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Reconstruction of $f(R)$ Gravity from a Polytropic Gas Scalar Field Dark Energy Model

Chandra Rekha Mahanta, Krishna Pandit*

Department of Mathematics, Gauhati University,
Guwahati-781014, Assam, India
*Corresponding author's E-mail: krishna13pandit@gmail.com

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Abstract. In this paper, we present a reconstruction of an $f(R)$ gravity cosmological model from polytropic scalar field. For the purpose of reconstruction, we consider an equivalence between the $f(R)$ gravity and scalar-tensor gravity and obtain exact forms of $f(R)$ functions and the scalar field potentials. Further, we study the universe under two asymptotic scenarios of early universe and late universe and obtain the forms of the function $f(R)$ and the corresponding potentials. We find that the functions $f(R)$ satisfy the viability conditions.

KEY WORDS: Cosmic acceleration, $f(R)$ gravity, Polytropic equation of state, Reconstruction, Scalar field.

1 Introduction

Einstein's General Theory of Relativity is the best theory of gravitation that we have today [1]. From the expansion of the universe to the merger of black holes, general relativity has been consistently able to explain a number of cosmological phenomena [2–4]. But there are some phenomena which have evaded convincing explanation - like dark matter [5, 6] and the accelerated expansion of the universe [7, 8]. Dark matter is generally believed to be an unknown form of non-luminous, very low interacting matter. Compared to dark matter which have been around for many decades, the phenomenon of cosmic acceleration is a relatively new discovery. Supernova Type Ia observations [7, 8] were the first to suggest that our universe is currently undergoing an accelerated expansion. Subsequently, a combination of results from Cosmic Microwave Background (CMB) and observations of galaxy clustering revealed more evidence in favour of it [5, 9]. Cosmic acceleration contradicts the conventional understanding of the gravity under whose effect all the objects in the universe attract each other. This renders cosmological models with previously unknown properties i.e. either a model of the universe where gravity is overcome at large scales by some unknown repulsive force or gravity behaves differently at large scales. The best model so far which explains the phenomenon of accelerated expansion is the ΛCDM model with equation of state parameter $\omega = -1$ but it lacks theoretical origins [10]. Now, it is accepted widely in the field of cosmology that as much as 2/3 of the total density of the universe is in a form which has large negative pressure and is known as the so called dark energy.

There have been other attempts also to explain the acceleration, namely, the dynamical dark energy and the modified gravity models [11, 12]. Dynamical dark energy models have a variable equation of state parameter ω. Some of the popular dynamical dark energy models are quintessence [13], k-essence [14], quintom dark energy [15], phantom dark energy [16], tachyon dark energy [17] and so on. In modified gravity models, one modifies the curvature part of the Einstein Field Equations which result in the observed acceleration. Some of the popular modified gravity models are $f(R)$ gravity [18], $f(R,T)$ gravity [20], $f(T)$ gravity [21], $f(G)$ gravity [22] and so on. $f(R)$ theory of gravity is the simplest modification to Einstein's General Theory of Relativity wherein the Ricci scalar R is replaced by a more general function $f(R)$ of R [18,19]. The field equations from the action in $f(R)$ theory of gravity can be obtained in three ways namely metric formalism, Palatini formalism and metric-affine formalism. In the metric formalism, the affine connection $\Gamma^{\alpha}_{\mu\nu}$ depends on $g_{\mu\nu}$ [24] and the matter is minimally coupled to the metric. In the Palatini formalism, $\Gamma^{\alpha}_{\mu\nu}$ and $g_{\mu\nu}$ are treated as independent variables and the action is varied with respect to both the metric and the connection [25]. In the metric-affine $f(R)$ gravity [23], one uses the Palatini variation but abandons the assumption that the matter action is independent of the connection. The modified field equations thus obtained are of fourth degree and result in different evolution of the universe than as predicted by general relativity. $f(R)$ gravity has a long history with the origins being loosely traced to Weyl's 1919 theory in which a term quadratic in the Weyl tensor was added to the Einstein-Hilbert Lagrangian [26]. Later authors like Eddington [27] and Buchdahl [24] also studied $f(R)$ gravity. Further, it was realized that quadratic corrections to the Einstein-Hilbert action were necessary to improve the renormalizability of general relativity [28]. A number of authors have also studied astrophysical implications of $f(R)$ gravity [29–32]. $f(R)$ gravity theories are also considered to be an important candidate as one can avoid the Ostrogradski instability [33]. $f(R)$ gravity theories have also shown to satisfy viability conditions in order to avoid instabilities and astrophysical constraints. Recently, Mishra and Sharma have obtained a new shape function for wormholes in $f(R)$ gravity and general relativity [34]. Malik and Shamir have studied the dynamics of some cosmological solutions in modified $f(R)$ gravity [35]. Capozziello et al have studied cosmological perfect-fluids in $f(R)$ gravity [36]. Capozziello et al have also derived the gravitational energy–momentum pseudotensor in metric $f(R)$ gravity and in teleparallel $f(T)$ gravity [37]. The modified field equations and the evolution equations resulting from a nonlinear $f(R)$ in the action can also be seen simply as the addition of a new scalar degree of freedom.

