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Certain Investigations on Bianchi-III Cosmological Model with FLVDP in $f(R,T)$ Theory

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Abstract. In this study, we have conducted certain investigations of Bianchi-III cosmological models with fractional linear varying deceleration parameter (FLVDP) within the framework of the $f(R,T)$ theory. The aim was to examine the behaviour of the expansion rate of the universe over time. We have assumed deceleration parameter (DP) as $q(t) = 1 - \alpha_1 t/(1+\alpha_2 t)$ where $\alpha_1, \alpha_2 \geq 0$ and $q(t) = -1 + n\beta/(\beta+t)^2$ where $n, \beta > 0$. In this article, we have examined the values of a number of constants that can be used to explain the universe as it moves from an early stage of cosmic deceleration to a later stage of cosmic acceleration. Additionally, physical and graphic behaviours, along with energy conditions and the jerk parameter, have also been explored in this study.

KEY WORDS: Bilinear varying deceleration parameter, Cosmological models, Dark energy, Deceleration parameter, $f(R,T)$ gravity theory, fractional linear varying deceleration parameter.

1 Introduction

Cosmological models in general aim to describe the evolutionary process and the large-scale structure of the cosmos. It also provides a mathematical framework that incorporates various physical theories. To describe the observed aspects of the cosmos, the Bianchi-type universes including Bianchi-type III universe plays a significant role for the expansion of the universe in cosmology. As it permit anisotropic expansion, this is in contrast to the homogeneous and isotropic expansion assumed in the FLRW model. By studying a Bianchi-type model, we can explore the effects of anisotropy on the dynamics and evolution of the universe. We also acknowledge the deceleration parameter (DP), through revealing details about the dynamics and evolution of the cosmos as it plays a crucial role in the investigation of the cosmological model. It is important because it allow us to comprehend the history of the cosmos expansion. By measuring the DP at different cosmic epochs or redshifts, we can determine how the expansion rate has changed over time. This information is vital for reconstructing the past behaviour of the universe and understanding its evolution.

According to the conventional cosmological model, the DP is directly linked to the the existence of dark energy (DE) and the acceleration of the cosmos. While the negative value of q (i.e., $q < 0$) results in rapid expansion, the positive value of q (i.e., $q > 0$) suggests decelerated expansion.The study of different kinds of DP helps us to differentiate between various cosmological models. Different theoretical frameworks predict different expansion histories and behaviours. We have chosen investigation on Bianchi-III cosmological models with fractional linear varying deceleration parameter (FLVDP) in $f(R,T)$ theory due to the significance of this parameter. Under this study we believe that by comparing different assumptions of the DP we can access their agreements with observational data and investigate their viability. It also provides information about the energy content of the universe which depends on the contributions from different comparisons such as radiation, matter, DE, and modification of gravity. In present communication, we have considered the $f(R,T)$ theory, which is also known as $f(R,T)$ gravity. It is an alternative theory of gravity that extends the general theory of relativity (GTR) to include additional terms involving the stress-energy tensor's trace (T) and scalar curvature (R). In this theory, gravitational action is modified to include additional terms which permits a more comprehensive discussion of gravitational phenomena. This theory has also been applied to the study of the early universe during the inflationary epoch. By incorporating $f(R,T)$ gravity into inflationary cosmological models, many cosmologists investigated its effects on the dynamics and predictions of the inflationary scenario. Many cosmologists, including our own research group, have published numerous cosmological models in $f(R,T)$ theory with certain additional constraints due to significance of $f(R,T)$ gravity [1–10].

The standard Einstein-Hilbert action is known to be replaced by an arbitrary function, $f(R)$, where R is the scalar curvature, as was previously mentioned. Bertolami et al. [11] rebuilt gravity as $f(R,\mathscr{L}_m)$ by coupling the matter Lagrangian density $\mathscr{L}_m$ with a scalar curvature R. Sotiriou and Faraoni [12] showed that excess force in the case of dust disappears in the $f(R)$ gravity by accounting for the matter Lagrangian density $\mathscr{L}_m^1 = p$, with p as the pressure. Additionally, it has been deduced that the more natural version of $\mathscr{L}_m$, $\mathscr{L}_m^{(2)} = -\rho$ denotes the existence of a force that might be regarded as an alternative to dark matter [13]. Furthermore, Harko et al. [2] noted that the gravitational Lagrangian and the stress-energy tensor in $f(R)$ theory with non minimal coupling depends on the type of coupling between geometry and matter. Nojiri and Obintsov [14] investigated the integration of DE and early inflation with cosmic late-time acceleration. Harko et al. [1] have suggested a new theory called the

$f(R,T)$ gravity theory, incorporating the dependency of the stress-energy tensor T_{kl}'s trace T. Haghani et al. [15] further updated the $f(R,T)$ theory by adding the additional term $R_{kl}T^{kl}$ to the Lagrangian, resulting in the $f(R,T,R_{kl},T^{kl})$ theory.

As discussed above, the variable DP has much importance in measuring expansion rate and it is also an essential parameter in studying cosmological models. It provides valuable insights into the universe's expansion history. Recent studies have highlighted the construction of various models using the time-dependent DP in $f(R,T)$ theory [16, 17]. A new type of variable DP has also been assumed by Mishra et al. [6] who formulated differently by taking the bilinear varying deceleration parameter (BVDP) and published the result [8]. Under the above motivation, we assumed here the fractional linear varying deceleration parameter (FLVDP) for the present communication. We believe that it may be important in cosmology as it reveal details about various aspects and contents of the cosmos, it may also indicate the dominance of different components such as radiation, matter, and DE at different cosmic epochs.

There are nine sections in this communication. In Section 2, we presented mathematical equations governing the $f(R,T)$ theory, while Section 3 explored the metric and computational equations underlying the cosmological model. The field equation's solutions and physical and geometric properties for both the cases of DP were presented in Sections 4 and 5, respectively. Section 6 provides information about the energy conditions of the considered models. In Section 7, we give the expression for the jerk parameter, whereas the results and discussion have been shown in Section 8. The concluding remarks of the study are presented in the last Section 9.

2 Mathematical Equations Governing the $f(R,T)$ Theory

As mentioned in Section 1, the $f(R,T)$ gravity theory is primary modification of the GTR, which was given by Harko et al. [1]. Here, an arbitrary function of scalar curvature R and stress energy tensor's trace T is used to express the gravitational Lagrangian. The following is the action equation for this theory:

$$S = \frac{1}{16\pi}\int f(R,T)\sqrt{-g}d^4x + \int \mathscr{L}_m\sqrt{-g}d^4x, \tag{1}$$

here $\mathscr{L}_m$ is a matter Lagrangian density and the stress-energy tensor of the matter is defined as

$$T_{kl} = \frac{-2}{\sqrt{-g}}\frac{\delta(\sqrt{-g}\mathscr{L}_m)}{\delta g^{kl}}, \tag{2}$$

and T is defined as $T = g^{kl}T_{kl}$. The equation (2) changes to

$$T_{kl} = g_{kl}\mathscr{L}_m - 2\frac{\partial \mathscr{L}_m}{\partial g^{kl}}, \tag{3}$$

when the Legrangian density $\mathscr{L}_m$ of matter is considered to depend simply on the components g_{kl} of the metric tensor. The gravitational field equation of $f(R,T)$ gravity is obtained by varying the action S in equation (1) w.r.t the components of the metric tensor g^{kl}.

$$f_R(R,T)R_{kl} - \frac{1}{2}f(R,T)g_{kl} \\ = 8\pi T_{kl} + (\nabla_k\nabla_l - g_{kl}\Box)\, f_R(R,T) - (T_{kl} + \Theta_{kl})\, f_T(R,T), \quad (4)$$

here

$$\Theta_{kl} = -2T_{kl} + g_{kl}\mathscr{L}_m - 2g^{kl}\frac{\partial^2 \mathscr{L}_m}{\partial g^{kl}\partial g^{\mu\nu}}, \quad (5)$$

$\Box = g^{kl}\nabla_l\nabla_k = \nabla^k\nabla_k$ is d'Alembert operator, $f_T(R,T) = \frac{\partial}{\partial T}f(R,T)$, $f_R(R,T) = \frac{\partial}{\partial R}f(R,T)$, and ∇_k is the covarient derivative.

