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Curvature Properties of Bardeen Black Hole Spacetime

A.A. Shaikh*, **S.K. Hui**, **M. Sarkar**

Department of Mathematics, University of Burdwan,
Golapbag, Burdwan-713104, West Bengal, India
*Corresponding author's E-mail: aashaikh@math.buruniv.ac.in

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Abstract. The Bardeen solution corresponding to Einstein field equations with a cosmological constant is a regular black hole. The main goal of this manuscript is to investigate the geometric structures in terms of curvature conditions admitted by this spacetime. It is found that this spacetime is pseudosymmetric and possesses several kinds of pseudosymmetries. Also, it is a manifold of pseudosymmetric Weyl curvature and the difference tensor $C \cdot R - R \cdot C$ linearly depends on the tensors $Q(g, C)$ and $Q(S, C)$. It is interesting to note that such a spacetime is weakly generalized recurrent manifold and satisfies special recurrent like structure. Further, it is an Einstein manifold of level 2 and Roter type. The energy momentum tensor of this spacetime is pseudosymmetric and finally a worthy comparison between the geometric properties of Bardeen spacetime and Reissner-Nordström spacetime is given.

KEY WORDS: Bardeen black hole metric, pseudosymmetric Weyl conformal curvature tensor, pseudosymmetric type curvature condition, Roter type manifold, Einstein manifold of level 2.

1 Introduction

The geometry of a space is described by the curvature, which plays a crucial role in differential geometry as the symmetry of the space is determined by the restriction on the curvature tensor R. There are several classes of manifolds with specific geometric structures, such as, the locally symmetric manifolds by Cartan [1] defined as $\nabla R = 0$, semisymmetric manifolds again by Cartan [2–5] defined as $R \cdot R = 0$, pseudosymmetric manifolds by Adamów and Deszcz [6]. Many authors have studied locally symmetric manifolds and introduced several generalized notions of manifolds, such as recurrent manifolds by Ruse [7–10], several classes of generalized recurrent manifolds by Shaikh and his coauthors [11–14], weakly symmetric manifolds by Tamássy and Binh [15, 16], pseudosymmetric manifolds by Chaki [17, 18], curvature 2-forms of recurrent manifolds by Besse [19–23] etc. We note that the notion of pseudosymmetry in the sense of Deszcz is important in pseudo-Riemannian geometry as well as in general relativity because several spacetimes are models of pseudosymmetric manifolds (see e.g. [24–32]). Also, the geometrical and physical significances of different kinds of pseudosymmetries have been studied by Haesen and Verstraelen [33–35]. We mention that pseudosymmetry in the sense of Deszcz and pseudosymmetry in the sense of Chaki are different concepts (see, [36]). The study of geometric structure of a certain spacetime gives us the idea about its geometry and provides knowledge about its physical nature.

It is known that spacetime singularity is a reflection of the incompleteness of general relativity. In 1968, Bardeen [37] obtained a solution of Einstein field equation in spherical symmetry, which describes a singularity free black hole spacetime. In fact, it is the first regular black hole model in general relativity. In spherical coordinates system (t, ρ, θ, ϕ), the metric of Bardeen spacetime is given as follows:

$$ds^2 = -\Big(1 - \frac{2M\rho^2}{(e^2+\rho^2)^{3/2}}\Big)dt^2 + \Big(1 - \frac{2M\rho^2}{(e^2+\rho^2)^{3/2}}\Big)^{-1} d\rho^2 + \rho^2\left(d\theta^2 + \sin^2\theta d\phi^2\right), \tag{1}$$

where M and e respectively denote the mass and magnetic charge of the non-linear self-gravity monopole [37]. If $e^2 \leq \frac{16}{27}M^2$, then the Bardeen model represents a black hole and a singularity-free structure [38]. When $e^2 = \frac{16}{27}M^2$, the horizons shrink into a single one that corresponds to an extreme black hole such as the extreme Reissner-Nordström solution. We note that the metric in (1) represents a particular case of the spherically symmetric doubly warped product metric

$$ds^2 = -b^2(\rho)dt^2 + a^2(t)[f_1^2(\rho)d\rho^2 + f_2^2(\rho)d\Omega_2^2], \tag{2}$$

which was covariantly characterized by Mantica and Molinari [39], where $d\Omega_2^2 = d\theta^2 + \sin^2\theta d\phi^2$. Especially, if $a(t) = 1$, $b(\rho) = [1 - \frac{2M\rho^2}{(e^2+\rho^2)^{3/2}}]^{\frac{1}{2}}$, $f_1(\rho) = [1 - \frac{2M\rho^2}{(e^2+\rho^2)^{3/2}}]^{-\frac{1}{2}}$ and $f_2(\rho) = \rho$, then the metric (2) turns out to the metric (1). In [40], Borde has studied Bardeen spacetime and proved that topology change is necessary for the existence of regular black holes satisfying the weak energy condition. In [38] Ayón-Beato and García reinterpreted this black hole as a gravitational field of a nonlinear magnetic monopole. However, the curvature properties of Bardeen spacetime are yet to known. Thus the present study is devoted to deduce the geometric structures of Bardeen black hole metric in terms of curvature restrictions.

Deriving the components of various curvature tensors, we investigate the curvature properties admitted by the Bardeen black hole metric (1). It is found that Bardeen black hole spacetime is not semisymmetric but pseudosymmetric manifold and satisfies various pseudosymmetric type curvature conditions. The difference tensor $C \cdot R - R \cdot C$ is linearly dependent with the tensors $Q(g, C)$ and $Q(S, C)$. It is also a weakly generalized recurrent manifold and satisfies special recurrent like structure. Further, it is Roter type and Einstein manifold of level 2. It is interesting to note that this spacetime has pseudosymmetric energy momentum tensor.

In this context, we mention that recently the curvature properties of interior black hole metric [41] were investigated and it is shown that such a interior black hole metric admits several geometric structures such as pseudosymmetry, Ricci pseudosymmetry and several kinds of pseudosymmetric type curvature conditions.

The present paper is composed in four sections: in Section 2 we review various curvature tensors and geometric structures as preliminaries. Section 3 determines the curvature restricted geometric structures admitted by the Bardeen black hole metric (1). Section 4 deals with some geometric structures of energy momentum tensor. In Section 5, we make a comparison between the geometric properties of Bardeen black hole spacetime and Reissner-Nordström spacetime.

2 Preliminaries

Let M, a smooth and connected manifold of dimension $n \geq 3$, be furnished with the Levi-Civita connection ∇ and also with a semi-Riemannian metric g. Let κ be the scalar curvature, S the Ricci curvature and R the Riemann curvature of M.