One of the most studied alternative theories of gravity is the scalar–tensor theory where the gravitational action contains, apart from the metric, a scalar field which describes matter part of the gravitational field [38]. In scalar-tensor theories, different degrees of freedom are present such as scalar field, the coupling constant and the cosmological constant as well. However, scalar-tensor theory is still a metric theory as the scalar field is not coupled directly to the matter and so matter responds only to the metric. The role of the scalar field is just to intervene in the generation of the space-time curvature. Scalar fields arise naturally in particle physics such as super-symmetric field theories and string theory. The relation between scalar–tensor theory and $f(R)$ gravity, and their possible equivalence, has been studied by a number of authors in the literature. The most commonly studied equivalence is that of the metric $f(R)$ gravity and Brans–Dicke theory [39–41]. The Brans–Dicke theory is one of the special classes of the scalar-tensor theory, where the coupling parameter $\omega(\phi)$ is supposed to be independent of the scalar field ϕ. It is considered to be constant and hence the name 'coupling constant'. $f(R)$ gravity can be written in terms of a scalar field by redefining the function $f(R)$ by using a convenient scalar field and then performing a conformal transformation.

To study the evolution of the universe in any theory, the field equations are generally solved. In the case of $f(R)$ theory of gravity, the field equations are of fourth order and thereby very difficult to solve analytically and numerically. But, there is another approach to study these theories called reconstruction method. In reconstruction method, one assumes that the history of expansion of the universe is known and then one can invert the field equations to find the form of the $f(R)$ gravity. In most cases, the reconstruction is done in the presence of an auxiliary scalar which may be excluded at the final step so that any FRW cosmology may be realized within specific reconstructed $f(R)$ gravity. Nojiri et al have developed a scheme for cosmological reconstruction of $f(R)$ gravity in terms of e-folding (or, redshift z) so that there is no need to use more complicated formulation with auxiliary scalar [42]. Odintsov and Oikonomou have introduced a bottom-up $f(R)$ gravity reconstruction technique, in which they fixed the observational indices and obtained the $f(R)$ gravity which may realize them [43]. Particularly, as an exemplification of this method, the authors assumed that the scalar to tensor ratio has a specific form, and from it, they reconstructed the $f(R)$ gravity that may realize it, focusing on special values of the parameters in order to obtain analytical results. In the present work, we consider polytropic gas as a scalar field and reconstruct an $f(R)$ gravity model. We obtain the energy density of the polytropic gas in terms of the scalar field and which in turn is obtained from the pressure and energy density of the polytropic gas. The same method has been used in [44] by Sami et al to obtain $f(R)$ gravity from a Chaplygin gas scalar field in de-Sitter spacetimes.