The stress-energy tensor T_{kl} for the perfect fluid in terms of pressure p and energy-density ρ is given by

$$T_{kl} = (\rho + p)u_k u_l - p g_{kl}. \quad (6)$$

Here u^k is the 4-velocity vector with $u^k = (0,0,0,1)$ in co-moving coordinates which satisfies the condition $u^k u_k = 1$ and $u^k\nabla_l u_k = 0$. To make equation (5) into

$$\Theta_{kl} = -2T_{kl} - p g_{kl}, \quad (7)$$

we use $\mathscr{L}_m = -p$. Now we have adopted the assumption of $f(R,T)$ as proposed by Harko et al. [1]

$$f(R,T) = R + 2f(T), \quad (8)$$

where $f(T)$ is an any mathematical function of the variable T. When we plug the previous two equations i.e., (7) & (8) into the field equation (4), we obtain

$$G_{kl} = 8\pi T_{kl} + 2f^{'}(T)T_{kl} + [2pf^{'}(T) + f(T)]g_{kl}, \quad (9)$$

where $G_{kl} = R_{kl} - \frac{1}{2}Rg_{kl}$. Furthermore, we would like to take into consideration the following choice of $f(T)$

$$f(T) = \eta T, \quad (10)$$

with η as arbitrary constant.

3 Metric and Computational Equations Guiding Cosmological Model

The Bianchi-III metric has been considered in the following form

$$ds^2 = dt^2 - R_1^2dx^2 - R_2^2e^{-2\alpha x}dy^2 - R_3^2dz^2, \quad (11)$$

where $\alpha > 0$ with a dimension of length's inverse and R_1, R_2, R_3 are the only functions of variable t. For the sake of simplicity, let's say that $\alpha = 1$. This assures that the universe model is anisotropic and spatially homogeneous.

Now we wish to discuss various physical terms for the metric (11) which is essential tool for such investigations:

- Spatial volume (V): It is the three-dimensional extent of a region in the universe, linked to the scale factor representing the universe's relative size over time. It is crucial for understanding the large-scale structure and dynamics of the cosmos.

$$V = R_1 R_2 R_3, \tag{12}$$

- Expansion scalar (θ): It characterizes how the size of the universe changes, indicating whether in undergoes expansion or contraction.

$$\theta = 3H = \sum_{i=1}^{3} \left[\frac{\dot{R}_i}{R_i} \right], \quad i = 1, 2, 3 \tag{13}$$

- Hubble parameter (H): As we have discussed in the introduction, the Hubble parameter represents the current rate of the universe's expansion. It quantifies the recession velocity of galaxies from each other due to the overall expansion of space.

$$H = \frac{1}{3} \sum_{i=1}^{3} \left[\frac{\dot{R}_i}{R_i} \right] = \frac{1}{3} \left(H_1 + H_2 + H_3 \right), \tag{14}$$

  where the x, y, and z directional Hubble parameters are given by $H_1 = \dot{R}_1/R_1$, $H_2 = \dot{R}_2, R_2$, and $H_3 = \dot{R}_3/R_3$, respectively.

- Mean anisotropic parameter (A_m): In cosmology A_m quantifies deviations from isotropy in the large-scale distribution of matter and energy in the universe, indicating preferred directions or spatial variations.

$$A_m = \frac{1}{3} \Big[\Big(\frac{H_1 - H}{H} \Big)^2 + \Big(\frac{H_2 - H}{H} \Big)^2 + \Big(\frac{H_3 - H}{H} \Big)^2 \Big], \tag{15}$$

- Shear scalar (σ): It measure the distortion of the region during the universe's expansion. It describes the relative changes in distances between cosmic structures, such as galaxies or galaxy clusters, in different directions. It helps to study the large-scale dynamics and evolution of cosmic structures in the expanding universe.

$$\sigma^2 = \frac{1}{2} \sigma_{kl} \sigma^{kl} = \frac{1}{2} \left[\sum_{i=1}^{3} H_i^2 - 3H^2 \right], \tag{16}$$

  where $\sigma_{kl} = u_{k;l} + \frac{1}{2}(u_{k;\mu} u^\mu u_l + u_{l;\mu} u^\mu u_k) + \frac{1}{3}\theta(g_{kl} + u_k u_l)$.

- Dimensionless deceleration parameter

$$q(t) = -\frac{a\ddot{a}}{\dot{a}^2}. \tag{17}$$

By utilizing equations (6) and (10), we can depict the field equations (9) corresponding to the metric (11) as

$$\frac{\ddot{R}_1}{R_1} + \frac{\ddot{R}_3}{R_3} + \frac{\dot{R}_1\dot{R}_3}{R_1R_3} = -(8\pi + 3\eta)p + \eta\rho, \tag{18}$$

$$\frac{\ddot{R}_2}{R_2} + \frac{\ddot{R}_3}{R_3} + \frac{\dot{R}_2\dot{R}_3}{R_2R_3} = -(8\pi + 3\eta)p + \eta\rho, \tag{19}$$

$$\frac{\ddot{R}_1}{R_1} + \frac{\ddot{R}_2}{R_2} + \frac{\dot{R}_1\dot{R}_2}{R_1R_2} - \frac{1}{R_1^2} = -(8\pi + 3\eta)p + \eta\rho, \tag{20}$$

$$\frac{\dot{R}_1\dot{R}_2}{R_1R_2} + \frac{\dot{R}_1\dot{R}_3}{R_1R_3} + \frac{\dot{R}_2\dot{R}_3}{R_2R_3} - \frac{1}{R_1^2} = (8\pi + 3\eta)\rho - \eta p, \tag{21}$$

$$\frac{\dot{R}_1}{R_1} - \frac{\dot{R}_2}{R_2} = 0. \tag{22}$$

Equation (22) yields

$$R_1 = \tau R_2, \tag{23}$$

where τ is the integration constant. Without loss of generality, we assume τ=1 which yields

$$R_1 = R_2. \tag{24}$$

Field equations (18)-(21) can be reduced to the following form by using this relation

$$\frac{\ddot{R}_1}{R_1} + \frac{\ddot{R}_3}{R_3} + \frac{\dot{R}_1\dot{R}_3}{R_1R_3} = -(8\pi + 3\eta)p + \eta\rho, \tag{25}$$

$$2\left(\frac{\ddot{R}_1}{R_1}\right) + \left(\frac{\dot{R}_1}{R_1}\right)^2 - \frac{1}{R_1^2} = -(8\pi + 3\eta)p + \eta\rho, \tag{26}$$

$$\left(\frac{\dot{R}_1}{R_1}\right)^2 + 2\frac{\dot{R}_1\dot{R}_3}{R_1R_3} - \frac{1}{R_1^2} = (8\pi + 3\eta)\rho - \eta p. \tag{27}$$

Now the equations (25)-(27) are three in number and contain four unknown parameters R_1, R_3, ρ, and p. Therefore, we require additional connections between these parameters in order to explicitly solve these equations. Since the expansion scalar θ and the component σ_1^1 of shear scalar tensor (σ_k^l) are proportional i.e. $\sigma_1^1 \propto \theta$. This condition leads to

$$R_2 = R_3^m, \tag{28}$$

here m is a constant with $m > 0$. The above computational analysis was made as proposed by Throne [18].

4 Solutions of the Field Equations

To arrive at the solution of the field equations (25)-(27), we have taken the following additional requirements, as presented under Sections 4.1 and 4.2.

