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Inversion Invariant Volume Element for Strings, Antistrings and Braneworlds*

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Abstract. The specific model studied is in the context of the modified measure formulation the string or branes, where tension appear as an additional dynamical degree of freedom. We then consider the signed reparametrization invariant volume element formulation of dynamical strings and branes and find that the dynamical tension can produce positive tensions or negative tensions, corresponding exactly to strings and branes and antistrings and antibranes respectively. The antistrings are realized when a scalar time that defines the modified measure runs in the opposite direction to the world sheet time. For strings with positive tension, both times run in the same direction. The situation resembles the situation in Relativistic Quantum Mechanics with positive and negative energies, proper time of particles running forward with respect of coordinate time, while for antiparticles proper tome runs opposite of coordinate time. An example, where string antistring pair creation takes place in analogy to the pair creation in an external electric field in QED background field, this time in the presence of a background scalar field that couples to the strings and locally changes the tension, the tension field. When many types of strings probing the same region of space are considered this tension scalar is constrained by the requirement of quantum conformal invariance. For the case of two types of strings probing the same region of space with different dynamically generated tensions, there are two different metrics, associated to the different strings. Each of these metrics have to satisfy vacuum Einstein's equations and the consistency of these two Einstein's equations determine the tension scalar. The universal metric, common to both strings generically does not satisfy Einstein's equation. The two string dependent metrics considered here are flat space in Minkowski space and Minkowski space after a special conformal transformation. The limit, where the two string tensions are the same, is studied, it leads to a well defined solution. If the string tension difference between the two types of strings is

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very small but finite, the approximately homogeneous and isotropic cosmological solution lasts for a long time, inversely proportional to the string tension difference and then the homogeneity and and isotropy of the cosmological disappears and the solution turns into an expanding brane world, where the strings are confined between two expanding bubbles separated by a very small distance at large times.

KEY WORDS: string theory, brane theory, modified measures, strings, antistrings, braneworlds, pair creation.

1 Introduction

String theories have been considered by many physicists for some time as the leading candidate for the theory everything, including gravity, the explanation of all the known particles that we know and all of their known interactions (and probably more) [1]. According to some, one unpleasant feature of string theory as usually formulated is that it has a dimension full parameter, in fact, its fundamental parameter, which is the tension of the string. This is when formulated the most familiar way. The consideration of the string tension as a dynamical variable, using the modified measures formalism, which was previously used for a certain class of modified gravity theories under the name of Two Measures Theories or Non Riemannian Measures Theories, see for example [2–9]. It is also interesting to mention that the modified measure approach has also been used to construct braneworld scenarios [10] and that Modified Measure Theories could be the effective Theories of Causal fermion systems [11]. Most recently, we have shown that modified measure gravity theories can be be formulated using an invariant volume element under Signed General Coordinate Invariance, that is for general coordinate transformations with negative Jacobian [12], which could provide for a framework for the quantum creation of a baby universe as formulated by Farhi, Guth and Guven [14] which requires negatives measures for space time and for Linde's Universe multiplication scenario [15], which also involves negative measures for space time in a parallel universe, which requires the measure of integration to become negative at certain regions of the tunneling solution.

When applying these principles to string theory, this leads to the modified measure approach to string theory, where rather than to put the string tension by hand it appears dynamically.

This approach has been studied in various previous works [16–23]. See also the treatment by Townsend and collaborators for dynamical string tension [23, 28, 29], which does not involve changing integration measures, so it cannot be used to achieve the goals presented in this paper however.

Here we will see that this modified volume element can play a similar role to the proper time, which for antiparticles runs in the opposite direction to laboratory time and how introducing appropriate background fields, string antistring pair creation could be possible.

When many types of strings probing the same region of space are considered this tension scalar is constrained by the requirement of quantum conformal invariance. For the case of two types of strings probing the same region of space with different dynamically generated tensions, there are two different metrics, associated to the different strings. Each of these metrics have to satisfy vacuum Einstein's equations and the consistency of these two Einstein's equations determine the tension scalar. The universal metric, common to both strings generically does not satisfy Einstein's equation. The two string dependent metrics considered here are flat space in Minkowski space and Minkowski space after a special conformal transformation. The limit, where the two string tensions are the same, is studied, it leads to a well defined solution. If the string tension difference between the two types of strings is very small but finite, the approximately homogeneous and isotropic cosmological solution lasts for a long time, inversely proportional to the string tension difference and then the homogeneity and and isotropy of the cosmological disappears and the solution turns into an expanding brane world, where the strings are confined between two expanding bubbles separated by a very small distance at large times.

2 The Modified Measure String Theory

The standard world sheet string sigma-model action using a world sheet metric is [30–32]

$$S_{\text{sigma-model}} = -T \int d^2\sigma \frac{1}{2}\sqrt{-\gamma}\gamma^{ab}\partial_a X^\mu \partial_b X^\nu g_{\mu\nu} \, . \tag{1}$$

Here γ^{ab} is the intrinsic Riemannian metric on the 2-dimensional string world sheet and $\gamma = det(\gamma_{ab})$; $g_{\mu\nu}$ denotes the Riemannian metric on the embedding spacetime. T is a string tension, a dimension full scale introduced into the theory by hand.

Now instead of using the measure $\sqrt{-\gamma}$, on the 2-dimensional world-sheet, in the framework of this theory two additional world sheet scalar fields φ^i ($i = 1, 2$) are considered. A new measure density is introduced:

$$\Phi(\varphi) = \frac{1}{2}\epsilon_{ij}\epsilon^{ab}\partial_a\varphi^i\partial_b\varphi^j \, . \tag{2}$$

There are no limitations on employing any other measure of integration different than $\sqrt{-\gamma}$. The only restriction is that it must be a density under arbitrary diffeomorphisms (reparametrizations) on the underlying spacetime manifold. The modified-measure theory is an example of such a theory.