A gas which obeys the polytropic equation of state is known as polytropic gas and is generally termed as a polytrope in astrophysics. The polytropic equation of state is obtained as a solution of the Lane-Emden equation which is a dimensionless form of Poisson's equation for the gravitational potential of a Newtonian self-gravitating, spherically symmetric polytropic fluid. It is a relation which expresses an assumption about the change of pressure with radius in terms of the change of density with radius. The polytropic equation of state is general enough and has been found to be useful in many problems other than that of polytropes. The polytropic equation of state originates from the polytropic process which is a thermodynamic process and can describe multiple expansion and compression processes. Polytropic gas models are employed to study various phenomena in astrophysics and cosmology. For example, the polytropic gas models are used to explain the equation of state of degenerate white dwarfs and neutron stars [45]. Another application of polytropic gas models is the case of main sequence stars where pressure and density are related adiabatically [46]. Mukhopadhyay and Ray first explored the idea of dark energy with polytropic equation of state in cosmology [47]. The polytropic gas is a phenomenological model of dark energy where the pressure p is a function of energy density ρ. In [48], the authors have considered a polytropic gas as a candidate for the interacting dark energy to investigate the validity of the generalized second law of thermodynamics in non-flat universe enveloped by the dynamical apparent horizon. Malekjani has investigated the polytropic gas dark energy model in the non flat universe and reconstructed the dynamics and the potential of the tachyon and k-essence scalar field models according to the evolutionary behavior of polytropic gas model [49]. Karami and Khaledian have reconstructed different $f(R)$ gravity models corresponding to the polytropic, standard Chaplygin, generalized Chaplygin, modified Chaplygin and modified variable Chaplygin gas dark energy models [50]. T. Azizi and P. Naserinia have reconstructed of f(G) gravity with polytropic and Chaplygin gas dark energy models. Karami and Abdolmaleki have studied the correspondence between the interacting new agegraphic dark energy and the polytropic gas model of dark energy in the non-flat FRW universe [51]. Setare and Adami have identified the thermodynamic parameters of a black hole with that of a polytropic gas, which obeys an integrability condition in [52]. Salti et al have reconstructed a variable form of the original polytropic gas and compared the variable polytropic gas with the original polytropic gas by focusing on recent observational data set given in literature including Planck 2018 results [53]. Khurshudyan et al have shown that dark energy can be parameterized as a varying polytropic fluid [54]. In [55], P. H. Chavanis has constructed models of universe with a generalized equation of state $p = (\alpha\rho + k\rho^{1+\frac{1}{n}})c^2$ having a polytropic component and a linear component. The author has shown that polytropic gas with positive pressure $(k > 0)$ leads to past or future singularities whereas a negative pressure $(k < 0)$ leads to non-singular models. These models exhibit phases of early and late inflation associated with a maximum density $\rho_{\max} = \rho_P$ (Planck density) corresponding to the vacuum energy in the past and a minimum density $\rho_{\min} = \rho_\Lambda$ (cosmological density) corresponding to the dark energy in the future. J. Solanki has developed a model of gravitational collapse of anisotropic compact stars in $f(R) = R^4$ theory of gravity [56]. Karami and Abdolmaleki have also established a correspondence between the holographic dark energy density and the polytropic gas scalar field model of dark energy [51]. M. Taji and M. Malekjani have established a correspondence between the holographic dark energy model and polytropic gas model of dark energy in the FRW universe. This correspondence allows one to reconstruct the potential and the dynamics for the scalar field of the polytropic model according to the evolution of holographic dark energy in the FRW universe [57]. M. Korunur, M. Salti, and O. Aydogdu make use of the 5-dimensional polytropic gas model and calculate exact relations for the tachyonic scalar field [58]. Inspired by these results, we consider polytropic gas as a scalar field and reconstruct the corresponding $f(R)$ gravity model.

The paper is organised as follows: In Section 2, equivalence between $f(R)$ gravity and the scalar-tensor gravity is presented. In Section 3, we discuss polytropic equation of state and the associated physical quantities of the polytropic gas. We present two asymptotic cosmological scenarios of the early universe and the late universe in Sections 4 and 5 respectively. We conclude the paper with a brief discussion of the results obtained in Section 6.

2 $f(R)$ Theory of Gravity

The action for $f(R)$ theory of gravity is given by

$$S = \int \sqrt{-g} d^4x \left\{ \frac{1}{2\kappa} f(R) + L_m \right\}, \tag{1}$$

where $\kappa = 8\pi G$, L_M is the matter Lagrangian, $f(R)$ is a function of the Ricci scalar R and the action (1) reduces to the action for general relativity for $f(R) = R$.

Variation of (1) with respect to metric tensor $g_{\mu\nu}$ yields the field equations

$$F(R)R_{\mu\nu} - \frac{1}{2} f(R) g_{\mu\nu} - \nabla_\mu \nabla_\nu F(R) + g_{\mu\nu} \Box F(R) = \kappa T_{\mu\nu} \,, \tag{2}$$

where

$$T_{\mu\nu} = \frac{-2}{\sqrt{-g}} \frac{\delta L_M}{\delta g^{\mu\nu}} \tag{3}$$

is the energy-momentum tensor, $F(R) = \dfrac{df(R)}{dR}$, ∇_μ is the covariant derivative associated with the Levi-Civita connection of the metric, and $\Box = \nabla^\mu \nabla \mu$ is the d'Alembert operator.