4.1 Model I: $q(t) = 1 - \alpha_1 t/(1 + \alpha_2 t)$

In this model, we will find the solution of field equations using the FLVDP proposed by [8] (see Figure 1)

$$q(t) = 1 - \frac{\alpha_1 t}{1 + \alpha_2 t}, \tag{29}$$

where $\alpha_1, \alpha_2 \geq 0$.

The assumption mentioned above is primarily motivated by the fact that our universe is currently accelearting after previously decelerating, according to recent observations of SNe Ia [19, 20]. The DP must show the flipping in its

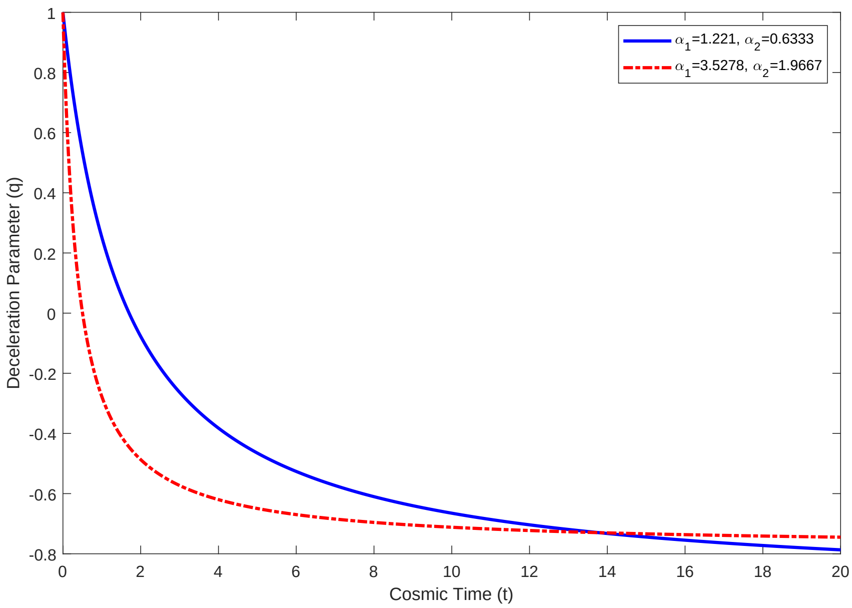

Figure 1. Graphical illustration of DP q with time t for model I.

sign [21–23] as a result of this transition (from the previous decelerating phase to the current accelerating phase). As a result, the DP is a time-varying quantity rather than a constant. We have also assumed a variable DP as

$$\frac{d}{dt}\left(\frac{1}{H}\right) = 1 + q(t), \tag{30}$$

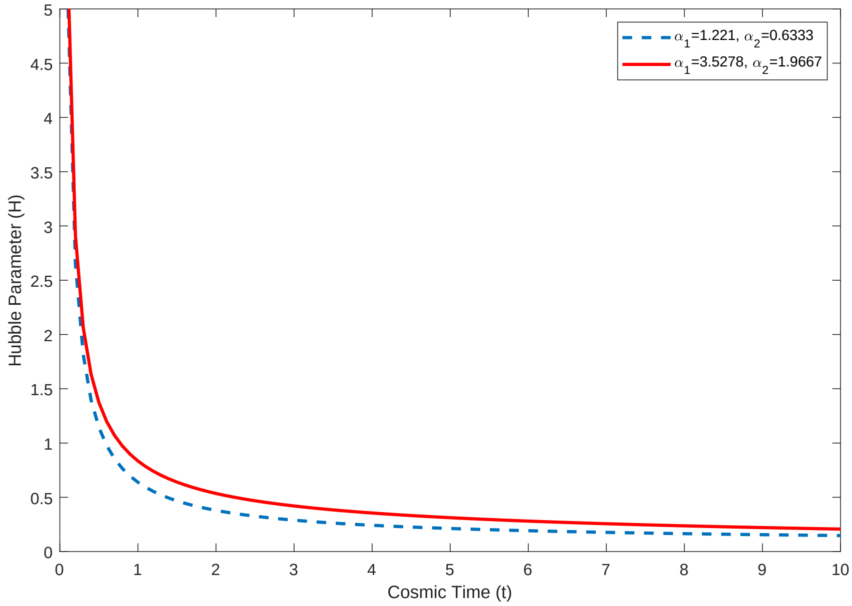

Figure 2. Graphical illustration of Hubble parameter H with time t for model I.

We acquire the Hubble parameter's expression, $H(t)$, by integrating our previous equation with respect to t (Figure 2).

$$H(t) = \frac{1}{\int(1 + q(t))dt + c_1}, \tag{31}$$

Here c_1 is a integrating constant.

$$H(t) = \frac{1}{\alpha_3 t + \alpha_4 log(1 + \alpha_2 t) + c_2}, \tag{32}$$

where $\alpha_3 = (2 - \alpha_1/\alpha_2)$, $\alpha_4 = \alpha_1/\alpha_2^2$, and c_2 are constants. For physical feasibility, c_2 is set to zero due to the behavior where as $t \to 0$, $H \to \infty$. During the early inflationary stage, the universe experiences a remarkably rapid expansion. Hence the rewritten expression for the Hubble parameter is

$$H = \frac{1}{\alpha_3 t + \alpha_4 \log(1 + \alpha_2 t)}. \tag{33}$$

After simplifying the above expression of H, we have

$$a(t) = a_0 \sqrt{t}.e^{Q_1(t)}, \tag{34}$$

where a_0 is integrating constant and

$$Q_1(t) = \frac{\alpha_1}{8}t + \frac{\alpha_1(3\alpha_1 - 8\alpha_2)}{192}t^2 + \frac{\alpha_1(3\alpha_1^2 - 16\alpha_1\alpha_2 + 24\alpha_2^2)}{1152}t^3 + \frac{\alpha_1(45\alpha_1^3 - 360\alpha_1^2\alpha_2 + 1040\alpha_1\alpha_2^2 - 1152\alpha_2^3)}{92160}t^4 + O(t^5).$$

Now from equations (12), (24), (28), and (34) we get

$$R_1 = R_2 = \left[a_0\sqrt{t}.e^{Q_1(t)})\right]^{\frac{3m}{2m+1}}, \tag{35}$$

and

$$R_3 = \left[a_0\sqrt{t}.e^{Q_1(t)}\right]^{\frac{3}{2m+1}}. \tag{36}$$

Using $a_0 = 1$ and $\alpha = 1$, the metric (11) can be written as

$$ds^2 = dt^2 - \left[\sqrt{t}.e^{Q_1(t)}\right]^{\frac{6m}{2m+1}} \left(dx^2 + e^{-2x}dy^2\right) - \left[\sqrt{t}.e^{Q_1(t)}\right]^{\frac{6}{2m+1}} dz^2. \tag{37}$$

4.2 Model II: $q(t) = -1 + n\beta/(\beta + t)^2$

The description of DP, where a is the average scale factor, is defined as $q = -a\ddot{a}/\dot{a}^2$ (see Figure 3). In order to resolve the three field equations (25)-(27)