Then the modified bosonic string action is (as formulated first in [16] and latter discussed and generalized also in [17]

$$S = -\int d^2\sigma \Phi(\varphi)(\frac{1}{2}\gamma^{ab}\partial_a X^\mu \partial_b X^\nu g_{\mu\nu} - \frac{\epsilon^{\text{ab}}}{2\sqrt{-\gamma}} F_{ab}(A)), \tag{3}$$

where F_{ab} is the field-strength of an auxiliary Abelian gauge field A_a: $F_{ab} = \partial_a A_b - \partial_b A_a$.

It is important to notice that the action (3) is invariant under conformal transformations of the internal metric combined with a diffeomorphism of the measure fields,

$$\gamma_{ab} \to j\gamma_{ab}\,, \tag{4}$$

$$\varphi^i \to \varphi'^i = \varphi'^i(\varphi^i)\,, \tag{5}$$

such that

$$\Phi \to \Phi' = j\Phi\,. \tag{6}$$

Here j is the Jacobian of the diffeomorphim in the internal measure fields which can be an arbitrary function of the world sheet space time coordinates, so this can called indeed a local conformal symmetry.

To check that the new action is consistent with the sigma-model one, let us derive the equations of motion of the action (3).

The variation with respect to φ^i leads to the following equations of motion:

$$\epsilon^{ab}\partial_b\varphi^i\partial_a(\gamma^{cd}\partial_c X^\mu \partial_d X^\nu g_{\mu\nu} - \frac{\epsilon^{cd}}{\sqrt{-\gamma}} F_{cd}) = 0\,. \tag{7}$$

Since $\det(\epsilon^{ab}\partial_b\varphi^i) = \Phi$, assuming a non degenerate case ($\Phi \neq 0$), we obtain

$$\gamma^{cd}\partial_c X^\mu \partial_d X^\nu g_{\mu\nu} - \frac{\epsilon^{cd}}{\sqrt{-\gamma}} F_{cd} = M = \text{const.} \tag{8}$$

The equations of motion with respect to γ^{ab} are

$$T_{ab} = \partial_a X^\mu \partial_b X^\nu g_{\mu\nu} - \frac{1}{2}\gamma_{ab}\frac{\epsilon^{cd}}{\sqrt{-\gamma}} F_{cd} = 0\,. \tag{9}$$

One can see that these equations are the same as in the sigma-model formulation. Taking the trace of (9) we get that $M = 0$. By solving $\frac{\epsilon^{cd}}{\sqrt{-\gamma}} F_{cd}$ from (8) (with $M = 0$) we obtain the standard string equations.

The emergence of the string tension is obtained by varying the action with respect to A_a

$$\epsilon^{ab}\partial_b(\frac{\Phi(\varphi)}{\sqrt{-\gamma}}) = 0\,. \tag{10}$$

Then by integrating and comparing it with the standard action, it is seen that

$$\frac{\Phi(\varphi)}{\sqrt{-\gamma}} = T\,. \tag{11}$$

That is how the string tension T is derived as a world sheet constant of integration opposite to the standard equation (1), where the tension is put ad hoc. Let us stress that the modified measure string theory action does not have any *ad hoc* fundamental scale parameters. associated with it. This can be generalized to incorporate super symmetry, see for example [17–20]. For other mechanisms for dynamical string tension generation from added string world sheet fields, see for example [28] and [29]. However the fact that this string tension generation is a world sheet effect and not a universal uniform string tension generation effect for all strings has not been sufficiently emphasized before.

Notice that each string in its own world sheet determines its own tension. Therefore the tension is not universal for all strings.

3 Elementary Review of Particles and Antiparticles and Pair Creation in QED

From the 'Feynman' perspective, negative energy waves propagating into the past are physically realized as the antiparticles propagating into the future. As he has shown [38] that from the classical equations of motion for a particle in an external field can be written as

$$m\frac{d^2z^\mu}{d\tau^2} = e\frac{dz_\nu}{d\tau}F^{\mu\nu}\,, \tag{12}$$

where τ is the proper time. If we note as Feynman has that if we allow $\tau \to -\tau$ the equation becomes

$$m\frac{d^2z^\mu}{d\tau^2} = -e\frac{dz_\nu}{d\tau}F^{\mu\nu}\,, \tag{13}$$

which is identical to the previous equation except that the particles charge has changed. In other words, as far as its charge is concerned, it has become the antiparticle. Thus, proper time running backward (i.e., $\tau \to -\tau$), while keeping the coordinate time unchanged. led to the particle becoming an antiparticle. Of course we can take the equivalent, but more suitable for our purposes transformation that we change the direction of coordinate time, while requiring that the proper time remains unchanged.

We are now in a position where we can discuss the scattering of a particle in an external field. Four possibilities are seen to exist: i) the scattering of the particle by the field, ii) the creation of a particle-antiparticle pair. iii) the annihilation particle-antiparticle pair iv) the scattering of an antiparticle by the field. Below, we review one of them and in the following section we draw conclusions concerning the dynamical string theory and the analogy of the modified measure in the string case with the proper time in the particle case.