Now, for two symmetric $(0, 2)$ tensors λ and τ, the Kulkarni-Nomizu product $\tau \wedge \lambda$ is defined as [42–44]

$$\begin{aligned}(\tau \wedge \lambda)(\zeta_1, \zeta_2, X, Y) = &\ \tau(\zeta_1, Y)\lambda(\zeta_2, X) - \tau(\zeta_1, X)\lambda(\zeta_2, Y) \\ &+ \tau(\zeta_2, X)\lambda(\zeta_1, Y) - \tau(\zeta_2, Y)\lambda(\zeta_1, X),\end{aligned}$$

where ζ_1, ζ_2, X, Y are smooth vector fields on M and throughout this paper we assume $\zeta_1, \zeta_2, X, Y \in \chi(M)$, $\chi(M)$ being the set of all smooth vector fields of M.

Now, we consider the following endomorphisms on M [45–48]:

$$\begin{aligned}
&(\zeta_1 \wedge_\tau \zeta_2)X = \tau(\zeta_2, X)\zeta_1 - \tau(\zeta_1, X)\zeta_2, \\
&\mu_{\mathscr{R}}(\zeta_1, \zeta_2)X = \left([\nabla_{\zeta_1}, \nabla_{\zeta_2}] - \nabla_{[\zeta_1,\zeta_2]}\right) X, \\
&\mu_{\mathscr{W}}(\zeta_1, \zeta_2)X = \mu_{\mathscr{R}}(\zeta_1, \zeta_2)X - \frac{\kappa}{n(n-1)} (\zeta_1 \wedge_g \zeta_2) X, \\
&\mu_{\mathscr{K}}(\zeta_1, \zeta_2)X = \mu_{\mathscr{R}}(\zeta_1, \zeta_2)X - \frac{1}{(n-2)} (\zeta_1 \wedge_S \mathscr{J}\zeta_2 + \mathscr{J}\zeta_1 \wedge_S \zeta_2) X,
\end{aligned}$$

$$\mu_{\mathscr{C}}(\zeta_1,\zeta_2)X = \mu_{\mathscr{R}}(\zeta_1,\zeta_2)X + \frac{\kappa}{(n-1)(n-2)}(\zeta_1 \wedge_g \zeta_2)X$$
$$- \frac{1}{(n-2)}\left(\zeta_1 \wedge_S \mathscr{J}\zeta_2 + \mathscr{J}\zeta_1 \wedge_S \zeta_2\right)X,$$
$$\mu_{\mathscr{P}}(\zeta_1,\zeta_2)X = \mu_{\mathscr{R}}(\zeta_1,\zeta_2)X - \frac{1}{(n-1)}\left(\zeta_1 \wedge_S \zeta_2\right)X,$$

where $\mathscr{J}$ is the Ricci operator and is defined by $g(\zeta_1, \mathscr{J}\zeta_2) = S(\zeta_1,\zeta_2)$.

Now, for an endomorphism $\mathscr{D}(\zeta_1,\zeta_2)$, the corresponding $(0,4)$-tensor D is given by

$$D(\zeta_1,\zeta_2,\zeta_3,\zeta_4) = g(\mathscr{D}(\zeta_1,\zeta_2)\zeta_3,\zeta_4). \tag{3}$$

In above relation, replacing $\mathscr{D}$ by $\mu_{\mathscr{C}}$ one can find the $(0,4)$-type Weyl conformal curvature C. Again, replacing $\mathscr{D}$ by $\mu_{\mathscr{R}}$, $\mu_{\mathscr{P}}$, $\mu_{\mathscr{W}}$ and $\mu_{\mathscr{K}}$ one can also look for $(0,4)$-type Riemann curvature R, projective curvature P, concircular curvature W and conharmonic curvature K respectively.

Let η be a $(0,l)$-tensor field on M, $l \geq 1$. Now, for a $(0,4)$-tensor D, we can define the $(0,l+2)$-tensor $D \cdot \eta$ [49–51] by

$$(D\cdot\eta)(\zeta_1,\zeta_2,\cdots,\zeta_l;X,Y) = (\mathscr{D}(X,Y)\cdot\eta)(\zeta_1,\zeta_2,\cdots,\zeta_l)$$
$$= -\eta(\mathscr{D}(X,Y)\zeta_1,\zeta_2,\cdots,\zeta_l) - \cdots - \eta(\zeta_1,\zeta_2,\cdots,\mathscr{D}(X,Y)\zeta_l).$$

In addition, if λ is a symmetric $(0,2)$-tensor, then we define the $(0,l+2)$-tensor $Q(\lambda,\eta)$, called Tachibana tensor [36,52,53], by

$$Q(\lambda,\eta)(\zeta_1,\zeta_2,\cdots,\zeta_l;X,Y) = ((X\wedge_\lambda Y)\cdot\eta)(\zeta_1,\zeta_2,\cdots,\zeta_l)$$
$$= -\eta((X\wedge_\lambda Y)\zeta_1,\zeta_2,\cdots,\zeta_l) - \cdots - \eta(\zeta_1,\zeta_2,\cdots,(X\wedge_\lambda Y)\zeta_l)$$
$$= \lambda(\zeta_1,X)\eta(Y,\zeta_2,\cdots,\zeta_l) + \cdots + \lambda(\zeta_l,X)\eta(\zeta_1,\zeta_2,\cdots,Y)$$
$$- \lambda(\zeta_1,Y)\eta(X,\zeta_2,\cdots,\zeta_l) - \cdots - \lambda(\zeta_l,Y)\eta(\zeta_1,\zeta_2,\cdots,X).$$

The local components $(D\cdot\eta)_{i_1 i_2\cdots i_l\alpha\beta}$ and $Q(\lambda,\eta)_{i_1 i_2\cdots i_l\alpha\beta}$ of the tensor $D\cdot\eta$ and the Tachibana tensor $Q(\lambda,\eta)$ can be written as follows:

$$(D\cdot\eta)_{i_1 i_2\ldots i_l\alpha\beta} = -g^{uv}[D_{\alpha\beta i_1 v}\eta_{u i_2\ldots i_l} + \cdots + D_{\alpha\beta i_l v}\eta_{i_1 i_2\ldots u}],$$
$$Q(\lambda,\eta)_{i_1 i_2\ldots i_l\alpha\beta} = \lambda_{i_1\beta}\eta_{\alpha i_2\ldots i_l} + \cdots + \lambda_{i_l\beta}\eta_{i_1 i_2\ldots\alpha}$$
$$- \lambda_{i_1\alpha}\eta_{\beta i_2\ldots i_l} - \cdots - \lambda_{i_l\alpha}\eta_{i_1 i_2\ldots\beta}.$$