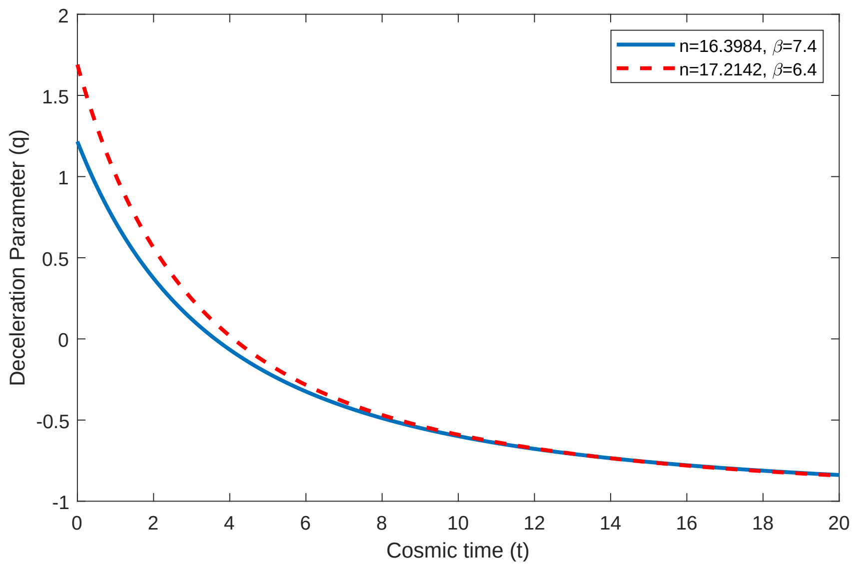

Figure 3. Graphical illustration of DP q with time t.

involving the four unknown parameters R_1, R_3, ρ and p. We have presented the time-varying deceleration parameter ($q \equiv q(t)$)

$$q(t) = -1 + \frac{n\beta}{(\beta+t)^2}, \tag{38}$$

here n and β are positive constants.

From equation (38) following constraints may be imposed, $q < 0$ for $t > \sqrt{n\beta} - \beta$, $q > 0$ for $t < \sqrt{n\beta} - \beta$. To validate this model it is necessary to mention here the DP's present value, which is governed by $q_0 = -1 + \beta/nH_0^2t_0^2$, where t_0 denotes the universe age and H_0 the current values of Hubble's parameter, respectively (Figure 4). From recent research findings, we know that for the suitable value of $n = 0.27\beta$ the present value of DP (q_0) lies in the range of $-1 \leq q \leq 0$ i.e. $q_0 \approx -0.73$ [24]. Using equation (38) into equation (31) and integrating with respect to time t, we arrive at the expression for the Hubble parameter

$$H(t) = \frac{1}{\frac{-n\beta}{\beta+t} + c_1} \tag{39}$$

where c_1 is an integration constant, and for computational simplification, we assume $c_1 = n$.

$$H(t) = \frac{1}{n}\left[1 + \frac{\beta}{t}\right]. \tag{40}$$

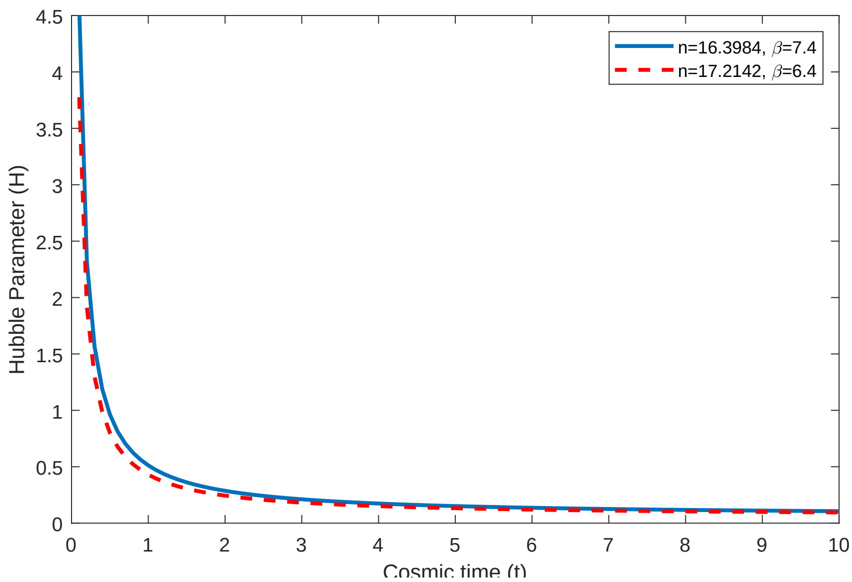

Figure 4. Graphical illustration of Hubble parameter H with time t for model II

Again, by integrating and simplifying the above equation, we obtained the scale factor $a(t)$ in terms of cosmic time t as

$$a(t) = (t^{\beta} e^{t})^{\frac{1}{n}} \tag{41}$$

Now, using equations (12), (24), (28), and (41), we determined the formula for directional potential functions as

$$R_1 = R_2 = (t^{\beta} e^{t})^{\frac{3m}{n(2m+1)}}, \tag{42}$$

$$R_3 = (t^{\beta} e^{t})^{\frac{3}{n(2m+1)}}. \tag{43}$$

The metric equation (11) for $\alpha_1 = 1$, after substituting the values of R_1, R_2, and R_3 becomes

$$ds^2 = dt^2 - \left(t^{\beta} e^{t}\right)^{\frac{6m}{n(2m+1)}} \left(dx^2 + e^{-2x} dy^2\right) - \left(t^{\beta} e^{t}\right)^{\frac{6}{n(2m+1)}} dz^2. \tag{44}$$

5 Physical and Geometric Properties

In this section, we have discussed the physical and geometric properties of both models under Sections 5.1 and 5.2. The exploration of these properties provide valuable insights into the dynamics and evolution of the cosmos within the context of our study on Bianchi-III cosmological models with FLVDP in the framework of the $f(R,T)$ theory. Section 5.1 focuses on the first model, $q(t) = 1 - \alpha_1 t/(1+\alpha_2 t)$, where we examine its physical implications and geometric behaviour. whereas, Section 5.2 delves into the second model, $q(t) = -1 + n\beta/(\beta+t)^2$, and its associated physical and geometric characteristics, providing valuable insights into the universe's expansion dynamics and the transitions between cosmic deceleration and acceleration.

5.1 Model I: $q(t) = 1 - \alpha_1 t/(1+\alpha_2 t)$

By examining the physical characteristics of the model described by equation (37) the additional physical parameter that are significant and helpful in the discussion of cosmological models can be determined. The value of R_1 and R_3 derived from (35) and (36) are utilised to calculate expression for H_i and $V(t)$, i.e., directional Hubble parameters, spatial volume and σ, $\theta(t)$, and A_m, i.e., shear scalar, expansion scalar, and mean anisotropic parameter.

$$H_1(t) = H_2(t) = \frac{3m}{2m+1} \left\{ \frac{1}{2t} + Q_2(t) \right\}, \tag{45}$$

$$H_3(t) = \frac{3}{2m+1} \left\{ \frac{1}{2t} + Q_2(t) \right\}, \tag{46}$$

where

$$Q_2(t) = \dot{Q}_1(t) = \frac{\alpha_1}{8} + \frac{\alpha_1(3\alpha_1 - 8\alpha_2)}{96}t + \frac{\alpha_1(3\alpha_1^2 - 16\alpha_1\alpha_2 + 24\alpha_2^2)}{384}t^2 + \frac{\alpha_1(45\alpha_1^3 - 360\alpha_1^2\alpha_2 + 1040\alpha_1\alpha_2^2 - 1152\alpha_2^3)}{23040}t^3 + O(t^4).$$

It is clear from these directional Hubble parameter values that they all tend to zero at $t \to \infty$.

$$V(t) = a_0^3 t^{\frac{3}{2}} e^{3Q_1(t)}, \tag{47}$$

$$\sigma^2 = \frac{3(m-1)^2}{(2m+1)^2}\left\{\frac{1}{\alpha_3 t + \alpha_4 log(1+\alpha_2 t)}\right\}^2, \tag{48}$$

$$A_m = \frac{2\,(m-1)^2}{(2m+1)^2}, \tag{49}$$

$$\theta = 3\left\{\frac{1}{2t} + Q_2(t)\right\}, \tag{50}$$

$$\frac{\sigma^2}{\theta^2} = \frac{1}{3}\frac{(m-1)^2}{(2m+1)^2}. \tag{51}$$

Both scalar expansion & shear scalar diverge at $t = 0$ and approaches zero as $t \to \infty$, as shown by the preceding equations. From equation (51), it is clear that for $m = 1$ the isotropic condition σ^2/θ^2 becomes zero. Simply, when $m = 1$, our model resembles a standard FLRW model. However, for $m > 0$ and $(m \neq 1)$, we get $\sigma^2/\theta^2 < 1/3 \simeq 0.3$. As the isotropic condition's value in this circumstance is infinitesimal, this means our model is on the edge of becoming isotropic, with a low level of anisotropy. The pressure p (Figure 5) and energy density ρ (Figure 6) formulas are determined by plugging the values of R_1 and R_3 from equations (35) and (36) into the field equations (25)-(27).