3.1 Pair creation in a strong uniform electric field in QED, importance of different directions of time and proper time for identifying particles and antiparticles

We shall present here a simplified version, by A.Vilenkin [39], of the calculation of pair production first performed by Schwinger [40] To begin our discussion, we consider a particle with charge e and mass m in a constant electric field. The general equation of a particle in a field is most convenient written in terms of the Maxwell tensor $F_{\mu\nu}$, where for a constant electric $\mathbf{E}$ in the $x-$direction, $F^{01} = -E$, $F^{10} = E$, $F^{0}_{1} = E$, ${F^1}_0 = E$. More explicitly, we have

$$F^{\mu}{}_{\nu} = \begin{bmatrix} 0 & E & 0 & 0 \\ E & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} . \tag{14}$$

The equation of motion for this particle in a constant electric field is

$$m\frac{d^2x^\mu}{d\tau^2} = eF^{\mu}{}_{\nu}\frac{dx^\nu}{d\tau} . \tag{15}$$

The formal solution of for $u^\mu = \dfrac{dx^\mu}{d\tau}$ is $u^\mu(\tau) = \exp\left[\dfrac{e}{m}F^{\alpha}{}_{\beta}\tau\right]^{\mu}_{\nu} u^\nu(0)$. The exponential can be expanded and we have

$$\exp[\frac{e}{m}F^{\alpha}{}_{\beta}\tau]^{\mu}_{\nu} = \delta^\mu_\nu + \frac{e}{m}\tau E\Delta^\mu_\nu + \frac{1}{2}(\frac{e}{m}\tau E\Delta^\mu_\nu)^2 + \cdots , \tag{16}$$

where

$$\Delta = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} . \tag{17}$$

Separating even and odd power in Eq. 16 we have

$$\begin{aligned} u^0 &= \cosh\left(\frac{eE\tau}{m}\right)u^0(0) + \sinh\left(\frac{eE\tau}{m}\right)u^1(0) , \\ u^1 &= \sinh\left(\frac{eE\tau}{m}\right)u^0(0) + \cosh\left(\frac{eE\tau}{m}\right)u^1(0) . \end{aligned} \tag{18}$$

Integrating with respect to τ yields (where we have dropped arbitrary constants of integration)

$$\begin{aligned} x^0 &= \frac{m}{eE}\left\{\sinh\left(\frac{eE\tau}{m}\right)u^0(0) + \cosh\left(\frac{eE\tau}{m}\right)u^1(0)\right\}, \\ x^1 &= \frac{m}{eE}\left\{\cosh\left(\frac{eE\tau}{m}\right)u^0(0) + \sinh\left(\frac{eE\tau}{m}\right)u^1(0)\right\}. \end{aligned} \tag{19}$$

We choose the following boundary conditions $u^1(0) = 0$, $u^0(0) = 1$ which leads to

$$\begin{aligned} x^0 &= \frac{m}{eE}\sinh\left(\frac{eE\tau}{m}\right), \\ x^1 &= \frac{m}{eE}\cosh\left(\frac{eE\tau}{m}\right), \end{aligned} \tag{20}$$

and for the boundary condition $u^1(0) = 0$, $u^0(0) = -1$ leads to

$$\begin{aligned} x^0 &= -\frac{m}{eE}\sinh\left(\frac{eE\tau}{m}\right), \\ x^1 &= -\frac{m}{eE}\cosh\left(\frac{eE\tau}{m}\right). \end{aligned} \tag{21}$$

The solution given by Eq. 20 represents a particle solution while the solution of Eq. 21 represents the anti-particle solution. Both solutions, together satisfy

$$(x^1)^2 - (x^0)^2 = \left(\frac{m}{eE}\right)^2. \tag{22}$$

At classical level, these solutions are distinct and one solution can not evolved into the other. Thus, a particle couldn't evolve to an anti-particle. However, the semi-classical approximation which consists of considering the classical equations of motion but with imaginary time. Then inserting $t = -it_E$ we obtain that the hyperbola of Eq. 22 becomes a circle

$$(x^1)^2 + t_E^2 = \left(\frac{m}{eE}\right)^2. \tag{23}$$

This tunneling solution can now interpolate between the anti-particle and particle solutions. In the imaginary time region, the action, $S = -iS_E$, where S_E is given by

$$S_E = \int dt_E \left\{ m\sqrt{1 + \left(\frac{dx}{dt_E}\right)^2} - eEx \right\}. \tag{24}$$

Introduction the angular variable θ, where $x = \frac{m}{eE}\cos\theta$, $t_E = \frac{m}{eE}\sin\theta$, we obtain that $S_E = \pi m^2/eE$. Since the probability is given (up to prefactors) by

$$\exp(-S_E) = \exp\left(-\frac{\pi m^2}{eE}\right) \qquad \text{for } eE > 0\,.$$

Notice that the distance of the particle and antiparticle at the moment of creation is $\Delta x = \dfrac{m}{eE}$ which has a physical interpretation is manifest in writing it as

$$W = eE\Delta x = 2mc^2 \,, \tag{25}$$

where we have restored the 'c' to make its physical meaning clearer. Thus we can create a pair of particle-antiparticle in a constant electric field by performing work, W in a distance Δx equal to the the sum of the rest masses of the particles, i.e. $2mc^2$. In the case for an electric field that doesn't extend through all of space, still it must be extended enough to perform the work equal to the sum rest masses of the two particles in order to create a pair.

4 Extension to Signed Reparametrization Invariant Modified Measure Volume Element as the Proper Volume of Strings

In [12], we discussed the generalization of general coordinate transformations to signed general coordinate transformations and found the modified volume which is invariant under signed general coordinate transformations, we then analyzed the application of signed reparametrization volume elements in string theory in [13], which will be the subject of this section.