To find equivalence with the $f(R)$ gravity, we take the scalar-tensor gravity action

$$S = \int \sqrt{-g} d^4x \left\{ \frac{1}{2\kappa} f(\phi(R)) + L_m \right\}, \tag{4}$$

as given in [59], where $f(\phi(R))$ is a function of $\phi(R)$. The scalar field ϕ is given by the relation

$$\phi = \frac{df}{dR} - 1\,, \tag{5}$$

as considered in [60]. In this paper, we use the natural units convention $(c = \kappa_B = 8\pi G = 1)$ and Latin indices a, b, $c, \ldots$ running from 0 to 3 and use the $(-,+,+,+)$ space-time signature. We consider the polytropic gas as a scalar field and obtain the energy density of the polytropic gas in terms of the scalar field, and from the polytropic gas property, we obtain the polytropic gas pressure.

3 Polytropic Gas Model of Dark Energy

The polytropic equation of state is given by

$$p = k\rho^{1+\frac{1}{n}}\,,$$

where p is the pressure and ρ is the density of the polytropic gas, k and n are constants called polytropic constant and polytropic index respectively. Cosmologically viable models can be constructed for both positive and negative values of the constants k and n. A negative polytropic pressure can exhibit the phases of early time inflation and late time cosmic acceleration. So, for a polytropic gas model of dark energy, it is convenient to take $k < 0$ and the polytropic equation of state in the form

$$p = -K\rho^{1+\frac{1}{n}}\,, \tag{6}$$

where K is a positive constant [50, 61].

For our present work, we consider the universe to be flat and use the Friedmann equation for flat geometry given as

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{\rho}{3} \tag{7}$$

with a being the isotropic scale factor. Assuming that the polytropic gas model of dark energy obeys the conservation law of perfect fluid

$$\dot{\rho} + 3\frac{\dot{a}}{a}(\rho + p) = 0\,, \tag{8}$$

the energy density, from equation (6) is obtained as

$$\rho = \left(Ca^{\frac{3}{n}} + K\right)^{-n}\,, \tag{9}$$

where C is a constant of integration.

The equation of state parameter $\omega = p/\rho$ for polytropic gas dark energy models is obtained as

$$\omega = -1 + \frac{Ca^{\frac{3}{n}}}{Ca^{\frac{3}{n}} + K}\,. \tag{10}$$

The energy density ρ_ϕ and the pressure p_ϕ corresponding to the scalar field ϕ are given by

$$\rho_\phi = \frac{1}{2}\dot{\phi}^2 + V(\phi)\,, \tag{11}$$

$$p_\phi = \frac{1}{2}\dot{\phi}^2 - V(\phi)\,. \tag{12}$$

Adding (11) and (12), the scalar field ϕ can be written in terms of the energy density and the pressure as

$$\dot{\phi}^2 = \rho_\phi + p_\phi\,. \tag{13}$$

Putting the pressure from (6) and the energy density from (9) in (13), we get

$$\dot{\phi} = \pm\frac{\sqrt{C}a^{\frac{3}{2n}}}{\left(Ca^{\frac{3}{n}} + K\right)^{\frac{n+1}{2}}}\,. \tag{14}$$

Also, we can express derivative of ϕ with respect to the scale factor a given as

$$\dot{\phi} = \phi'\dot{a}\,, \tag{15}$$

where prime denotes derivative with respect to the scale factor a. Substituting $\dot{a}$ from (7) in (15), we obtain

$$\phi' = \pm\frac{1}{a}\sqrt{\frac{3}{\rho}}\dot{\phi}\,, \tag{16}$$

which upon substituting $\dot{\phi}$ from (14) results in

$$\phi' = \pm\frac{\sqrt{3C}a^{\frac{3}{2n}-1}}{\left(Ca^{\frac{3}{n}} + K\right)^{\frac{1}{2}}}\,. \tag{17}$$

On integrating (17), we get the scalar field ϕ in terms of the scale factor a as

$$\phi(a) = \pm\frac{n}{\sqrt{3}}\log\left|Ca^{\frac{3}{n}} + \frac{K}{2} + \sqrt{C^2a^{\frac{6}{n}} + CKa^{\frac{3}{n}}}\right| + c_1\,, \tag{18}$$

where c_1 is a constant of integration. From (11) and (12), the potential $V(\phi)$ can be obtained as

$$V(\phi) = \frac{\rho_\phi - p_\phi}{2}\,. \tag{19}$$

Similarly, we can obtain the potential in terms of the scale factor a using (6) and (9) as

$$V(a) = \frac{\left(Ca^{\frac{3}{n}} + K\right)^{-n-1}\left(Ca^{\frac{3}{n}} + 2K\right)}{2}\,. \tag{20}$$

For our present work, we consider a constant negative deceleration parameter q given by

$$q = -\frac{a\ddot{a}}{\dot{a}^2} = -m\,, \tag{21}$$

where $0 < m < 1$. Integrating (21) twice, we get

$$a(t) = \mu\,(t - t_0)^{\frac{1}{1-m}}\ , \tag{22}$$

where μ and t_0 are constants of integration.