$$p = \frac{1}{(8\pi+3\eta)^2 - \eta^2}\left[\frac{9m\left\{2\eta - 8m(3\pi+\eta)\right\}}{(2m+1)^2}H^2 - \frac{6m(8\pi+3\eta)}{2m+1}\dot{H} + (8\pi+2\eta)\left\{a_0\sqrt{t}.e^{Q_1(t)}\right\}^{\frac{-6m}{2m+1}}\right], \tag{52}$$

here $\dot{H}$ is the time derivative of H,

$$\rho = \frac{1}{(8\pi+3\eta)^2 - \eta^2}\left[\frac{9m\left\{8\pi(m+2) + 6\eta\right\}}{(2m+1)^2}H^2 - \frac{6\eta m}{2m+1}\dot{H} - (8\pi+2\eta)\left\{a_0\sqrt{t}.e^{Q_1(t)}\right\}^{\frac{-6m}{2m+1}}\right], \tag{53}$$

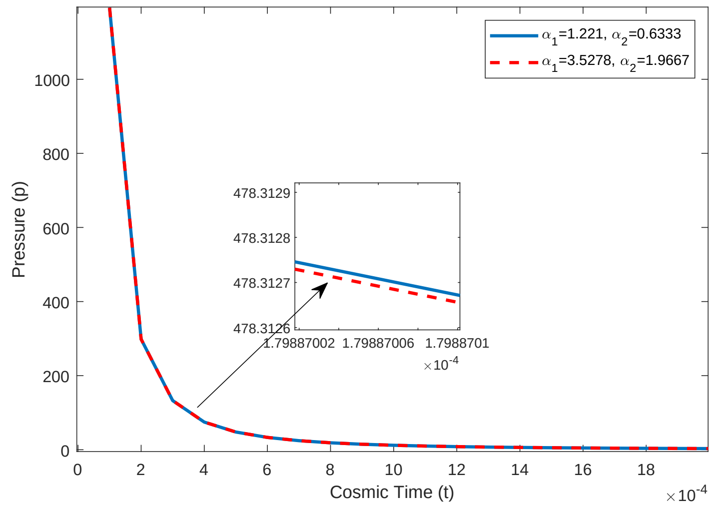

Figure 5. Graphical illustration of pressure p with time t for $m = 0.001$, $\eta = 0.00001$ and $a_0 = 1$ for model I.

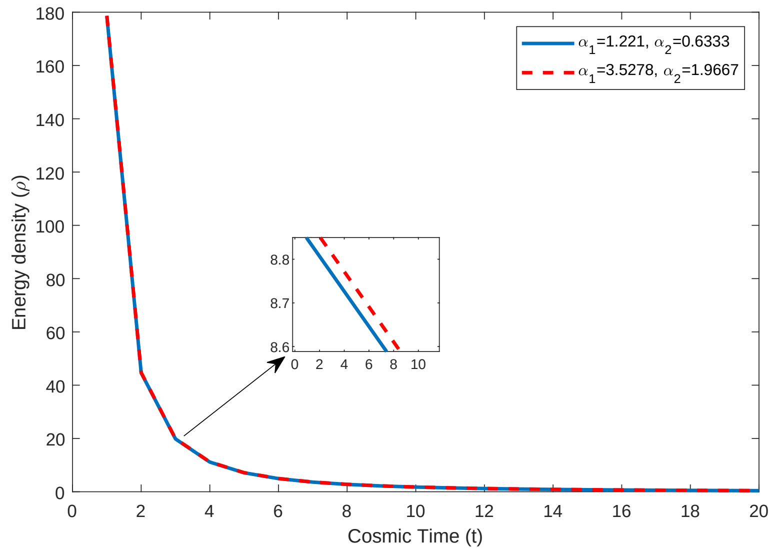

Figure 6. Graphical illustration of energy density ρ versus time t for $m = 0.001$, $\eta = 0.00001$ and $a_0 = 1$ for model I.

$$\omega = \frac{1}{\rho\left((8\pi+3\eta)^2-\eta^2\right)}\left[\frac{9m\left\{2\eta-8m(3\pi+\eta)\right\}}{(2m+1)^2}H^2 - \frac{6m(8\pi+3\eta)}{2m+1}\dot{H} + (8\pi+2\eta)\left\{a_0\sqrt{t}.e^{Q_1(t)}\right\}^{\frac{-6m}{2m+1}}\right]. \quad (54)$$

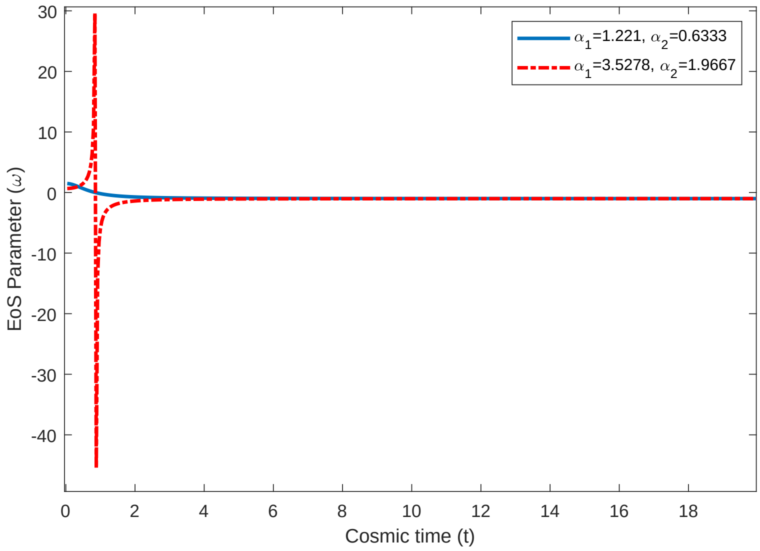

Figure 7. Graphical illustration of EoS parameter $\omega(t)$ with time t for model I.

As we know that Ricci scalar R is a fundamental component of Einstein's field equations, which describe how matter and energy influence the shape and geometry of the cosmos at both scales i.e., at small and large scale. The expression for Ricci scalar R is computed in the following manner:

$$R = 2\left[2\frac{\ddot{R}_1}{R_1} + \frac{\ddot{R}_3}{R_3} + \left(\frac{\dot{R}_1}{R_1}\right)^2 + 2\frac{\dot{R}_1\dot{R}_3}{R_1R_3} - \frac{1}{R_2^1}\right] \tag{55}$$

$$R = 6\dot{H} + \frac{18\left(3m^2+2m+1\right)}{(2m+1)^2}H^2 - 2\left\{a_0\sqrt{t}.e^{Q_1(t)}\right\}^{\frac{-6m}{2m+1}} \tag{56}$$

In the condition where $R \to \infty$ as $t \to 0$ and $R \to 0$ as $t \to \infty$, it suggest that at the beginning of the universe $(t = 0)$, there was a singularity with infinite curvature, possibly resembling the Big Bang. As time progresses to infinity, the curvature of spacetime flattens to zero, indicating a potentially asymptotically flat or Euclidean spacetime. Such a scenario could signify the universe's evolution from an initial singularity to a spatially flat.