For the case of strings now notice that under a signed general reparametrization transformation of the world sheet, that could for example change the direction of time,

$$d^2\sigma \to J d^2\sigma \,,$$

while

$$\sqrt{-\gamma} \to \mid J \mid^{-1} \sqrt{-\gamma} \,,$$

where J is the Jacobian of the transformation and $\mid J \mid$ is the absolute value of the Jacobian of the transformation. Therefore,

$$d^2\sigma\sqrt{-\gamma} \to \frac{J}{\mid J \mid} d^2\sigma\sqrt{-\gamma} \,,$$

so invariance of the volume element is achieved only for $J = \mid J \mid$, that is if $J > 0$, that is signed general coordinate transformations are excluded if we want to define signed invariant reparametrization volume element.

Notice that we are not discussing invariance of the action here, which is a different matter, since under a signed reparametrization transformation the time direction may change for example. Then the limits of integration change and when restoring back the limits of integration we obtain anothe minus sign and the action restores its invariance.

However what we want to know is, if we now can have a measure that does not change sign, even if we change for example the direction of the coordinate time, which will be the corresponding generalization of the proper time, which continues to increase as the particle goes backwards in time and now in the case of the strings, the modified volume can be monotonically increasing even as the strings go backwards in time, Furthermore, this will imply that there are effects that cannot be described in the standard formalism, like string antistring pair creation, which would require the generalization of the proper time, which continues to increase as the string goes backwards in time.

Indeed, replacing measure $\sqrt{-\gamma}$ by the modified measure Φ provides us with such measure since Φ transforms according to J^{-1} instead of $\mid J \mid^{-1}$. So we can have now a positive defined measure for strings propagating backwards in the string sheet time in exact analogy to the QED case.

Notice that

$$\Phi(\varphi) d^2\sigma = d\varphi_1 d\varphi_2 \,.$$

So the integration domain in the $\varphi_1\varphi_2$ space is invariant under these signed reparametrizations, since $\varphi_1\varphi_2$ are both scalars and do not transform under inversions. So we can take the time of the string, say σ_1 to go in the opposite direction to that of the coordinate time, thus defining an antistring, the same way as the antiparticle was defined when the proper time and the coordinate time ran in different directions. φ_2 could be taken as the spacial coordinate of the string, periodic for a closed sting for example, and will not play a role in the discussion on the difference between strings and antistrings. Generically, the consideration of the $\varphi_1\varphi_2$ space as opposed to the $\sigma_1\sigma_2$ space enlarges the space of possible configurations and solutions and allows for example the consideration of pair creation or pair annihilation, which requires non trial mappings between the $\sigma_1\sigma_2$ space and the $\sigma_1\sigma_2$ spaces. Furthermore if we want the full action density to be invariant under signed reparametrizations, in (3), we have to correct the term

$$\Phi(\varphi) \frac{\epsilon^{ab}}{2\sqrt{-\gamma}} F_{ab}(A) \,,$$

since $\Phi(\varphi)$ and $\sqrt{-\gamma}$ differ by a sign under signed reparametrization transformation, To cure this, we replace this by

$$\Phi(\varphi)^2 \frac{\epsilon^{ab}}{2(-\gamma)} F_{ab}(A) \,,$$

obtaining the action invariant under signed reparametrization invariant, where both the domain of integration and the lagrangian density are invariant under signed reparametrizations transformations

$$S = -\int d^2\sigma \Phi(\varphi)(\frac{1}{2}\gamma^{ab}\partial_a X^\mu \partial_b X^\nu g_{\mu\nu} - \Phi(\varphi)\frac{\epsilon^{ab}}{2(-\gamma)}F_{ab}(A))\,. \tag{26}$$

Now the equation of motion that comes from the variation of the internal gauge field is

$$\epsilon^{ab}\partial_b(\frac{\Phi^2(\varphi)}{(-\gamma))}) = 0\,, \tag{27}$$

which is integrated to,

$$\frac{\Phi^2(\varphi)}{(-\gamma)} = T^2\,, \tag{28}$$

where without loosing generality we can take T positive. The above equation implies that positive tensions are accompanied with negative ones,

$$\frac{\Phi(\varphi)}{\sqrt{-\gamma}} = \pm T\,. \tag{29}$$

The negative tension strings we associate with the "antistrings". We can see that the results obtained from the signed reparametrization invariant action are (26) to those obtained in the previous section, except for the possibility that there are positive and negative tensions. To start with, the equation that replaces (7) is

$$\epsilon^{ab}\partial_b\varphi^i\partial_a(-\gamma^{cd}\partial_c X^\mu \partial_d X^\nu g_{\mu\nu} + 2\Phi\frac{\epsilon^{cd}}{(-\gamma)}F_{cd}) = 0\,, \tag{30}$$

which again can be integrated, assuming non singular measure Φ

$$-\gamma^{cd}\partial_c X^\mu \partial_d X^\nu g_{\mu\nu} + \frac{\Phi\epsilon^{cd}}{(-\gamma)}F_{cd} = M\,. \tag{31}$$

The variation with respect to γ^{ab} gives now, for $\Phi \neq 0$ as

$$T_{ab} = \partial_a X^\mu \partial_b X^\nu g_{\mu\nu} - \Phi\gamma_{ab}\frac{\epsilon^{cd}}{(-\gamma)}F_{cd} = 0. \tag{32}$$

Taking the trace of (31) and comparing with (32) we obtain $M = 0$ then solving $\frac{\Phi\epsilon^{cd}}{(-\gamma)}F_{cd}$, from (31) now for $M = 0$ and inserting in (32) we recover the standard Polyakov equation for a string, but now we have two species of strings, with positive and negative tensions.