Definition 2.1. *The linear dependency of the tensor $D\cdot\eta$ with $Q(\lambda,\eta)$ defines an η-pseudosymmetric manifold [6, 51, 54–58] due to D, i.e., on this manifold we have $D\cdot\eta = \varrho_\eta Q(\lambda,\eta)$ where ϱ_η is a smooth function on $\{x\in M : Q(\lambda,\eta)_x \neq 0\}$. If $D\cdot\eta = 0$ holds then the manifold M is an η-semisymmetric manifold due to D [2–5, 59].*

For $D = R$, $\eta = R$ and $\lambda = g$ (resp., S) the manifold is simply called as pseudosymmetric manifold or Deszcz's pseudosymmetric manifold (resp., Ricci generalized pseudosymmetric manifold). Again a semisymmetric manifold is obtained for $D = R$ and $\eta = R$, i.e., $R \cdot R = 0$. Similarly, several kinds of pseudosymmetric and semisymmetric type curvature conditions can be obtained for other curvature tensors.

If $S = \alpha g$ holds on M i.e., the Ricci tensor and the metric tensor are linearly dependent, then it is called an Einstein manifold and in this case $\alpha = \frac{\kappa}{n}$. In a quasi-Einstein manifold the Ricci tensor takes the form $S = \beta A \otimes A + \alpha g$ (A being some 1-form). This notion is important in general relativity as a connected Lorentzian manifold of dimension 4 is a spacetime of perfect fluid if it is quasi-Einstein and vice versa. In a quasi-Einstein manifold rank$(S - \alpha g) = 1$, and if rank$(S - \alpha g) = 2$ or 3 then the manifold is called respectively 2 or 3 quasi-Einstein manifold [60–62]. It may be mentioned that Kaigorodov spacetime [32] is Einstein, Robertson Walker spacetime [63–65] is quasi-Einstein, whereas Kantowski-Sachs spacetime [46] and point-like global monopole spacetime [66] are 2-quasi-Einstein. Again, a Ricci simple manifold is a special case of quasi-Einstein manifold for $\alpha = 0$. We note that Morris-Thorne spacetime [67] and Vaidya metric [68] are examples of Ricci simple spacetimes. Another generalization of Einstein manifolds is given below:

Definition 2.2. *([19, 48, 51]) A manifold M satisfying the relation*

$$\beta_1 S^4 + \beta_2 S^3 + \beta_3 S^2 + \beta_4 S + \beta_5 g = 0, \ (\beta_1 \neq 0)$$

is called an Einstein manifold of level 4, where β_i are smooth functions on M. If $\beta_1 = 0$ but $\beta_2 \neq 0$ (resp., $\beta_1 = \beta_2 = 0$ but $\beta_3 \neq 0$), then it turns into an Einstein manifold of level 3 (resp. level 2).

It may be noted that Vaidya-Bonner spacetime [69], Som-Raychaudhuri spacetime [70], Sultana-Dyer spacetime [71], Lifshitz spacetime [72] are $Ein(3)$, whereas Hayward spacetime [73], Melvin magnetic spacetime [30], Robinson-Trautman spacetime [29], point-like global monopole spacetime [66] and the warped product metric appeared in [74] are $Ein(2)$.

Definition 2.3. *A manifold M is called a generalized Roter type manifold [48, 75–80] if its Riemann curvature tensor fulfills*

$$R = s_{22}(g \wedge g) + (s_{11} S + s_{12} g) \wedge S + (s_{00} S^2 + s_{01} S + s_{02} g) \wedge S^2,$$

s_{ij} being some smooth functions on M. For $s_{00} = s_{01} = s_{02} = 0$, R is linearly dependent on $g \wedge g$, $S \wedge g$ and $S \wedge S$, and in this case M is called a Roter type manifold [76, 81–84].

It is worthy to note that Melvin magnetic spacetime [30] and Robinson-Trautman spacetime [85] are Roter type, whereas Lifshitz spacetime [72] and Lemaître-Tolman-Bondi spacetime [31] are generalized Roter type.

Definition 2.4. *A super generalized recurrent manifold M is defined by curvature condition [14]*

$$\nabla R = \Pi \otimes R + A \otimes (g \wedge g) + \bar{A}(S \wedge g) + \bar{\bar{A}}(S \wedge S)\,,$$

where $A, \bar{A}, \bar{\bar{A}}$ are some 1-forms on M. A weakly generalized recurrent manifold [86, 87] (resp., hyper generalized recurrent manifold [11, 13]) is obtained for $A = \bar{A} = 0$ (resp., $A = \bar{\bar{A}} = 0$).

Definition 2.5. *Tamássy and Binh [15, 16] defined the notion of weak symmetry by*

$$\begin{aligned}(\nabla_X R)(\zeta_1, \zeta_2, \zeta_3, \zeta_4) = \Pi(X) \otimes R(\zeta_1, \zeta_2, \zeta_3, \zeta_4) &+ \bar{B}(\zeta_4) \otimes R(\zeta_1, \zeta_2, \zeta_3, X) \\ &+ B(\zeta_3) \otimes R(\zeta_1, \zeta_2, X, \zeta_4) + \bar{A}(\zeta_2) \otimes R(\zeta_1, X, \zeta_3, \zeta_4) \\ &+ A(\zeta_1) \otimes R(X, \zeta_2, \zeta_3, \zeta_4),\end{aligned}$$

where $A, \bar{A}, B, \bar{B}$ are some 1-forms on M. For $\Pi = A/2 = \bar{A}/2 = B/2 = \bar{B}/2$, it is a Chaki pseudosymmetric manifold [17, 18].

Definition 2.6. *[20–23, 88] Let D be a $(0, 4)$-type tensor and λ be a $(0, 2)$-type symmetric tensor on M. Then $\Omega^m_{(D)l}$ [89], the curvature 2-forms, are recurrent if*

$$\underset{\zeta_1,\zeta_2,\zeta_3}{\mathcal{S}} (\nabla_{\zeta_1} D)(\zeta_2, \zeta_3, X, Y) = \underset{\zeta_1,\zeta_2,\zeta_3}{\mathcal{S}} A(\zeta_1) \otimes D(\zeta_2, \zeta_3, X, Y),$$

$\mathcal{S}$ being the cyclic sum over ζ_1, ζ_2 and ζ_3, holds on M. In addition, the 1-forms $\iota_{(\lambda)l}$ [89] are recurrent if

$$(\nabla_{\zeta_1}\lambda)(\zeta_2, X) - (\nabla_{\zeta_2}\lambda)(\zeta_1, X) = A(\zeta_1) \otimes \lambda(\zeta_2, X) - \bar{A}(\zeta_2) \otimes \lambda(\zeta_1, X)$$

holds on M, where A, $\bar{A}$ are smooth 1-forms.