The expression for trace T of stress-energy tensor is obtained as

$$T = \rho - 3p = \frac{1}{(8\pi+3\eta)-\eta^2}\left[\frac{72m(10m\pi+3m\eta+2\pi)}{(2m+1)^2}H^2 + \frac{48m(3\pi+\eta)}{2m+1}\dot{H} - 4(8\pi+2\eta)\left\{a_0\sqrt{t}.e^{Q_1(t)}\right\}^{\frac{-6m}{2m+1}}\right]. \tag{57}$$

5.2 Model II: $q(t) = -1 + n\beta/(\beta+t)^2$

By examining the physical behaviour of the model described by equation (44), the additional physical parameter that are essential and helpful in the discussion of cosmological models can be determined. The values of R_1 and R_3 from equations (42) and (43) are used to calculate expressions for cosmological parameters such as H_i and $V(t)$ i.e., directional Hubble parameters, spatial volume and σ, $\theta(t)$, and A_m i.e., shear scalar, expansion scalar, and mean anisotropic parameter.

$$H_1 = H_2 = \frac{3m}{n(2m+1)}\Big[1+\frac{\beta}{t}\Big], \tag{58}$$

$$H_3 = \frac{3}{n(2m+1)}\Big[1+\frac{\beta}{t}\Big], \tag{59}$$

$$V = \left(t^{\beta}e^{t}\right)^{\frac{3}{n}}, \tag{60}$$

$$\sigma^2 = \frac{3(m-1)^2}{n^2(2m+1)^2}\left[1+\frac{\beta}{t}\right]^2, \tag{61}$$

$$\theta = \frac{3}{n}\left[1+\frac{\beta}{t}\right], \tag{62}$$

$$A_m = \frac{2(m-1)^2}{(2m+1)^2}, \tag{63}$$

$$\frac{\sigma^2}{\theta^2} = \frac{(m-1)^2}{3(2m+1)^2}. \tag{64}$$

It is important to note that the q's sign indicates whether the cosmos is expanding faster or slower. It is obvious from equations (60) and (62) that the spatial volume (V) is zero at $t=0$, whereas the expansion scalar (θ) becomes infinite, indicating that the cosmos started out with zero volume and infinite rate of expansion at $t=0$. Additionally, the scale factor $a(t)$ is zero at $t=0$, indicating that our mathematical model features the singularity at the beginning.

As t increases, the expansion scalar falls and the spatial volume rises, implying that the rate of expansion slows over time. The preceding study strongly favors the Big-Bang scenario.

We deduce the expressions for energy density ρ (Figure 8), pressure p (Figure 9), and EoS parameter $\omega(t)$ (Figure 10) by plugging the values of R_1 and R_3 from equations (42) and (43) via the field equations (25)-(27).

$$\rho = \frac{1}{(8\pi+3\eta)^2-\eta^2}\left[\left\{\frac{6m\,(12m\pi+24\pi+9\eta)}{(2m+1)^2}\right\}H^2 - \left\{\frac{6m\eta}{2m+1}\right\}\dot{H} - (8\pi+2\eta)(t^{\beta}e^{t})^{\frac{-6m}{n(2m+1)}}\right], \tag{65}$$

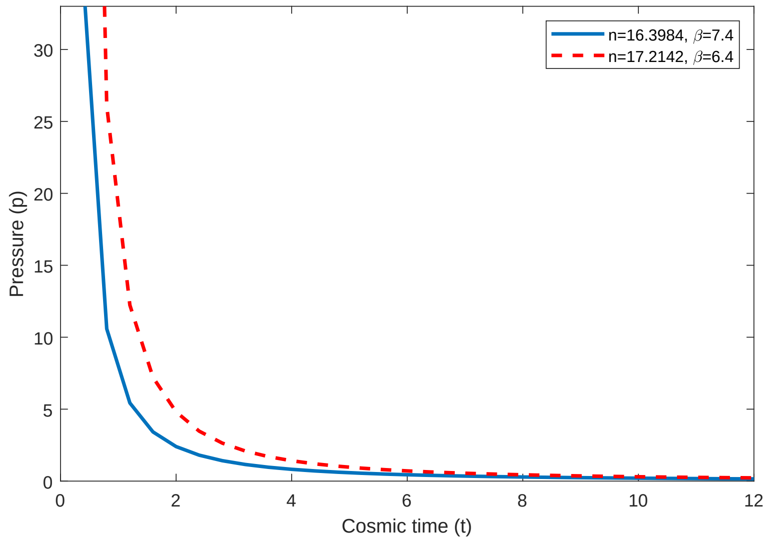

Figure 8. Graphical illustration of pressure p with time t for $m = 8.332$ and $\eta = 0.0501$ for model II.

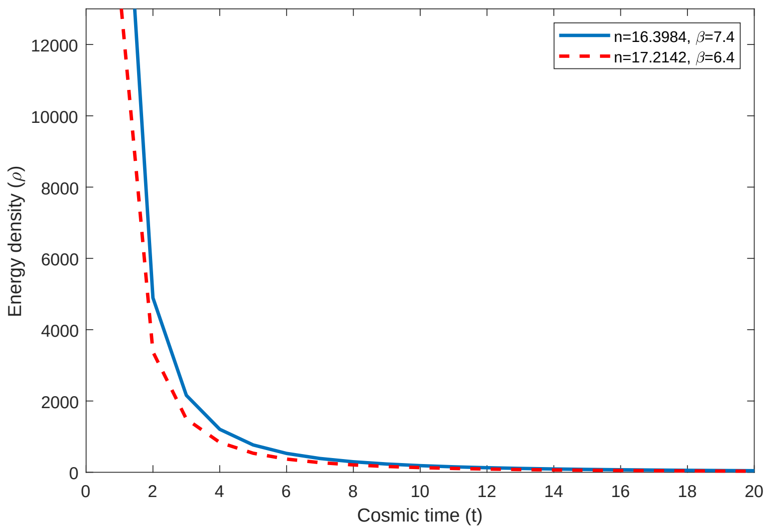

Figure 9. Graphical illustration of energy density ρ with time t for $m = 8.332$ and $\eta = 0.0501$ for model II.

$$p = \frac{1}{(8\pi+3\eta)^2-\eta^2}\left[\left\{\frac{18m\left(-12m\pi-4m\eta+\eta\right)}{(2m+1)^2}\right\}H^2 - \left\{\frac{6m(8\pi+3\eta)}{(2m+1)}\right\}\dot{H} + (8\pi+2\eta)(t^{\beta}e^{t})^{\frac{-6m}{n(2m+1)}}\right], \quad (66)$$

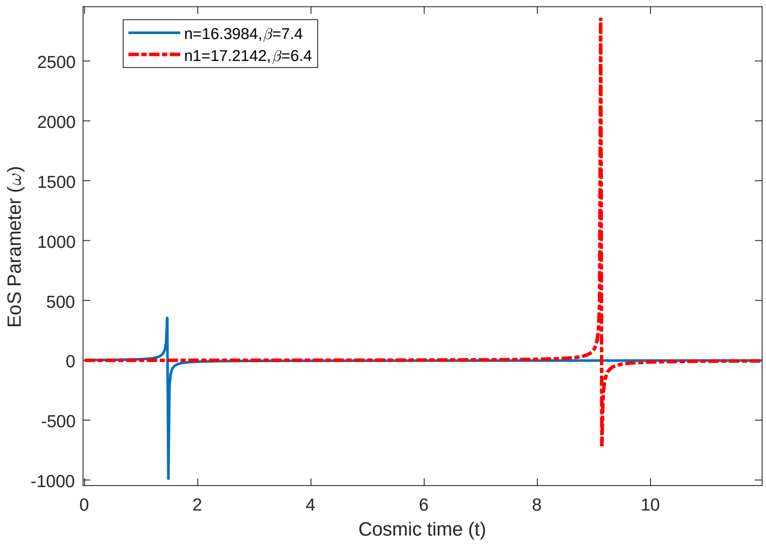

Figure 10. Graphical illustration of EoS parameter $\omega(t)$ with time t for model II.