Notice however that from a "practical" point of view, it is not necessary to work with an action density that is completely invariant under the signed reparametrizations, in particular, we can still work with the original action (3), which avoids quadratic equations for the tension, but still allows for positive and negative tensions, just as in the modified version that has complete signed reparametrization invariance and is simple to manipulate, since everything becomes linear. We now consider possible ways to scatter strings from an external field so that the tension may change, still in the most simple formalism.

5 Introducing a New Background Field. The Tension Field

Schwinger [41] had an important insight and understood that all the information concerning a field theory can be studied by understanding how it reacts to sources of different types. Different types of background bulk fields, like the dilaton field, the two index antisymmetric gauge field have been discussed in the text book by Polchinski for example [34]. Here, instead of the traditional background fields usually considered in conventional string theory, one may consider another scalar field that induces currents in the string world sheet and since the current couples to the world sheet gauge fields, this produces a dynamical tension controlled by the external scalar field as shown at the classical level in [33]. In the next two subsections we will study how this comes about in two steps, first we introduce world sheet currents that couple to the internal gauge fields in Strings and Branes and second we consider a coupling to an external scalar field by defining a world sheet current that is induced by such external scalar field.and then coupling this current to the internal gauge fields of the Strings. After this is done, we argue that such external field could lead to string antistring pair creation, in analogy to the QED process of electron antielectron pair production. So the tension field probes exactly the string - antistring structure of the modified measure string theory.

5.1 Introducing world sheet currents that couple to the internal gauge fields and locally change the tension

If to the action of the string we add a coupling to a world-sheet current j^a, i.e. a term

$$S_{\text{current}} = \int d^2\sigma A_a j^a, \tag{33}$$

then the variation of the total action with respect to A_a gives

$$\epsilon^{ab}\partial_a\left(\frac{\Phi}{\sqrt{-\gamma}}\right) = j^b. \tag{34}$$

We thus see indeed that, in this case, the dynamical character of the brane is crucial here.

5.2 How a world sheet current can naturally be induced by a bulk scalar field. The tension field

Suppose that we have an external scalar field $\phi(x^\mu)$ defined in the bulk. From this field we can define the induced conserved world-sheet current

$$j^b = e\partial_\mu\phi\frac{\partial X^\mu}{\partial\sigma^a}\epsilon^{ab} \equiv e\partial_a\phi\epsilon^{ab}, \tag{35}$$

where e is some coupling constant. The interaction of this current with the world sheet gauge field is also invariant under local gauge transformations in the world sheet of the gauge fields $A_a \rightarrow A_a + \partial_a \lambda$.

For this case, (34) can be integrated to obtain

$$T = \frac{\Phi}{\sqrt{-\gamma}} = e\phi + T_0 \,. \tag{36}$$

The constant of integration T_0 could represent the initial value of the tension before the string enters the region where the tension field is present. Notice that the interaction is metric independent since the internal gauge field does not transform under the the conformal transformations. This interaction does not therefore spoil the world sheet conformal transformation invariance in the case the field ϕ does not transform under this transformation. Therefore, one way to dynamically change the string tension tension dynamically is through the introduction this new background field, that changes the value of the the tension locally in the world sheet [33]. This background field was then called the "tension field" and used for many different scenarios [23], [24], the results of which have been summarized in [25] and most recently for the construction of braneworld scenarios in dynamical string tension theories [26] and [27], and as we will discuss in the next section, this tension field could provide a way to communicate between strings and antistrings.

6 Scattering of a Dynamical Tension String Field Leading to String Antistring Pair Creation in Analogy with the Analog QED Process

The QED example explained before in this paper inspires an analogous model for a possible scenario for a string-antistring creation process. Here instead of the the external electric field we can have the tension field.

We specify the initial state according to the φ_1 time as the antistring, where the time φ_1 and the world sheet coordinate time σ_1 (running in the same direction as the coordinate time t) run in opposite directions, so the tension is therefore negative. Assume T_0 that could represent the initial value of the tension before the string enters the region where the tension field is present is negative,

Then as the φ_1 time continues to increase, and the world sheet time continues to decrease, we encounter a region where the tension field ϕ is present, and over this region the tension field suffers a a positive jump $e\Delta\phi$ which can be bigger than $-T_0$, turning the tension from negative to positive. This is the analog of the condition for pair creation in the QED case, that is that the electric field is extended id a region of space with minimum extension given by $W = eE\Delta x = 2mc^2$. This again, as in the QED case, cannot happen classically, there should be a tunneling region that connects the antistring solution going backwards in coordinate time with the string solution going forward in coordinate time. Unfortunately we cannot yet present full details as compared with the QED case, because on the much higher complexity of the string dynamics as compared with the particle dynamics of the QED problem, but we plan to present more details in a future publication.

In the case we were to use the full signed reparametrization formulation the equation 36 has to be modified and if the coupling to the tension field is the same, now becomes,

$$T^2 = \frac{\Phi^2}{(-\gamma)} = e\phi + T_0\,. \tag{37}$$

As we see, there is a difference with the simpler case studied before, since the above equation involves the square of the dynamical tension, so there must be a boundary for the evolution of the tension field giver by $e\phi + T_0 = 0$ and after this point the tension field must bounce in the opposite direction, and if particle antiparticle is desired, when bouncing in the opposite direction the opposite sign of the tension must be taken. If the motion of the tension field is symmetric around the bounce, the string and antistring will have exactly opposite tensions, something that is not necessary in the simplified model which is not signed reparametrization invariant.