Definition 2.7. *The Ricci tensor of M is of Codazzi type [90, 91] (resp., cyclic parallel [92–95]) if*

$$(\nabla_{\zeta_1} S)(\zeta_3, \zeta_2) = (\nabla_{\zeta_2} S)(\zeta_3, \zeta_1)$$

$$(resp., \underset{\zeta_1,\zeta_2,\zeta_3}{\mathcal{S}} (\nabla_{\zeta_1} S)(\zeta_2, \zeta_3) = 0)$$

holds on M,

We mention that the Ricci tensor of the $(t - z)$-type plane wave metric is of Codazzi type [96] and the Ricci tensor is cyclic parallel in Gödel spacetime [45].

Definition 2.8. *([78, 97–101]) The Ricci tensor S of M is called Riemann compatible if the relation*

$$\underset{\zeta_1,\zeta_2,\zeta_3}{\mathcal{S}} R(\mathscr{J}\zeta_1, X, \zeta_2, \zeta_3) = 0$$

holds.

This notion of compatibility can be extended to the curvatures C, P, W and K to define the corresponding curvature compatibility.

Definition 2.9. (*[102, 103]) Let D be a symmetric $(0,4)$-type tensor of M. If the 1-forms Θ satisfying the relation*

$$\underset{\zeta_1,\zeta_2,\zeta_3}{\mathcal{S}} \Theta(\zeta_1) \otimes D(\zeta_2, \zeta_3, X, Y) = 0$$

generates a k-dimensional vector space with $k \geq 1$, then M is a D-space by Venzi.

3 Geometric Properties Admitted by Bardeen Black Hole Metric

The components of metric (1) are

$$g_{11} = -\Big(1 - \frac{2M\rho^2}{\rho_1^3}\Big), \qquad g_{22} = \Big((1 - \frac{2M\rho^2}{\rho_1^3})^{-1}\Big),$$
$$g_{33} = \rho^2\,, \qquad g_{44} = \rho^2 \sin^2\theta\,, \qquad g_{ij} = 0 \text{ otherwise},$$

where $\rho_1^2 = e^2 + \rho^2$.

Now, the components of various curvature tensors of Bardeen black hole metric (1) are calculated in a straight forward manner.

The non-zero components of second kind Christoffel symbols Γ^h_{ij} of g are given by

$$\begin{aligned}
\Gamma^2_{11} &= -\frac{M\rho(\rho_1^2 - 3e^2)(2M\rho^2 - \rho_1^3)}{\rho^8};\\
\Gamma^1_{12} &= \frac{M\rho(\rho_1^2 - 3e^2)}{\rho_1^2(-2M\rho^2 + \rho_1^3)} = -\Gamma^2_{22};\\
\Gamma^3_{23} &= \frac{1}{\rho} = \Gamma^4_{24};\\
\Gamma^2_{33} &= -\rho + \frac{2M\rho^3}{\rho_1^3};\\
\Gamma^4_{34} &= \cot\theta;\\
\Gamma^2_{44} &= \rho\Big[-1 + \frac{2M\rho^2}{\rho_1^3}\Big]\sin^2\theta;\\
\Gamma^3_{44} &= -\cos\theta\sin\theta.
\end{aligned} \tag{4}$$

The non-zero components R_{hijk} and S_{ij} of R and S of the Bardeen metric (1) along with its scalar curvature k are

$$
\begin{aligned}
R_{1212} &= \frac{M(15\rho^2 e^2 - 2\rho_1^4)}{\rho_1^7},\\
R_{1313} &= \frac{M\rho^2(\rho_1^2 - 3e^2)(-2M\rho^2 + \rho_1^3)}{\rho_1^8} = \frac{1}{\sin^2\theta} R_{1414},\\
R_{2323} &= -\frac{M\rho^2(\rho_1^2 - 3e^2)}{\rho_1^2(-2M\rho^2 + \rho_1^3)} = \frac{1}{\sin^2\theta} R_{2424},\\
R_{3434} &= \frac{2M\rho^4 \sin^2\theta}{\rho_1^3}; \qquad (5)\\
S_{11} &= \frac{3Me^2(2\rho_1^2 - 3e^2)(-2M\rho^2 + \rho_1^3)}{\rho_1^{10}},\\
S_{22} &= \frac{3e^2 M(5\rho^2 - 2\rho_1^2)}{\rho_1^4(-2M\rho^2 + \rho_1^3)},\\
S_{33} &= -\frac{6Me^2\rho^2}{\rho_1^5} = \frac{1}{\sin^2\theta} S_{44}; \text{ and } k = \frac{6Me^2(5\rho^2 - 4\rho_1^2)}{\rho_1^7}.
\end{aligned}
$$

It may be mentioned that in [39] (see eq. (49)), Mantica and Molinari obtained the covariant form of the Ricci tensor of a spherically symmetric doubly warped spacetime. Hence the Ricci tensor of (1) can be inferred when $a(t) = 1$ (i.e., $\phi = \xi = 0$ in eq. (49) of [39]).

Let $L^1 = (g \wedge g)$, $L^2 = (g \wedge S)$ and $L^3 = (S \wedge S)$. Then the non-zero components of these Kulkarni-Nomizu products are obtained as follows:

