, 3]. Other references include [13], where, in particular, Theorem 1 is found. A systematic study of dual number conserving actions of classical groups and number non-conserving actions of Lie superalgebras, which include Lie algebras as special cases, was initiated by Howe [14] (preprint 1976) and continued in work including [15–19]. The *classical groups* are the general linear groups $\mathrm{GL}(n)$, the orthogonal groups $\mathrm{O}(n)$, the symplectic groups $\mathrm{Sp}(n)$ and their subgroups. Some special cases of Howe's general duality theorem [14] are presented in [1, 3]. Evidently, every such duality implies by restriction a duality involving the Lie algebra of the pertinent group.

A basic tool in Howe's analysis [14, 15] is a set of results in classical invariant theory which Weyl calls the *first main theorems* of the classical groups [11]. Howe and successors [12, 19] call them first fundamental theorems. According to its first main theorem, the algebra of invariants of a classical group is generated by its quadratic invariants. Wang [16] suggests dual actions of groups $\mathrm{Pin}(n)$ and certain infinite-dimensional Lie algebras, invoking similarity with arguments in [15] which employ the classical first main theorems. Similarly, Cheng, Kwon and Wang [18] propose dual actions of groups $\mathrm{Pin}(n)$ and either Lie superalgebras or certain infinite-dimensional Lie algebras, invoking similarity with arguments by Cheng and Zhang [17], who, in turn, refer to [14]. However, Howe does not list the groups $\mathrm{Pin}(n)$ among those with known first main theorems [20].

For comparison my proof of Theorem 2 is based on Theorem 1, which can be obtained without recourse to a first main theorem. Anyway the purely fermionic case of Theorem A.1 in [18] is equivalent to the odd d case of my Theorem 2. The representations considered there are obtained, indeed, from the present ones by writing $a_{p\tau}$ and $a^\dagger_{p\tau}$ as $a^\dagger_{-p,-\tau}$ and $a_{-p,-\tau}$ for $p \le 0$. This renders neither representation of $\mathfrak{o}(d)$ nor $\mathfrak{o}(2k)$ number conserving in general. My result thus provides a proof of the purely fermionic case of the theorem in [18] that does not rely on a first main theorem.

8 Summary

The result of my discussion is Theorem 2, which establishes an orthogonal-Pin duality of Fock space representations. The proof is based on Theorem 1, which was obtained in [4] by a calculation of characters analogous to Helmers's proof of a symplectic-symplectic duality [21]. Theorem 2 is closely analogous to the theorem of $\mathrm{O}(d)$-$\mathfrak{o}(2k)$ duality discussed in [1, 3] by establishing a 1–1 correspondence between representations of a Lie algebra and a group. Both differ in this respect from Theorem 1, which establishes a 2–1 or 1–2 correspondence between representations of Lie algebras. Together, these three theorems present of remarkably symmetric pattern. In [3], the theorem of $\mathrm{O}(d)$-$\mathfrak{o}(2k)$ duality is derived from Theorem 1 like Theorem 2 was done above. Both theorems thus follow from Theorem 1, and inversely, both of them obviously imply Theorem 1. This renders the three theorems equivalent and ultimately based on a calculation of characters. In this respect, the proof of the theorem of $\mathrm{O}(d)$-$\mathfrak{o}(2k)$ duality in [3] differs from a derivation as a special case of a general duality theorem due to Howe whose proof is based on a set of results in classical invariant theory which Weyl calls the first main theorems on invariants of classical groups. (Howe and successors call them first fundamental theorems.) Theorem 2 could not be obtained in this way due to a lack of first main theorems for Pin groups. My result provides a proof that does not rely on such a theorem of the purely fermionic case of a theorem on dual Fock space representations of a Lie superalgebra and a Pin group formulated by Cheng, Kwon and Wang.

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