7 Signed Inversion Invariant Volume Element for Brane Theories. Analogy with Relativistic QM Positive and Negative Energies

The d dimensional extended object ($d = 1$ corresponds to the string case) has an action that provides for a dynamically generated brane tension

$$S = -\int d^{d+1}\sigma\Phi(\varphi)\left(\frac{1}{2}\gamma^{ab}\partial_a X^\mu \partial_b X^\nu g_{\mu\nu} - \frac{\epsilon^{a_1 a_2 \ldots\ldots a_{d+1}}}{2\sqrt{-\gamma}} F_{a_1 a_2 \ldots\ldots a_{d+1}}(A)\right), \tag{38}$$

where $F_{a_1 a_2 \ldots\ldots a_{d+1}}(A)$ is a field strength deriving from the totally antisymmetric derivative of a totally antisymmetric potential with d indices. Also the measure is constructed from the derivatives of $d + 1$ scalars. The variation with respect to those totally antisymmetric potential with d indices gives the equation that the derivative of $\Phi(\varphi)/\sqrt{-\gamma}$ is a constant T, which we identify with the tension. (38) is not however invariant under signed reparametrization transformations, in order to do this $\sqrt{-\gamma}$ sould not appear, only $-\gamma$, so the c signed invariant generalization would be,

$$S = -\int d^{d+1}\sigma\Phi(\varphi)\left(\frac{1}{2}\gamma^{ab}\partial_a X^\mu \partial_b X^\nu g_{\mu\nu} - \frac{\Phi(\varphi)\epsilon^{a_1 a_2 \ldots\ldots a_{d+1}}}{2(-\gamma)} F_{a_1 a_2 \ldots\ldots a_{d+1}}(A)\right), \tag{39}$$

from the variation of the internal gauge fields and integration of the resulting equation, we get now $\Phi^2(\varphi)/(-\gamma) = T^2$, just as in the string case, which again leads to positive and negative tensions, i.e. $\Phi(\varphi)/\sqrt{-\gamma} = \pm T$. As in the modified measure string, it is not strictly necessary to work with (39), (38) also works.

As in relativistic quantum mechanics [35] one may attempt to eliminate negative solutions, which in the case of relativistic quantum mechanics are negative energy solutions, that attempt, known as the Relativistic Schroedinger equation fails [35] and such equation cannot even describe the scattering by a square well potential. As pointed out by Feynman, this is related to the fact that the space of positive energy solutions is not large enough to represent an arbitrary initial condition. The correct interpretation of the Klein paradox [35] must involve particles and antiparticles, the alternative of eliminating the negative energy solutions does not work.

We introduced a new background field,the tension field, which is specially designed to probe the string antistring dynamics of the modified measure string theories. The process of string antistring creation can be studied qualitatively for a simpler formulation that is not totally invariant under signed reparametrizations. in this case the resulting pair of string anti and string may not have exactly opposite tensions. In the signed reparametrization invariant formulation, we find that zero tension squared is a special point from which it must bounce back and transition from negative to positive values is possible, in most likelihood this bounce could be symmetric, leading to pair creation with exactly opposite tensions for the string and the antistring.

Concerning simultaneous use of positive and negative brane tensions, we can point out that this has been used extensively in brane world models, see for example [36]. The interpretation of the negative tension branes as antibranes could be useful.

One interesting aspect of the dynamical strings and brane, in that due to the influence of the tension field a part of the string or brane could be antistring or antibrane while the rest could be string or brane, i.e., have positive tension. One may have an antibrane bubble in the midst of a brane environment, since the transition for brane to antibrane or the reverse due to the tension field is local.

Finally, the tension field that locally changes the value of the string tension has been introduced as a background field, so far just an external field. To have a complete picture, more work should be done to determine it dynamically Some ideas in this directions were discussed for a system containing two types of strings with different integration constants T_1 and T_2 and imposing separately conformal invariance on the two string types separately, which lead to an algebraic equation for the tension field, that indicates the formation of a braneworld at large times [26, 27]. An interesting possibility could be if those two types of strings could be strings and antistrings generated by a pair creation process like the one that we have discussed here, this may complete the picture described in those paper, including the quantum formation of such braneworlds.

A different approach to dynamical string tension and the possibility of negative string tension has been discussed in [37], the relation of the approach of this paper is not straightforward, but could exist at the level of potentially similar effects.

8 The Case of Two Different String Tensions with the Constraint of Quantum Conformal Invariance

If we have a scalar field coupled to a string or a brane in the way described in the sub section above, i.e. through the current induced by the scalar field in the extended object, according to eq. (36), so we have two sources for the variability of the tension when going from one string to the other: one is the integration constant T_0, which we now generalize to T_i which varies from string to string (labelled by i) and the other the local value of the scalar field, which produces also variations of the tension even within the string or brane world sheet. As we discussed in the previous section, we can incorporate the result of the tension as a function of scalar field ϕ, given as $e\phi + T_i$, for a string with the constant of integration T_i by defining the action that produces the correct equations of motion for such string, adding also other background fields, the antisymmetric two index field $A_{\mu\nu}$ that couples to $\epsilon^{ab}\partial_a X^\mu \partial_b X^\nu$ and the dilaton field φ.

$$S_i = -\int d^2\sigma (e\phi + T_i)\frac{1}{2}\sqrt{-\gamma}\gamma^{ab}\partial_a X^\mu \partial_b X^\nu g_{\mu\nu} + \int d^2\sigma A_{\mu\nu}\epsilon^{ab}\partial_a X^\mu \partial_b X^\nu + \int d^2\sigma \sqrt{-\gamma}\varphi R \,. \quad (40)$$