Now, to obtain the form of the function $f(R)$, we need a relation between the Ricci scalar R and the scale factor a. For a spatially flat FLRW metric, the Ricci scalar R is related to the scale factor a by

$$R = 6\left(\frac{\dot{a}^2}{a^2} + \frac{\ddot{a}}{a}\right)\ . \tag{23}$$

Substituting (22) in (23), we get

$$a = \gamma R^{\frac{1}{2(m-1)}}\ , \tag{24}$$

where $\gamma = \left\{\frac{\mu^{2(m-1)}(1-m)^2}{6(1+m)}\right\}^{\frac{1}{2(m-1)}}$. For the scale factor (24), the energy density ρ, the equation of state parameter ω, the scalar field ϕ and the potential V are obtained as

$$\rho = \left\{C\gamma^{\frac{3}{n}}R^{\frac{3}{2n(m-1)}} + K\right\}^{-n}\ , \tag{25}$$

$$\omega = -1 + \frac{1}{1 + \frac{K}{C}\gamma^{-\frac{3}{n}}R^{-\frac{3}{2n(m-1)}}}\ , \tag{26}$$

$$\phi(R) = \pm\log\left|C\gamma^{\frac{3}{n}}R^{\frac{3}{2n(m-1)}} + \frac{K}{2} + \sqrt{C^2\gamma^{\frac{6}{n}}R^{\frac{3}{n(m-1)}} + CK\gamma^{\frac{3}{n}}R^{\frac{3}{2n(m-1)}}}\right| + c_1 \tag{27}$$

$$\text{and}\quad V(R) = \frac{C\gamma^{\frac{3}{n}}R^{\frac{3}{2n(m-1)}} + 2K}{2\left\{C\gamma^{\frac{3}{n}}R^{\frac{3}{2n(m-1)}} + K\right\}^{n+1}}\ . \tag{28}$$

4 Early Universe

In the early universe $a \approx 0$. Taking $n > 1$, $\frac{2C}{K} < 1$, we get $\frac{2\sqrt{C}a^{\frac{3}{n}}}{\sqrt{K}} \ll 1$. As a result, equation (18) gives

$$\phi(a) = \pm\frac{2n\sqrt{C}}{\sqrt{3K}}a^{\frac{3}{2n}} + c_2\,, \tag{29}$$

where c_2 is a constant.

And (19) gives

$$V(a) = K^{-n}\,. \tag{30}$$

Substituting the value of a from (24) in (29), we get

$$\phi(R) = \pm\frac{2n\sqrt{C}}{\sqrt{3K}}\gamma^{\frac{3}{2n}}R^{\frac{3}{4n(m-1)}} + c_2\,. \tag{31}$$

Using the relation (5), the function $f(R)$ is obtained as

$$f(R) = \pm\frac{8n^2\sqrt{C}(m-1)}{\sqrt{3K}(3+4mn-4n)}\gamma^{\frac{3}{2n}}R^{\frac{3+4mn-4n}{4n(m-1)}} + c_2R + c_3\,. \tag{32}$$

c_3 being a constant of integration.

Further, from equation (29), we obtain the scale factor a in terms of the scalar field ϕ as

$$a^{\frac{3}{n}} = \frac{3K}{4n^2C}\left\{\phi(a) - c_2\right\}^2\,. \tag{33}$$

Substituting $a^{\frac{3}{n}}$ from above in equation (20), we obtain the potential as

$$V(a) = K^{-n}\left\{1 + \frac{1}{4n^2}\left(\phi(a) - c_2\right)^2\right\}^{-n-1}\left\{1 + \frac{1}{8n^2}\left(\phi(a) - c_2\right)^2\right\}. \tag{34}$$

Also, equating equations (30) and (34), we obtain a constraint relation for the scalar field ϕ as

$$\left\{1 + \frac{1}{4n^2}\left(\phi(a) - c_2\right)^2\right\}^{n+1} = \left\{1 + \frac{1}{8n^2}\left(\phi(a) - c_2\right)^2\right\}. \tag{35}$$