$$\omega = \frac{1}{\rho\left((8\pi+3\eta)^2-\eta^2\right)}\left[\left\{\frac{18m\left(-12m\pi-4m\eta+\eta\right)}{(2m+1)^2}\right\}H^2 - \left\{\frac{6m(8\pi+3\eta)}{(2m+1)}\right\}\dot{H} + (8\pi+2\eta)(t^{\beta}e^{t})^{\frac{-6m}{n(2m+1)}}\right]. \quad (67)$$

The value of Ricci scalar R for this model is computed by using the equation (55) in the following manner

$$R = \frac{9(2m^2+2m+1)}{(2m+1)^2}H^2 + \frac{3(m+1)}{2m+1}\dot{H} - \left(t^{\beta}e^{t}\right)^{-\frac{6m}{n(2m+1)}}, \quad (68)$$

Here we can predict the same behaviour of Ricci scalar R as we did in the previous section i.e., Section 5.1, under equation (56).

The expression for trace T of stress-energy tensor is obtained as

$$T = \rho - 3p = \frac{1}{(8\pi+3\eta)-\eta^2}\left[\frac{72m(10m\pi+3m\eta+2\pi)}{(2m+1)^2}H^2 + \frac{48m(3\pi+\eta)}{2m+1}\dot{H} - 4(8\pi+2\eta)\left\{t^{\beta}e^{t}\right\}^{\frac{-6m}{n(2m+1)}}\right]. \quad (69)$$

Now we wish to discuss the energy conditions in a separate section, section 6

6 Energy Condition

An energy conditions (EC) is a generalization of the concept that the energy density of an area of space can't be negative in a relativistically-phrased computational formulation in relativistic classical field theories of gravity, particularly GTR (see Figures 11 and 12). These boundary conditions were established mathematically in an attempt to express the idea that energy should be positive. These provide an explanation of how null, time-like, and space-like geodesics behave. In the literature, it can be seen that EC are defined in many ways, including physically, where they are expressed using the stress-energy tensor T_{kl}, geometrically, where they are expressed using scalar curvature R_{kl}, and effectively, where they are expressed using ρ and p. In this study, we built our models in an effective way, which is defined as follows:

- Null energy conditions (NEC): For each future-pointing null vector field $\vec{n}$, the NEC state that $T_{kl}n^k n^l \geq 0$ must be true.
- Weak energy conditions (WEC): These energy conditions imply that the matter density remains non-negative, or $T_{kl}z^k z^l \geq 0$, for each time-like vector field $\vec{z}$.
- Strong energy conditions (SEC): Here, for each time-like vector field $\vec{z}$, the trace of tidal tensor is always non-negative i.e., $(T_{kl} - \frac{1}{2}Tg_{kl})z^k z^l \geq 0$.

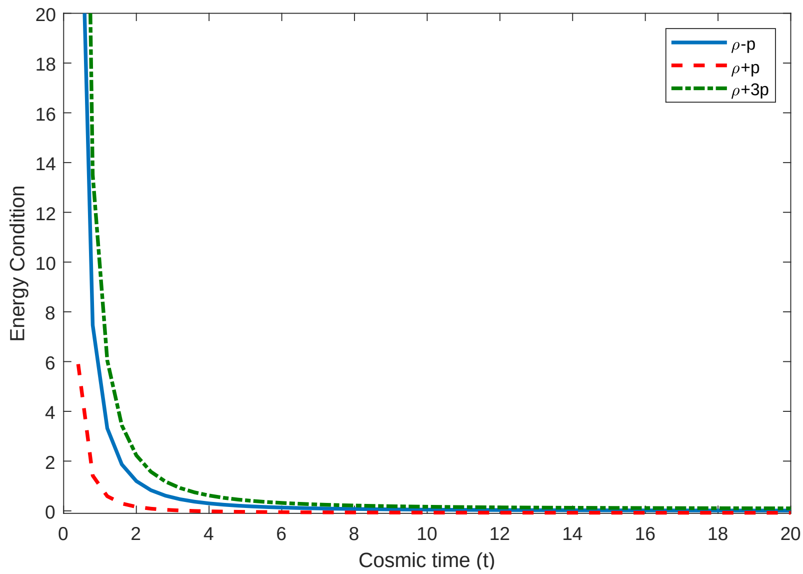

Figure 11. Graphical illustration of energy condition with time t for model I.

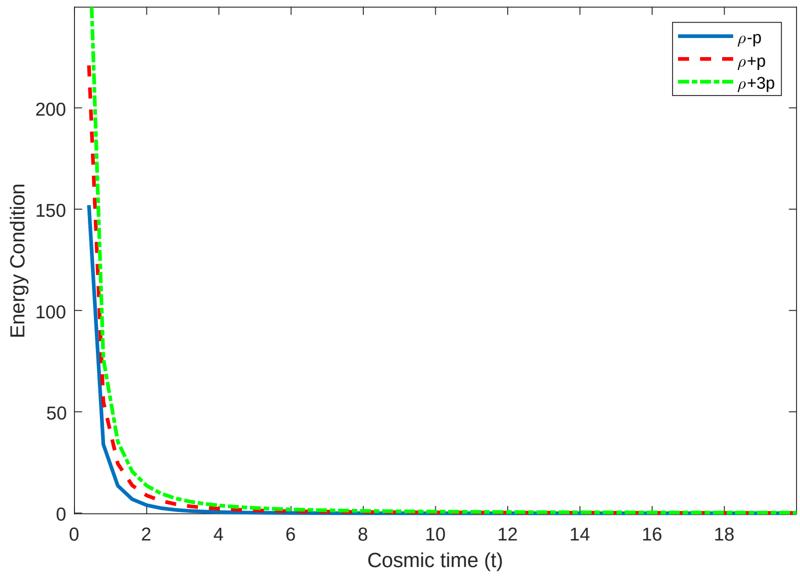

Figure 12. Graphical illustration of energy condition with time t for model II.

The behaviour of the energy condition (EC) in the case of a perfect fluid is $\rho + p \geq 0$ for the null energy condition (NEC), $\rho \geq 0$, and $\rho - p \geq 0$ for the weak energy condition (WEC), and $\rho + 3p \geq 0$ for the strong energy condition (SEC).

Now for deeper and more accurate understanding of the dynamics and evolution of the universe, we have discussed the jerk parameter in Section 7.

7 Expression for Jerk Parameter

In order to study the dynamics and evolution of the universe, there is another approach that enables a cosmologist to observe this phenomenon using the scale factor $a(t)$. This includes a brief study of kinematic parameters, i.e., the jerk parameter $j(t)$.This non-dimensional parameter is derived from the third derivative of the cosmic scale factor $a(t)$, and its mathematical expression is given by

$$j \equiv j(t) = \frac{a^2 \dddot{a}}{\dot{a}^3} = \frac{\dddot{a}}{aH^3}. \quad (70)$$

From the literature survey, it can be concluded that at the present epoch, the value of the jerk parameter of the universe is one [25]. For the construction of the above models as defined in previous sections, we may obtain the expression for j(t) using equation (70) as follows:

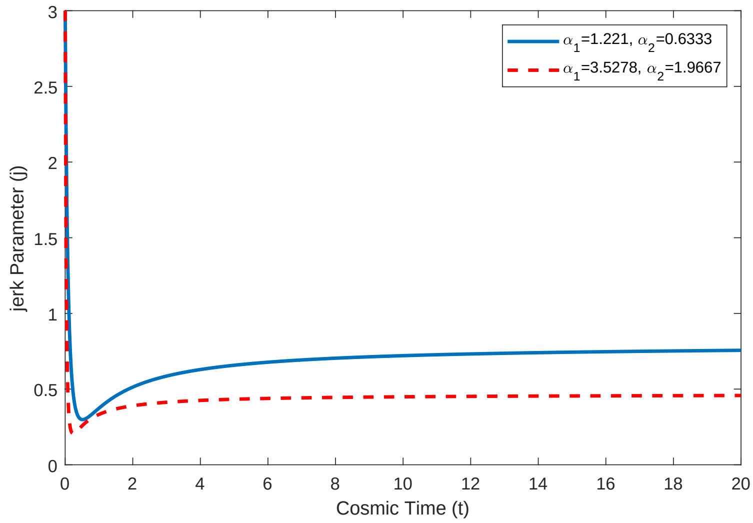

Figure 13. Graphical illustration of jerk parameter j with time t for model I.