Notice that if we had just one string, or if all strings will have the same constant of integration $T_i = T_0$. We will take cases where the dilaton field is a constant or zero, and the antisymmetric two index tensor field is pure gauge or zero, then the demand of conformal invariance for $D = 26$ becomes the demand that all the metrics

$$g^i_{\mu\nu} = (e\phi + T_i)g_{\mu\nu} \quad (41)$$

will satisfy simultaneously the vacuum Einstein's equations. The interesting case to consider is when there are many strings with different T_i, let us consider the simplest case of two strings, labeled 1 and 2 with $T_1 \neq T_2$, then we will have two Einstein's equations, for $g^1_{\mu\nu} = (e\phi + T_1)g_{\mu\nu}$ and for $g^2_{\mu\nu} = (e\phi + T_2)g_{\mu\nu}$,

$$R_{\mu\nu}(g^1_{\alpha\beta}) = 0 \quad (42)$$

and, at the same time,

$$R_{\mu\nu}(g^2_{\alpha\beta}) = 0 \,. \quad (43)$$

These two simultaneous conditions above impose a constraint on the tension field ϕ, because the metrics $g^1_{\alpha\beta}$ and $g^2_{\alpha\beta}$ are conformally related, but Einstein's equations are not conformally invariant, so the condition that Einstein's equations hold for both $g^1_{\alpha\beta}$ and $g^2_{\alpha\beta}$ is highly non trivial. Then for these situations, we have,

$$e\phi + T_1 = \Omega^2(e\phi + T_2)\,, \tag{44}$$

which leads to a solution for $e\phi$

$$e\phi = \frac{\Omega^2 T_2 - T_1}{1-\Omega^2}\,, \tag{45}$$

which leads to the tensions of the different strings to be

$$e\phi + T_1 = \frac{\Omega^2(T_2 - T_1)}{1-\Omega^2} \tag{46}$$

and

$$e\phi + T_2 = \frac{(T_2 - T_1)}{1-\Omega^2}\,. \tag{47}$$

Both tensions can be taken as positive if $T_2 - T_1$ is positive and Ω^2 is also positive and less than 1.

8.1 Flat space in Minkowski coordinates and flat space after a special conformal transformation

The flat spacetime in Minkowski coordinates is,

$$ds_1^2 = \eta_{\alpha\beta}dx^\alpha dx^\beta\,, \tag{48}$$

where $\eta_{\alpha\beta}$ is the standard Minkowski metric, with $\eta_{00} = 1$, $\eta_{0i} = 0$ and $\eta_{ij} = -\delta_{ij}$. This is of course a solution of the vacuum Einstein's equations.

We now consider the conformally transformed metric

$$ds_2^2 = \Omega(x)^2\eta_{\alpha\beta}dx^\alpha dx^\beta\,, \tag{49}$$

where conformal factor coincides with that obtained from the special conformal transformation

$$x\prime^\mu = \frac{(x^\mu + a^\mu x^2)}{(1 + 2a_\nu x^\nu + a^2x^2)} \tag{50}$$

for a certain D vector a_ν. which gives $\Omega^2 = \dfrac{1}{(1 + 2a_\mu x^\mu + a^2x^2)^2}$ In summary, we have two solutions for the Einstein's equations, $g^1_{\alpha\beta} = \eta_{\alpha\beta}$ and

$$g^2_{\alpha\beta} = \Omega^2\eta_{\alpha\beta} = \frac{1}{(1 + 2a_\mu x^\mu + a^2x^2)^2}\eta_{\alpha\beta}\,. \tag{51}$$

We can then study the evolution of the tensions using $\Omega^2 = \dfrac{1}{(1 + 2a_\mu x^\mu + a^2x^2)^2}$. We will consider the cases where $a^2 \neq 0$.

9 The Homogeneous and Isotropic Universe in Dynamical String Tension Theories

We now consider the case when a^μ is not light like and we will find that for $a^2 \neq 0$, irrespective of sign, i.e. irrespective of whether a^μ is space like or time like, we will have thick braneworlds where strings can be constrained between two concentric spherically symmetric bouncing higher dimensional spheres and where the distance between these two concentric spherically symmetric bouncing higher dimensional spheres approaches zero at large times. The string tensions of the strings one and two are given by

$$
\begin{aligned}
e\phi + T_1 &= \frac{(T_2 - T_1)(1 + 2a_\mu x^\mu + a^2 x^2)^2}{(1 + 2a_\mu x^\mu + a^2 x^2)^2 - 1} \\
&= \frac{(T_2 - T_1)(1 + 2a_\mu x^\mu + a^2 x^2)^2}{(2a_\mu x^\mu + a^2 x^2)(2 + 2a_\mu x^\mu + a^2 x^2)}\,, \qquad (52) \\
e\phi + T_2 &= \frac{(T_2 - T_1)}{(1 + 2a_\mu x^\mu + a^2 x^2)^2 - 1} \\
&= \frac{(T_2 - T_1)}{(2a_\mu x^\mu + a^2 x^2)(2 + 2a_\mu x^\mu + a^2 x^2)}\,. \qquad (53)
\end{aligned}
$$

Let us by consider the case where a^μ is time like, then without loosing generality we can take $a^\mu = (A, 0, 0, \ldots, 0)$. Now, in order to get homogeneous and isotropic cosmological solutions we must consider the limit $A \to 0$ and $(T_2 - T_1) \to 0$, in such a way that $(T_2 - T_1)/A = K$, where K is a constant. In that case the spatial dependence in the tensions (52) and (53) drops out and we get,

$$
e\phi + T_1 = e\phi + T_2 = \frac{K}{4t}\,. \qquad (54)
$$

The embedding metric can now be solved.

$$
g_{\mu\nu} = \frac{1}{(e\phi + T_1)} g^1_{\mu\nu} = \frac{4t}{K}\eta_{\mu\nu}\,, \qquad (55)
$$

which is not a vacuum metric, as opposed to $\eta_{\mu\nu}$ because of the conformal factor $4t/K$.