$$
\begin{aligned}
L^1_{1212} &= 2, \quad L^1_{1313} = 2(\rho^2 - \frac{2M\rho^4}{\rho_1^3}) = \frac{1}{\sin^2\theta} L^1_{1414},\\
L^1_{2323} &= \frac{2\rho^2\rho_1^3}{2M\rho^2 - \rho_1^3} = -\frac{1}{\sin^2\theta} L^1_{2424}, \quad L^1_{3434} = -2\rho^2\sin^2\theta;\\
L^2_{1212} &= \frac{6Me^2(5\rho^2 - 2\rho_1^2)}{\rho_1^7},\\
L^2_{1313} &= -\frac{3Me^2\rho^2(4\rho_1^2 - 5\rho^2)(-2M\rho^2 + \rho_1^3)}{\rho_1^{10}} = \frac{1}{\sin^2\theta} L^2_{1414},\\
L^2_{2323} &= \frac{3Me^2\rho^2(4\rho_1^2 - 5\rho^2)}{\rho_1^4(-2M\rho^2 + \rho_1^3)} = \frac{1}{\sin^2\theta} L^2_{2424}, \qquad (6)\\
L^2_{3434} &= \frac{12Me^2\rho^4\sin^2\theta}{\rho_1^5}; \quad L^3_{1212} = \frac{18M^2e^4(2\rho_1^2 - 5\rho^2)^2}{\rho_1^{14}},\\
L^3_{1313} &= \frac{36M^2e^4\rho^2(2\rho_1^2 - 5\rho^2)(-2M\rho^2 + \rho_1^3)}{\rho_1^{15}} = \frac{1}{\sin^2\theta} L^3_{1414},\\
L^3_{2323} &= -\frac{36M^2e^4\rho^2(2\rho_1^2 - 5\rho^2)}{\rho_1^9(-2M\rho^2 + \rho_1^3)} = \frac{1}{\sin^2\theta} L^3_{2424},\\
L^3_{3434} &= -\frac{72M^2e^4\rho^4\sin^2\theta}{\rho_1^{10}}.
\end{aligned}
$$

From (6) we can decompose Riemann tensor explicitly as follows:

$$R = \varrho_1 L^1 + \varrho_2 L^2 + \varrho_3 L^3, \tag{7}$$

where $\varrho_1 = \dfrac{M(18\rho_1^2 - 25\rho^2)}{25\rho^2\rho_1^3}$, $\varrho_2 = \dfrac{\rho_1^2(6\rho_1^2 - 5\rho^2)}{25e^2\rho^2}$ and $\varrho_3 = \dfrac{(3\rho_1^2 - 5\rho^2)\rho_1^7}{150Me^4\rho^2}$.
Contracting the relation (7) the following relation is entailed:

$$S^2 + \beta S + \bar{\beta} g = 0, \tag{8}$$

where $\beta = \dfrac{3Me^2(4\rho_1^2 - 5\rho^2)}{\rho_1^7}$ and $\bar{\beta} = \dfrac{18M^2e^4(2\rho_1^2 - 5\rho^2)}{\rho_1^{12}}$.

Proposition 3.1. *The Bardeen metric (1) is not Einstein manifold but it is (i) Einstein manifold of level 2 and (ii) fulfills Roter type condition.*

Corollary 3.1. *Since the Bardeen metric is Roter type, from Theorem 6.7 of [42] we obtain the following geometric structures:*

(i) $R \cdot R = \varrho_R Q(g, R)$;
$$\varrho_R = \frac{1}{2\varrho_2^3}(2((\varrho_2)^2 - 4\varrho_3\varrho_1) - 2\varrho_1) = -\frac{M(2\rho_1^2 - 3\rho^2)^{5/2}}{\rho_1^2};$$

(ii) $R \cdot C = \varrho_R Q(g, C)$;

(iii) $C \cdot R = \varrho_C Q(g, R)$;
$$\varrho_C = \varrho_R - \left(\frac{\kappa}{3} + \frac{\varrho_2}{2\varrho_3}\right) + \frac{1}{4\varrho_3} = -\frac{M\rho^2(3\rho_1^2 - 5\rho^2)}{2\rho_1^7};$$

(iv) $C \cdot C = \varrho_C Q(g, C)$,

(v) $R \cdot R = Q(S, R) + \varrho Q(g, C)$; $\quad \varrho = \varrho_R + \dfrac{\varrho_2}{2\varrho_3} = \dfrac{2M(6\rho_1^2 - 7\rho^2)}{(3\rho_1^2 - 5\rho^2)\rho_1^3}$.

The conformal curvature components of the metric (1) are:

$$\begin{aligned}
C_{1212} &= \frac{M\rho^2(3\rho_1^2 - 5\rho^2)}{\rho_1^7} = -\frac{1}{\rho^4 \sin^2\theta} C_{3434};\\
C_{1313} &= -\frac{M\rho^4(3\rho_1^2 - 5\rho^2)(-2M\rho^2 + \rho_1^3)}{2\rho_1^5} = \frac{1}{\sin^2\theta} C_{1414};\\
C_{2323} &= \frac{M\rho^4(3\rho_1^2 - 5\rho^2)}{2\rho_1^4(-2M\rho^2 + \rho_1^3)} = \frac{1}{\sin^2\theta} C_{2424}.
\end{aligned} \tag{9}$$

Let $N^1_{abcd,f} = \nabla_f R_{abcd}$ and $N^2_{abcd,f} = \nabla_f C_{abcd}$. Then the non-vanishing components of the covariant derivatives of R and C are calculated and presented in below:

$$\begin{cases} N^1_{1212,2} = \dfrac{3M\rho(12e^4 - 21e^2\rho^2 + 2\rho^5)}{\rho_1^9}; \\ N^1_{1213,3} = -\dfrac{3M\rho^3(5\rho^2 - 4\rho_1^2)(-2M\rho^2 + \rho_1^3)}{\rho_1^5} = -N^1_{1313,2} \\ \qquad = \dfrac{1}{\sin^2\theta} N^1_{1214,4} = \dfrac{1}{\sin^2\theta} N^1_{1414,2}; \\ N^1_{2323,2} = \dfrac{3M\rho^3(5\rho^2 - 4\rho_1^2)}{\rho_1^4(-2M\rho^2 + \rho_1^3)} = \dfrac{1}{\sin^2\theta} N^1_{2424,2}; \\ N^1_{2334,4} = \dfrac{3M\rho^5 \sin^2\theta}{\rho_1^5} = -N^1_{2434,3} = -2N^1_{3434,2}; \end{cases} \tag{10}$$

$$\begin{cases} N^2_{1212,2} = \dfrac{\rho M(6e^4 - 23e^2\rho^2 + 6\rho^5)}{\rho_1^9} = -\dfrac{1}{\rho^4 \sin^2\theta} N^2_{3434,2}; \\ N^2_{1213,3} = \dfrac{3M\rho^3(3\rho_1^2 - 5\rho^2)(-2M\rho^2 + \rho_1^3)}{2\rho_1^5} = \dfrac{1}{\sin^2\theta} N^2_{1214,4}; \\ N^2_{1313,2} = -\dfrac{M\rho^3(6e^4 - 23e^2\rho^2 + 6\rho^4)(-2M\rho^2 + \rho_1^3)}{2\rho_1^6} \\ \qquad = -\dfrac{1}{\sin^2\theta} N^2_{1414,2}; \\ N^2_{2323,2} = \dfrac{M\rho^3(6e^4 - 23e^2\rho^2 + 6\rho^4)}{2\rho_1^6(-2M\rho^2 + \rho_1^3)} = \dfrac{1}{\sin^2\theta} N^2_{2424,2}; \\ N^2_{2334,4} = -\dfrac{3M\rho^5(3\rho_1^2 - 5\rho^2)\sin^2\theta}{2\rho_1^7} = -N^2_{2434,3}. \end{cases} \tag{11}$$