5 Late Universe

In the late universe, $a \to \infty$. So, for $n > 1$ and $\frac{2C}{K} < 1$, we have from (18)

$$\phi(a) = \pm\sqrt{3}\log a + c_4\,, \tag{36}$$

where c_4 is a constant of integration and

$$V(a) = \frac{1}{2C^n a^3}\,. \tag{37}$$

Substituting a from (24) in (36), we get

$$\phi(R) = \pm\sqrt{3}\log|\gamma R^{\frac{1}{2(m-1)}}| + c_4\,. \tag{38}$$

Using the relation (5) and equation (38), we get

$$f(R) = \pm\sqrt{3}\left\{R\left(\log\gamma - \frac{1}{2(m-1)}\right) + \frac{R\log R}{2(m-1)}\right\} + c_4 R + c_5\,, \quad (39)$$

where c_5 is a constant of integration. Also, from equation (36), the scale factor a is obtained as

$$a = e^{\pm\frac{1}{\sqrt{3}}(\phi(a)-c_4)}\,. \quad (40)$$

Hence, the potential in terms of the scalar field ϕ is obtained as

$$V(\phi) = \lambda e^{\pm\sqrt{3}\phi}\,, \quad (41)$$

where $\lambda = \frac{e^{\pm\sqrt{3}c_1}}{2C^n}$. From the above equation, we can observe that the potential is in exponential form which corresponds to a de-Sitter evolution of the universe at late times.

6 Results and Discussion

In the present work, we reconstruct $f(R)$ gravity cosmological models from polytropic gas scalar field. The reconstruction is done by considering the fact that $f(R)$ gravity can be considered as a type of scalar-tensor gravity, particularly Brans-Dicke theory of gravity. For our purpose, we consider a function $f(\phi(R))$ which is a function of a scalar field ϕ and the relation $\phi = \frac{df}{dR} - 1$. A polytropic gas with pressure $p = -K\rho^{1+\frac{1}{n}}$ is considered and obtained the energy density, the equation of state parameter, the scalar field and the potential for the cosmological model. All the parameters are shown as a function of the scale factor a. We study the model so obtained in asymptotic cases of early universe i.e. $a \to 0$ and the late time universe i.e. $a \to \infty$. For early universe, the scalar field is obtained and used to obtain the function $f(R)$. In addition, the potential and a constraint relation for the scalar field are obtained. For the very late time scenario, similarly we reconstruct the function $f(R)$ using the polytropic scalar field obtained. The scalar field obtained is a logarithmic function of the scale factor a and so is the function $f(R)$.

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References

[1] A. Ashtekar (Ed. in Chief), B.K. Berger, J. Isenberg, M. MacCallum (2015) "*General relativity and gravitation: a centennial perspective*" (Cambridge University Press).

[2] M.A.H. MacCallum (2015) Milestones of general relativity: Hubble's law (1929) and the expansion of the universe. *Class. Quantum Grav.* **32** 124002.

[3] R.X. Adhikari (2014) Gravitational radiation detection with laser interferometry. *Rev. Mod. Phys.* **86** 121.

[4] J. Centrella, J.G. Baker, B.J. Kelly, J.R. van Meter (2010) Black-hole binaries, gravitational waves, and numerical relativity. *Rev. Mod. Phys.* **82** 3069.

[5] F. Zwicky (1933) The redshift of extragalactic nebulae. *Helv. Phys. Acta* **6** 110-127.

[6] V.C. Rubin, W.K. Ford Jr. (1970) Rotation of the Andromeda nebula from a spectroscopic survey of emission regions. *Astrophys. J.* **159** 379.

[7] A.G. Riess, A.V. Filippenko, P. Challis, A. Clocchiatti, A. Diercks, P.M. Garnavich, R.L. Gilliland, C.J. Hogan, S. Jha, R.P. Kirshner, et al. (1998) Observational evidence from supernovae for an accelerating universe and a cosmological constant *Astronom. J.* **116** 1009.

[8] S. Perlmutter, G. Aldering, G. Goldhaber, R.A. Knop, P. Nugent, P.G. Castro, S. Deustua, S. Fabbro, A. Goobar, D.E. Groom, E. Donald, et al. (1999) Measurements of Ω and Λ from 42 high-redshift supernovae. *Astrophys. J.* **517** 565.

[9] D.N. Spergel, R. Bean, O. Doré, M.R. Nolta, C.L. Bennett, J. Dunkley, G. Hinshaw, N. Jarosik, E. Komatsu, L. Page, et al. (2007) Three-year Wilkinson Microwave Anisotropy Probe (WMAP) observations: implications for cosmology. *Astrophys. J. Suppl.* **170** 377.