9.1 Life of the homogeneous and isotropic Universe and emergence of a braneworld at large times

One should notice that the homogeneous and isotropic solution has been obtained only in the limit $A \to 0$ and $(T_2 - T_1) \to 0$, in such a way that $(T_2 - T_1)/A = K$, where K is a constant. If A and $T_2 - T_1$ are small but finite, then for large times, of the order of $1/A$. We can formulate this as an uncertainty principle,

$$
(T_2 - T_1)\Delta t \approx \text{const.}\,, \qquad (56)
$$

where we have used that A is of the order of $(T_2 - T_1)$. So a small uncertainty in the tension $(T_2 - T_1)$ leads to a long lived homogeneous and isotropic phase, while a big uncertainty in the tension $(T_2 - T_1)$ leads to short lived homogeneous and isotropic phase.

In fact in these situations, for finite $(T_2 - T_1)$ and A, it is the case that the string tensions can only change sign by going first to infinity and then come back from minus infinity. We can now recognize at those large times the locations where the string tensions go to infinity, which are determined by the conditions

$$2a_\mu x^\mu + a^2 x^2 = 0\,, \tag{57}$$

or

$$2 + 2a_\mu x^\mu + a^2 x^2 = 0\,. \tag{58}$$

Let us start by considering the case where a^μ is time like, then without loosing generality we can take $a^\mu = (A, 0, 0, \ldots, 0)$. In this case the denominator in (52), (53) is

$$\begin{aligned}(2a_\mu x^\mu + a^2 x^2)&(2 + 2a_\mu x^\mu + a^2 x^2) \\ &= (2At + A^2(t^2 - x^2))(2 + 2At + A^2(t^2 - x^2))\,.\end{aligned} \tag{59}$$

The condition (57), if $A \neq 0$ implies then that

$$x_1^2 + x_2^2 + x_3^2 + \cdots + x_{D-1}^2 - (t + \frac{1}{A})^2 = -\frac{1}{A^2}\,. \tag{60}$$

If $A \to 0$, it is more convenient to write this in the form

$$A(x_1^2 + x_2^2 + x_3^2 + \cdots + x_{D-1}^2) - At^2 - 2t = 0\,, \tag{61}$$

which for the limit $A \to 0$ gives us the single singular point $t = 0$, which is the origin of the homogeneous and isotropic cosmological solution.

The other boundary of infinite string tensions is, (58) is given by,

$$x_1^2 + x_2^2 + x_3^2 + \cdots + x_{D-1}^2 - (t + \frac{1}{A})^2 = \frac{1}{A^2}\,. \tag{62}$$

This has no limit for $A \to 0$, all these points disappear from the physical space (they go to infinity).

For $A \neq 0$ we see that (62) represents an exterior boundary which has an bouncing motion with a minimum radius $1/A$ at $t = -1/A$, The denominator (59) is positive between these two bubbles. So for $T_2 - T_1$ positive the tensions are positive and diverge at the boundaries defined above.

The internal boundary (60) exists only for times t smaller than $-2/A$ and bigger than 0, so in the time interval $(-2/A, 0)$ there is no inner surface of infinite tension strings. This inner surface collapses to zero radius at $t = -2/A$ and emerges again from zero radius at $t = 0$.

For large positive or negative times, the difference between the upper radius and the lower radius goes to zero as $t \to \infty$

$$\sqrt{\frac{1}{A^2} + (t + \frac{1}{A})^2} - \sqrt{-\frac{1}{A^2} + (t + \frac{1}{A})^2} \to \frac{1}{tA^2} \to 0\,, \tag{63}$$

of course the same holds $t \to -\infty$. This means that for very large early or late times the segment where the strings would be confined (since they will avoid having infinite tension) will be very narrow and the resulting scenario will be that of a brane world for late or early times, while in the bouncing region the inner surface does not exist. Notice that this kind of braneworld scenario is very different to the ones previously studied, in particular both gravity (closed strings) and gauge fields (open strings) are treated on the same footing, since the mechanism that confines the strings between the two surfaces relies only on the string tension becoming very big.

We can ignore the part of the solution where $t < -2/A$ and instead take $t = 0$ as the origin of the Universe and only consider positive values of cosmic time because the part of the solution with $t < -2/A$ is disconnected, at least at the classical level from the part of the solution with positive cosmic time.

We see then that for the exact limit of $\Delta T \to 0$ and $A \to 0$ we get a perfect homogeneous and isotropic cosmology, but as ΔT and A are deformed to be small but finite, the scenario is modified at large times into a braneworld scenario.

10 Euclidean Extension and the Quantum Creation of a Braneworld from Nothing

The surface (62) allows an Euclidean extension, by defining an Euclidean time $(t_E + 1/A) = -i(t + 1/A)$, which replaced into (62) gives the spherical euclidean region

$$x_1^2 + x_2^2 + x_3^2 ..... + x_{D-1}^2 + (t_E + \frac{1}{A})^2 = \frac{1}{A^2} \tag{64}$$

The Euclidean region defined above and the surface (62) can be smoothly matched at $(t_E + 1/A) = (t + 1/A) = 0$. The quantum mechanical interpretation of the euclidean region (64) is the tunneling region that eventually leads by matching to the creation of the braneworld. The second surface (60) cannot be extended to Euclidean space and it starts at $t = 0$ from zero size after the other surface has been created. Then the two surfaces approach leading to the braneworld senario.

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