From (6), (10) and (11) we get the following:

Proposition 3.2. *The Bardeen metric* (1) *admits the following geometric structures:*

(i) $\nabla R = \Pi \otimes R + A \otimes (S \wedge S)$ *where* $\Pi = \left\{0, \dfrac{6\rho(8M - 5\rho_1)}{5(-2M\rho^2 + \rho_1^3)}, 0, 0\right\}$ *and*

$$A = \left\{0, -\frac{\rho(29e^4 + e^2(53\rho^2 - 8M\rho_1) + 24(\rho^2 - 2M\rho_1\rho^2))}{30M(-2M\rho^2 + \rho_1^3)}, 0, 0\right\},$$

(ii) $\nabla R = A \otimes (g \wedge S)$ *where* $A = \left\{0, \dfrac{2\rho(8M - 5\rho_1)}{5(-2M\rho^2 + \rho_1^3)}, 0, 0\right\}$,

(iii) $\underset{U,V,X}{\mathcal{S}} \nabla_U C(V, X, Y, Z) = \underset{U,V,X}{\mathcal{S}} A(U) \otimes C(V, X, Y, Z)$ *where*

$$A = \left\{0, \frac{5e^2(3\rho_1^2 - 7\rho^2)}{\rho\rho_1^2(3\rho_1^2 - 5\rho^2)}, 0, 0\right\}.$$

Let $B^1 = R \cdot C$, $B^2 = C \cdot R$, $F^1 = Q(g,R)$, $F^2 = Q(S,R)$, $F^3 = Q(g,C)$ and $F^4 = Q(S,C)$. Then, the non-vanishing components of the tensors B^1, B^2, F^1, F^2, F^3 and F^4 are given by (upto symmetry)

$$\begin{cases} B^1_{122313} = -\dfrac{3M^2\rho^4}{2\rho_1^{12}}(2\rho_1^2 - 3\rho^2)(3\rho_1^2 - 5\rho^2) \\ \qquad = \dfrac{1}{\sin^2\theta}B^1_{122414} = -B^1_{121323} = -\dfrac{1}{\sin^2\theta}B^1_{121424}, \\ B^1_{143413} = -\dfrac{3M^2\rho^6}{2\rho_1^{15}}(2\rho_1^2 - 3\rho^2)(3\rho_1^2 - 5\rho^2)(-2M\rho^2 + \rho_1^2)\sin^2\theta \\ \qquad = -B^1_{133414}, \\ B^1_{243423} = \dfrac{3M^2\rho^6(2\rho_1^2 - 3\rho^2)(3\rho_1^2 - 5\rho^2)\sin^2\theta}{2\rho_1^9(-2M\rho^2 + \rho_1^3)} = -B^1_{233424}; \end{cases} \tag{12}$$

$$\begin{cases} B^2_{1223,13} = \dfrac{3M^2\rho^6(-4\rho_1^2 + 5\rho^2)(3\rho_1^2 - 5\rho^2)}{2\rho_1^{14}} \\ \qquad = -B^2_{1213,23} = \dfrac{1}{\sin^2\theta}B^2_{1224,14} = -\dfrac{1}{\sin^2\theta}B^2_{1214,24}, \\ B^2_{1434,13} = \dfrac{3M^2\rho^8}{2\rho_1^{15}}(3\rho_1^2 - 5\rho^2)(-2Mr^2 + \rho_1^3) = -B^2_{1334,14}, \\ B^2_{2434,23} = -\dfrac{3M^2\rho^8(3\rho_1^2 - 5\rho^2)}{2\rho_1^9(-2M\rho^2 + \rho_1^3)} = -B^2_{2334,24}; \end{cases} \tag{13}$$

$$\begin{cases} F^1_{1223,13} = -\dfrac{3M\rho^4}{\rho_1^7}(-4\rho_1^2 + 5\rho^2) \\ \qquad = -\dfrac{1}{\sin^2\theta}F^1_{1214,24} = \dfrac{1}{\sin^2\theta}F^1_{1224,14} = -F^1_{1213,23}, \\ F^1_{1434,13} = \dfrac{3M\rho^6}{\rho_1^4}(2M\rho^2 - \rho_1^3)\sin^2\theta = -F^1_{1334,14}, \\ F^1_{2434,23} = \dfrac{3M\rho^6\sin^2\theta}{\rho_1^2(-2M\rho^2 + \rho_1^3)} = -F^1_{2334,24}; \end{cases} \tag{14}$$

$$\begin{cases} F^2_{1223,13} = \dfrac{3M^2e^2\rho^4}{\rho_1^{12}}(14\rho_1^2 + 13\rho^2) \\ \qquad = -F^2_{1213,23} = \dfrac{1}{\sin^2\theta}F^2_{1224,14} = -\dfrac{1}{\sin^2\theta}F^2_{1214,24}, \\ F^2_{1434,13} = -\dfrac{12M^2e^2\rho^6}{\rho_1^{13}}(-2M\rho^2 + \rho_1^3)\sin^2\theta = -F^2_{1334,14}, \\ F^2_{2434,23} = \dfrac{12M^2e^2\rho^6\sin^2\theta}{\rho_1^7(-2M\rho^2 + \rho_1^3)} = -F^2_{2334,24}; \end{cases} \tag{15}$$

$$\begin{cases} F^3_{1223,13} = \dfrac{3M\rho^4}{2\rho_1^7}(3\rho_1^2 - 5\rho^2) \\ \qquad\qquad = -F^3_{1213,23} = \dfrac{1}{\sin^2\theta}F^3_{1224,14} = -\dfrac{1}{\sin^2\theta}F^3_{1214,24}, \\ F^3_{1434,13} = \dfrac{3M\rho^6}{2\rho_1^{10}}(3\rho_1^2 - 5\rho^2)(-2M\rho^2 + \rho_1^3)\sin^2\theta = -F^3_{1334,14}, \\ F^3_{2434,23} = -\dfrac{3M\rho^6(3\rho_1^2 - 5\rho^2)\sin^2\theta}{2\rho_1^4(-2M\rho^2 + \rho_1^3)} = -F^3_{2334,24}; \end{cases} \tag{16}$$