[10] P.J.E. Peebles, Bharat Ratra (2003) The cosmological constant and dark energy. *Rev. Mod. Phys.* **75** 559.

[11] U. Alam, V. Sahni, A.A. Starobinsky (2004) The case for dynamical dark energy revisited. *JCAP* **2004 06** 008.

[12] S. Tsujikawa (2010) Modified gravity models of dark energy. In: "*Lectures on Cosmology*" pp. 99-145.

[13] L. Amendola (2000) Coupled quintessence. *Phys. Rev. D* **62** 043511.

[14] C. Armendariz-Picon, V. Mukhanov, P.J. Steinhardt (2001) Essentials of k-essence. *Phys. Rev. D* **63** 103510.

[15] Yi-Fu Cai, Mingzhe Li, Jian-Xin Lu, Yun-Song Piao, Taotao Qiu, Xinmin Zhang (2007) A string-inspired quintom model of dark energy. *Phys. Lett. B* **651** 1-7.

[16] R.R. Caldwell (2002) A phantom menace? Cosmological consequences of a dark energy component with super-negative equation of state. *Phys. Lett. B* **545** 23-29.

[17] T. Padmanabhan (2002) Accelerated expansion of the universe driven by tachyonic matter. *Phys. Rev. D* **66** 021301.

[18] F.A. De, T. Shinji (2010) $f(R)$ theories. *Living Rev. Relativ.* **13**.

[19] T.P. Sotiriou, V. Faraoni (2010) $f(R)$ theories of gravity. *Rev. Mod. Phys.* **82** 451.

[20] T. Harko, F.S.N. Lobo, S. Nojiri, S.D. Odintsov (2011) $f(R,T)$ gravity. *Phys. Rev. D* **84** 024020.

[21] B. Li, T.P. Sotiriou, J.D. Barrow (2011) $f(T)$ gravity and local Lorentz invariance. *Phys. Rev. D* **83** 064035.

[22] N.M. García, F.S.N. Lobo, J.P. Mimoso, T. Harko (2011) $f(G)$ modified gravity and the energy. *J. Phys.: Conf. Ser.* **314** 012056.

[23] T.P. Sotiriou, S. Liberati (2007) The metric-affine formalism of $f(R)$ gravity. *J. Phys. Conf. Ser.* **68** 39.

[24] H.A. Buchdahl (1970) Non-linear Lagrangians and cosmological theory. *Mon. Notices Royal Astron. Soc.* **150** 1-8.

[25] A. Palatini (1919) Deduzione invariantiva delle equazioni gravitazionali dal principio di Hamilton. *Rendiconti del Circolo Matematico di Palermo (1884-1940)* **43** 203-212.

[26] H. Weyl (1918) Reine Infinitesimalgeometrie. *Mathematische Zeitschrift* **2** 384-411.

[27] A.S. Eddington (1923) "*The mathematical theory of relativity*" (The University Press).

[28] R. Utiyama, B.S. DeWitt (1962) Renormalization of a classical gravitational field interacting with quantized matter fields. *J. Math. Phys.* **3** 608-618.

[29] Y.-S. Song, W. Hu, I. Sawicki (2007) Large scale structure of $f(R)$ gravity. *Phys. Rev. D* **75** 044004.

[30] A.V. Astashenok, S. Capozziello, S.D. Odintsov (2013) Further stable neutron star models from $f(R)$ gravity. *JCAP* **2013** 040.

[31] A. Borisov, B. Jain, P. Zhang (2012) Spherical collapse in $f(R)$ gravity. *Phys. Rev. D* **85** 063518.

[32] G.G.L. Nashed, S. Nojiri (2021) Analytic charged BHs in $f(R)$ gravity. *Phys. Lett. B* **820** 136475.

[33] R. Woodard (2007) Avoiding dark energy with 1/r modifications of gravity. In: Papantonopoulos, L. (eds) "*The Invisible Universe: Dark Matter and Dark Energy*" pp. 403-433 (Lecture Notes in Physics, vol 720. Springer, Berlin, Heidelberg).

[34] A.K. Mishra, U.K. Sharma (2021) A new shape function for wormholes in $f(R)$ gravity and general relativity. *New Astron.* **88** 101628.

[35] A. Malik, M.F. Shamir (2021) Dynamics of some cosmological solutions in modified $f(R)$ gravity. *New Astron.* **82** 101460.

[36] S. Capozziello, C.A. Mantica, L.G. Molinari (2019) Cosmological perfect-fluids in $f(R)$ gravity. *Int. J. Geom. Methods Mod. Phys.* **16** 1950008.