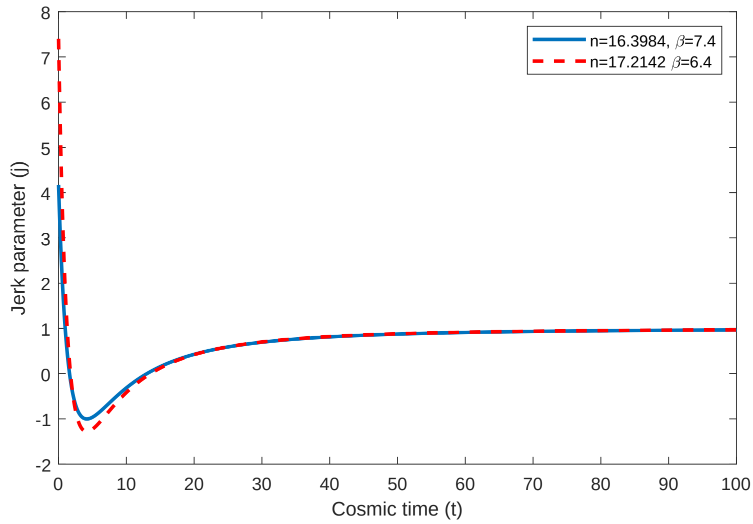

Figure 14. Graphical illustration of jerk parameter j with time t for model II.

• When $q = 1 - \alpha_1 t/(1 + \alpha_2 t)$, we obtain

$$j(t) = 1 - \frac{\alpha_1 t}{1 + \alpha_2 t} + \frac{2(1 + \alpha_2 t - \alpha_1 t)^2 + \alpha_1 \alpha_3 t + \alpha_1 \alpha_4 \log(1 + \alpha_2 t)}{(1 + \alpha_2 t)^2}, \quad (71)$$

where $\alpha_3 = 2 - \alpha_1/\alpha_2$ and $\alpha_4 = \alpha_1/\alpha_2^2$.

• When $q = -1 + n\beta/(\beta+1)^2$, we get

$$j(t) = 1 - \frac{3n\beta}{(\beta+t)^2} + \frac{2n^2\beta}{(\beta+t)^3}, \tag{72}$$

8 Results and Discussion

In this study, we have conducted an investigation on the Bianchi-III cosmological model, incorporating the FLVDP assumption within the framework of $f(R,T)$ theory and considering the formula $f(R,T) = R + f(T)$. Under this theory, we have also obtained the solution of EFE by taking into account additional requirements, which are mentioned under model-I $q(t) = 1 - \alpha_1 t/(1+\alpha_2 t)$ and model-II $q(t) = -1 + n\beta/(\beta+t)^2$. The proposed assumption is driven by the fact that our cosmos is now accelerating after a decelerating period, as is detailed on page 7 under Section 4.1. In such a situation, DP must show the flipping nature. It means that it indicates a transitional phase from decelerating to the current accelerating phase. In both the assumptions in sections 4.1 and 4.2, we have obtained the solution by solving the EFE, and the result has already been mentioned as and where required in this section. For greater clarity of the results, we have also investigated other physical and geometrical parameters and presented them under Section 5 along with the physical and geometrical properties of cosmological models.The following are the model's primary characteristics:

- Pictorial representation of deceleration parameter q may be summarized as under in Figure 1 and Figure 3, where it indicates that q is showing the transitional phase of the universe w.r.t time t.

- In Figures 2 and 4, we can observe the variation of the Hubble parameter H concerning cosmic time t. The results indicate that H is a positive decreasing function as time progresses. As $t \to \infty$, the Hubble parameter asymptotically approaches zero.These empirical findings shed light on the dynamics of the universe's expansion and offer significant implications for cosmological investigations.

- Initially i.e., at $t = 0$ the spatial volume (V) was zero and with the passage of time $V \to \infty$ according to equations (47) and (60). This demonstrate that both of our universe models started with a zero volume (the singularity at $t = 0$) and grew over the duration of cosmic time. Therefore, our study is consistent with the generally accepted Big Bang explanation of cosmos.

- Figure 6 and Figure 9 in the reference depict the energy density profile against time t. As can be observed, energy density converges to zero as

$t \to \infty$, as predicted and maintains a positive value throughout the universe's history. We also looked at the pressure p's time-varying behaviour, as seen in Figure 5 and Figure 8. It has been observed that pressure is similarly a positively sloping time function that converges to zero at $t \to \infty$.

- Figures 13 and 12 represent the jerk parameter j. As we know, a positive value indicates an accelerating universe driven by dark energy, while a negative value suggests deceleration due to matter dominance. It is clear from these figures that both the models comply with an accelerating universe, providing valuable insights into the role of dark energy.

- The behavior of the EoS parameter $\omega(t)$ for model-I is depicted in Figure 7. It can be seen that $\omega(t)$ begins out positively before altering its sign, eventually falling into the quintessence region at the moment and the Λ CDM model later. Figure 10 of model-II clearly shows the cause of the universe's transition from decelerating to an accelerating state.

- In Figures 11 and 12, corresponding to model-I and model-II respectively, demonstrate the satisfaction of all the energy conditions (NEC, WEC, and SEC) throughout the evolution of the universe. Additionally, it emphasizes that these graphs are helpful in understanding the properties of matter and energy in the universe and their influence on the cosmic evolution.

9 Concluding Remarks

This research work focuses on the development of the Bianchi-III cosmological model using the $f(R,T)$ theory. A key aspect of our investigation involves incorporating the computation of a FLVDP, which offers valuable insights into the dynamics of the universe. Through meticulous analysis and computation, we investigated profound insight into the dynamics of the cosmos at various phases of its evolution as mentioned above under result and discussion section. Our findings shed light on the interplay between modified gravity theories and the geometry of spacetime, which itself reflects the significance of considering alternative theories in cosmological investigations. The introduction of FLVDP enriched our understanding of the universe's expansion history. It is also evident under investigation that, the $f(R,T)$ theory is augmented by FLVDP, which presents a compelling avenue for unraveling the mystery of cosmic evolution. This work not only contributes to the theoretical framework but also sets the stage for future tests and observational validation.

In addition to above we have also investigated calculated the jerk parameter, as we believe that it is important for deeper and more accurate understanding of the dynamics and evolution of the universe.The jerk parameter reflects how the cosmic scale factor's acceleration changes with cosmic time. Understanding acceleration helps us discern whether the universe expands or contracts. Moreover, it influences essential cosmological parameters, like the equation of state

(EoS) of dark energy and the Hubble constant, which profoundly shape the universe's evolution and structure formation. This knowledge plays a pivotal role in advancing our understanding of the cosmos.

During this research, we thoroughly investigated three essential energy conditions i.e., null energy conditions (NEC), weak energy conditions (WEC), and strong energy conditions (SEC), because we believe they are essential for constructing a physically meaningful and mathematically consistent model of the universe. These conditions provide insight into the nature of matter and energy distribution, the stability of space and time, and the occurrence of singularities.

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