$$\begin{cases} F^4_{1223,13} = \dfrac{3M^2e^2\rho^4}{2\rho_1^{14}}(3\rho_1^2 - 5\rho^2)(6\rho_1^2 - 7\rho^2) \\ \qquad\qquad = \dfrac{1}{\sin^2\theta}F^4_{1224,14} = -F^4_{1213,23} = \dfrac{1}{\sin^2\theta}F^4_{1214,24}, \\ F^4_{1434,13} = -\dfrac{3M^2e^2\rho^6}{\rho_1^{17}}(3\rho_1^2 - 5\rho^2)^2(-2M\rho^2 + \rho_1^3)\sin^2\theta \\ \qquad\qquad = -F^4_{1334,14}, \\ F^4_{2434,23} = \dfrac{3M^2e^2\rho^6(3\rho_1^2 - 5\rho^2)}{\rho_1^{11}(-2M\rho^2 + \rho_1^3)} = F^4_{2334,24}. \end{cases} \tag{17}$$

Proposition 3.3. *From* (12)– (17) *we obtain the following pseudosymmetric type curvature relations for the metric* (1)*:*

$$C \cdot R - R \cdot C = \bar{\varrho}_2\, Q(S,R) + \bar{\varrho}_1\, Q(g,R)\,, \tag{18}$$

where

$$\bar{\varrho}_1 = -\frac{M(3\rho_1^2 - 5\rho^2)(\rho^2(6\rho_1^2 - 7\rho^2) - (2\rho_1^2 - 3\rho^2)^2)}{2(6\rho_1^2 - 7\rho^2)\rho_1^7},$$
$$\bar{\varrho}_2 = 1 - \frac{3}{14}e^2\Big(\frac{12}{6\rho_1^2 - 7\rho^2} + \frac{5}{\rho_1^2}\Big)$$

and

$$C \cdot R - R \cdot C = Q(S,C) + \bar{\varrho}_3\, Q(g,C)\,, \tag{19}$$

where
$$\bar{\varrho}_3 = \frac{2M(4\rho_1^2 - 5\rho^2)e^2}{\rho_1^7}\,.$$

Let $B^3 = W \cdot R$ and $B^4 = K \cdot R$. Then the non-zero components of the tensors $W \cdot R$ and $K \cdot R$ are given by

$$\begin{cases} B^3_{1223,13} = \dfrac{3M^2\rho^6}{2\rho_1^{14}}(-4\rho_1^2+5\rho^2)(3\rho_1^2-5\rho^2) \\ \qquad = -B^3_{1213,23} = \dfrac{1}{\sin^2\theta}B^3_{1224,14} = -\dfrac{1}{\sin^2\theta}B^3_{1214,24}, \\ B^3_{1434,13} = \dfrac{3M^2\rho^8}{2\rho_1^{15}}(3\rho_1^2-5\rho^2)(-2M\rho^2+\rho_1^3)\sin^2\theta \\ \qquad = -B^3_{1334,14}, \\ B^3_{2434,23} = -\dfrac{3M^2\rho^8(3\rho_1^2-5\rho^2)\sin^2\theta}{2\rho_1^9(-2M\rho^2+\rho_1^3)} = -B^3_{2334,24}; \end{cases} \tag{20}$$

$$\begin{cases} B^4_{1223,13} = -\dfrac{3M^2\rho^4}{2\rho_1^{14}}(-4\rho_1^2+5\rho^2)(8e^4-5e^2\rho^2+2\rho^4) \\ \qquad = -B^4_{1213,23} = \dfrac{1}{\sin^2\theta}B^4_{1224,14} = -\dfrac{1}{\sin^2\theta}B^4_{1214,24}, \\ B^4_{1434,13} = -\dfrac{3M^2\rho^6}{2\rho_1^{15}}(8e^4-5e^2\rho^2+2\rho^4)(-2M\rho^2+\rho_1^3)\sin^2\theta \\ \qquad = -B^4_{1334,14}, \\ B^4_{2434,23} = \dfrac{3M^2\rho^6(8e^4-5e^2\rho^2+2\rho^4)\sin^2\theta}{2\rho_1^9(-2M\rho^2+\rho_1^3)} = -B^4_{2334,24}. \end{cases} \tag{21}$$

From (14), (20) and (21) we get the following:

Proposition 3.4. *The Bardeen metric* (1) *fulfills the curvature conditions*

$$W \cdot R = -\frac{M\rho^2(3\rho_1^2-5\rho^2)}{2\rho_1^7}Q(g,R)$$

and

$$K \cdot R = \frac{M(8e^4-5e^2\rho^2+2\rho^4)}{2\rho_1^7}Q(g,R).$$

From propositions 3.1–3.4 and from corollary 3.1 we can conclude about the curvature properties of the Bardeen spacetime metric (1) as follows:

Theorem 3.1. *The Bardeen metric* (1) *admits the following curvature restricted geometric structures:*

(i) it is a pseudosymmetric spacetime, and as a result it realizes Ricci pseudosymmetry, conformal pseudosymmetry, concircular pseudosymmetry, conharmonic pseudosymmetry and projective pseudosymmetry,

(ii) it is also pseudosymmetric due to conformal curvature, concircular curvature and conharmonic curvature,

(iii) *the difference tensor $C \cdot R - R \cdot C$ is linearly dependent with the tensors $Q(g, R)$ and $Q(S, R)$ as well as it is also linearly dependent with the tensors $Q(g, C)$ and $Q(S, C)$,*

(iv) *the pseudosymmetric type condition*

$$R \cdot R - Q(S, R) = \varrho\, Q(S, C)$$

is possessed by this spacetime, and also, it is equipped with pseudosymmetric Weyl tensor, where ϱ is a smooth scalar function given in corollary 3.1,

(v) *it is a weakly generalized recurrent manifold satisfying special recurrent like structure $\nabla R = A \otimes (g \wedge S)$,*

(vi) *its conformal 2 forms are recurrent,*

(vii) *it is a Roter type spacetime and is an Einstein manifold of level 2,*

(viii) *Ricci tensor is Weyl compatible as well as Riemann compatible.*

Remark 3.1. *From the components of various curvatures we conclude that the following geometric structures are not admitted by the Bardeen metric (1):*

(i) *any semisymmetric type conditions for C, P, W, K, S,*

(ii) *Ricci generalized pseudosymmetry,*

(iii) *D-Venzi space for $E = C, R, P, W, K$,*

(iv) *Einstein or quasi-Einstein condition,*

(v) *curvature 2-forms recurrence,*

(vi) *Codazzi type Ricci tensor or cyclic parallel Ricci tensor,*

(vii) *Chaki pseudosymmetry,*

(viii) *Weak symmetry.*

4 Energy Momentum Tensor of Bardeen Black Hole Metric

In general theory of relativity, Einstein describes the physics of a spacetime in terms of geometry by the system of equations

$$S - \frac{k}{2} g + \Lambda g = \frac{8\pi G}{c^4} T\,,$$

where k is the scalar curvature, S is the Ricci curvature and T is the energy momentum tensor of the spacetime. Also Λ represents the cosmological constant, G is the gravitational constant, c is the speed of light in vacuum.