[37] S. Capozziello, M. Capriolo, M. Transirico (2017) The gravitational energy-momentum pseudo-tensor of higher-order theories of gravity. *Annalen der Physik* **529** 1600376.

[38] Yasunori Fujii, Kei-ichi Maeda (2003) The scalar-tensor theory of gravitation. *Class. Quantum Grav.* **20** 4503.

[39] J.D. Barrow, S. Cotsakis (1988) Inflation and the conformal structure of higher-order gravity theories. *Phys. Lett. B* **214** 515-518.

[40] S. Capozziello, S. Nojiri, S.D. Odintsov (2006) Dark energy: the equation of state description versus scalar-tensor or modified gravity. *Physics Letters B* **634** 93-100.

[41] Kei-ichi Maeda (1989) Towards the Einstein-Hilbert action via conformal transformation. *Phys. Rev. D* **39** 3159.

[42] S. Nojiri, S.D. Odintsov, D. Sáez-Gómez (2009) Cosmological reconstruction of realistic modified F(R) gravities. *Phys. Lett. B* **681** 74-80.

[43] S.D. Odintsov, V.K. Oikonomou (2018) Reconstruction of slow-roll F(R) gravity inflation from the observational indices. *Ann. Phys.* **388** 267-275.

[44] H. Sami, N. Namane, J. Ntahompagaze, M. Elmardi, A. Abebe (2018) Reconstructing $f(R)$ gravity from a Chaplygin scalar field in de Sitter spacetimes. *Int. J. Geom. Methods Mod. Phys.* **15** 1850027.

[45] J. Christensen-Dalsgaard (2008) ASTEC—the Aarhus stellar evolution code. *Astrophys. Space Sci.* **316** 13-24.

[46] K. Karami, S. Ghaffari, J. Fehri (2009) Interacting polytropic gas model of phantom dark energy in non-flat universe. *Eur. Phys. J. C* **64** 85-88.

[47] U. Mukhopadhyay, S. Ray, S.B. Dutta Choudhury (2008) Dark energy with Polytropic equation of state. *Mod. Phys. Lett. A* **23** 3187-3198.

[48] K. Karami, S. Ghaffari (20101) The generalized second law of thermodynamics for the interacting polytropic dark energy in non-flat FRW universe enclosed by the apparent horizon. *Phys. Lett. B* **688** 125-128.

[49] M. Malekjani (2013) Polytropic gas scalar field models of dark energy. *Int. J. Theor. Phys.* **52** 2674-2685.

[50] K. Karami, M.S. Khaledian (2012) Polytropic and Chaplygin $f(R)$-gravity models. *Int. J. Mod. Phys. D* **21** 1250083.

[51] K. Karami, A. Abdolmaleki (2010) Reconstructing interacting new agegraphic polytropic gas model in non-flat FRW universe. *Astrophys. Space Sci.* **330** 133-136.

[52] M.R. Setare, H. Adami (2015) Polytropic black hole. *Phys. Rev. D* **91** 084014.

[53] M. Salti, E.E. Kangal, O. Aydogdu (2019) Variable Polytropic gas cosmology. *Ann. Phys.* **407** 166-178.

[54] M. Khurshudyan, M. Hakobyan, As. Khurshudyan (2018) Nonlinear logarithmic interactions and a varying polytropic gas. *Mod. Phys. Lett. A* **33** 1850189.

[55] P.-H. Chavanis (2014) Models of universe with a polytropic equation of state: II. The late universe. *Eur. Phys. J. Plus* **129** 222.

[56] J. Solanki (2022) Gravitational collapse of compact stars in $f(R) = \xi R^4$ gravity. *Int. J. Geom. Methods Mod. Phys.* **19** 2250021.

[57] M. Taji, M. Malekjani (2013) Interacting Holographic Polytropic Gas Model of Dark Energy. *Int. J. Theor. Phys.* **52** 3405-3412.

[58] M. Korunur, M. Salti, O. Aydogdu (2019) Polytropic tachyon scalar field model in 5-dimensional universe. *World Scientific News* **115** 138-148.

[59] A. Amare (2005) "*Beyond Concordance Cosmology*" Scholars' Press.

[60] A.V. Frolov (2008) Singularity problem with $f(R)$ models for dark energy. *Phys. Rev. Lett.* **101** 061103.

[61] T. Azizi, P. Naserinia (2017) Reconstruction of f(G) gravity with polytropic and Chaplygin gas dark energy models. *Eur. Phys. J. Plus* **132** 1-11.