Taking $8\pi G/c^4 = 1$ the components of Energy momentum tensor are given by

$$
\begin{aligned}
T_{11} &= -\frac{(-2M\rho^2 + \rho_1^3)(6Me^2 + \rho_1^5\Lambda)}{\rho_1^8}\,,\\
T_{22} &= \frac{6Me^2 + \rho_1^5\Lambda}{\rho_1^2(-2M\rho^2 + \rho_1^3)}\,,\\
T_{33} &= \frac{\rho^2(3Me^2(2\rho_1^2 - 5\rho^2) + \rho_1^7\Lambda)}{\rho_1^7} = \frac{1}{\sin^2\theta}T_{44}\,.
\end{aligned}
$$

We note that the covariant form of the energy momentum tensor of a spherically symmetric doubly warped spacetime in Einstein gravity was given in [39] (see eq.(55)). Hence the energy momentum tensor of the metric (1) in Einstein gravity can be inferred as the particular case when $a(t) = 1$ (i.e., $\phi = \xi = 0$ in eq.(55) of [39]). It represents an anisotropic imperfect fluid.

The non-zero components of the tensor $R \cdot T$ are

$$
\begin{aligned}
(R\cdot T)_{1313} &= \frac{15M^2e^2\rho^4}{8\rho_1^{15}}(-2\rho_1^2 + 3\rho^2)(-2M\rho^2 + \rho_1^3) = \frac{1}{\sin^2\theta}(R\cdot T)_{1414},\\
(R\cdot T)_{2323} &= -\frac{15M^2e^2\rho^4(-2\rho_1^2 + 3\rho^2)}{8\rho_1^9(-2M\rho^2 + \rho_1^3)} = \frac{1}{\sin^2\theta}(R\cdot T)_{2424}\,.
\end{aligned}
$$

The non-zero components of the tensor $Q(g,T)$ are

$$
\begin{aligned}
Q(g,T)_{1313} &= \frac{15Me^2\rho^4}{8\rho_1^{10}}(-2M\rho^2 + \rho_1^3) = \frac{1}{\sin^2\theta}Q(g,T)_{1414}\,,\\
Q(g,T)_{2323} &= -\frac{15Me^2\rho^4}{8\rho_1^4(-2M\rho^2 + \rho_1^3)} = \frac{1}{\sin^2\theta}Q(g,T)_{2424}\,.
\end{aligned}
$$

Also the non-zero components of the tensor $C \cdot T$ are

$$
\begin{aligned}
(C\cdot T)_{1313} &= -\frac{15Me^2\rho^6(3\rho_1^2 - 5\rho^2)(-2M\rho^2 + \rho_1^3)}{2\rho_1^{17}} = \frac{1}{\sin^2\theta}(C\cdot T)_{1414}\,,\\
(C\cdot T)_{2323} &= -\frac{15Me^2\rho^6(3\rho_1^2 - 5\rho^2)}{2\rho_1^{11}(-2M\rho^2 + \rho_1^3)} = \frac{1}{\sin^2\theta}(C\cdot T)_{2424}\,.
\end{aligned}
$$

We can state the following, in view the above components:

Theorem 4.1. *The Bardeen spacetime* (1) *has the energy momentum tensor concurring the following properties:*

(i) $R \cdot T = -\frac{M(2\rho_1^2 - 3\rho^2)}{\rho_1^5} Q(g, T)$ *i.e., the energy momentum tensor is pseudosymmetric,*

(ii) $C \cdot T = -\frac{M\rho^2(3\rho_1^2 - 5\rho^2)}{2\rho_1^7} Q(g, T)$ *i.e., the energy momentum tensor is pseudosymmetric due to Weyl tensor,*

(iii) *the energy momentum tensor is Riemann compatible as well as Weyl compatible.*

5 Bardeen Black Hole Metric and Reissner-Nordström Metric

Reissner-Nordström metric [28] is a stationary solution of Einstein-Maxwell field equations with zero cosmological constant. Physically, it represents the exterior gravitational field of a charged black hole. In spherical coordinates (t, ρ, θ, ϕ), the Reissner-Nordström metric is given by ([104], see p.361)

$$ds^2 = g_{ij}dx^i dx^j = -\Big(1 - \frac{2m}{\rho} + \frac{q^2}{\rho^2}\Big)dt^2 + \Big(1 - \frac{2m}{\rho} + \frac{q^2}{\rho^2}\Big)^{-1} d\rho^2 + \rho^2 d\Omega_2^2\,, \tag{22}$$

where t is the time coordinate, ρ is the radial coordinate, the parameters m is the mass of the body and q is the charge of the body. We note that Bardeen spacetime is also a model of a charged black hole. Unlike Reissner-Nordström metric it does not bear curvature singularity. Hence we compare their curvature properties as this comparison compare a charge black hole with curvature singularity free black hole, the Bardeen black hole.

Similarities:

(i) both the black holes are Roter type,

(ii) both spacetimes describe $Ein(2)$ manifolds,

(iii) both the black holes are pseudosymmetric,

(iv) conformal curvature 2-forms are recurrent for both,

(v) both the spacetimes have Riemann compatible and Weyl compatible Ricci tensor.

However, they have the following dissimilar properties:

Dissimilarities:

(i) scalar curvature of Reissner-Nordström spacetime vanishes while it doesn't for the Bardeen spacetime,

(ii) the Bardeen spacetime comes out with a weakly generalized recurrent manifold while Reissner-Nordström spacetime doesn't,

(iii) also the Bardeen spacetime admits special recurrent like structure $\nabla R = A \otimes (g \wedge S)$ (A being some 1-form) but Reissner-Nordström spacetime does not admit such recurrence.

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