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An Almost Pseudo Symmetric Energy Momentum Spacetime

Ajit Barman*

Department of Mathematics, Ramthakur College, Agartala, Dist-West Tripura, Tripura, India

*Corresponding author E-mail: ajitbarmanaw@yahoo.in

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Abstract. The purpose of the present paper is to study an almost pseudo-symmetric energy-momentum tensor with a perfect fluid spacetime that satisfies Einstein's field equation without cosmological constant. Many geometric features of spacetime with different forms of almost pseudo-symmetric energy-momentum tensors have been studied.

KEY WORDS: Energy momentum tensor, Almost pseudo symmetric energy-momentum tensor, Einstein's field equation, Perfect fluid spacetime, Quasi-Einstein manifold, Levi Civita connection.

1 Introduction

In applied physics, especially in relativity and cosmology, there are many applications of the semi-Riemannian manifold that are part of modern differential geometry. The Lorentzian manifold is a special area of the semi-Riemannian manifold. Signature of the metric of a Lorentzian manifold is $(-,+,+,+,+,+,...,+)$. During the study of the Lorentzian manifold, the causal character of the vector field plays an important role, and thus it becomes a convenient choice for researchers to study relativity and cosmological theory.

In 2008, De and Gazi [1] proposed a kind of non-flat Riemannian manifold that discusses the manifold (M^n, g), $n \geq 2$ whose curvature tensor meets (R_{cur}) of type $(1,3)$ condition

$$\begin{aligned}(\nabla^*_X R_{\text{cur}})(Y,Z)W &= [A(X)+B(X)]R_{\text{cur}}(Y,Z)W \\ &\quad + A(Y)R_{\text{cur}}(X,Z)W + A(Z)R_{\text{cur}}(Y,X)W \\ &\quad + A(W)R_{\text{cur}}(Y,Z)X + g(R_{\text{cur}}(Y,Z)W,X)u_1\,, \quad (1)\end{aligned}$$

where A and B are two non-zero 1-form and u_1 and u_2 are two vector fields defined by $A(X) = g(X,u_1)$ and $B(X) = g(X,u_2)$ respectively for all vector fields X, ∇^* denotes the operator of covariant differentiation with respect to the metric g. This type of manifold was called an almost pseudo-symmetric manifold and was denoted by $(APS)_n$.

A non-flat Lorentzian manifold $(n > 3)$, is called an almost pseudo-Ricci-symmetric manifold if its Ricci tensor (S_{ric}) of type $(0,2)$ is not identically zero and which accepts the following condition:

$$(\nabla^*_X S_{\text{ric}})(Y,Z) = [A(X)+B(X)]S_{\text{ric}}(Y,Z) + A(Y)S_{\text{ric}}(X,Z) + A(Z)S_{\text{ric}}(Y,X)\,. \quad (2)$$

The almost pseudo symmetric manifolds are also studied by Barman [2, 3], De and Barman [4], De and Gazi [5] and many others.

The General relativity given by Einstein's equation [6] is the lower form of flow as follows :

$$S_{\text{ric}}(X,Y) - \frac{1}{2}r^* g(X,Y) + \lambda^* g(X,Y) = \kappa^* T_{\text{energy}}(X,Y)\,, \quad (3)$$

where r^* is the scalar curvature, $T_{\text{energy}}(X,Y)$ is the energy-momentum tensor of type $(0,2)$, λ^* is the cosmological constant and κ^* is the gravitational constant. Einstein's gave the equation without the cosmological constant as follows:

$$S_{\text{ric}}(X,Y) - \frac{1}{2}r^* g(X,Y) = \kappa^* T_{\text{energy}}(X,Y)\,. \quad (4)$$

From Einstein's equations (3) and (4), we conclude that "matter determines the geometry of spacetimes and conversely that the motion of matter is determined by the metric tensor of the space which is not flat" [7].

The energy-momentum tensor describes a Perfect fluid [6] if

$$T_{\text{energy}}(X,Y) = (\sigma^* + p^*)A(X)A(Y) + p^* g(X,Y)\,, \quad (5)$$

where σ^* is the energy density and p^* is the isotropic pressure of the fluid.

A memory carries many authors on the energy-momentum tensors of a semi-Riemannian manifold (M^n, g) have been studied by Barman [8, 9], Ozen [10], Guler and Altay [11], Mallick, Suh and De [12, 13] and many others.

Inspired by the above research paper, the present paper is organised as follows: After the introduction, in Section 2, we study for which condition an almost pseudo-symmetric energy momentum spacetime will be an almost Ricci-symmetric spacetime and the 4-dimensional matter distribution of the perfect fluid. Section 3 and Section 4 deal with Codazzi type and Quadratic killing energy-momentum tensors on the almost pseudo-symmetric energy-momentum spacetimes, respectively. Finally, we characterise recurrent energy momentum tensor on an almost pseudo-symmetric energy momentum spacetime.

2 An Almost Pseudo Symmetric Energy-Momentum Spacetime

Definition:- A non-flat Lorentzian manifold will be an almost pseudo-symmetric energy-momentum spacetime if its energy-momentum tensor (T_{energy}) type $(0,2)$ is not identically zero and it accepts the following conditions:

$$(\nabla^*_X T_{\text{energy}})(Y,Z) = [A(X)+B(X)]T_{\text{energy}}(Y,Z) + A(Y)T_{\text{energy}}(X,Z) + A(Z)T_{\text{energy}}(Y,X)\,. \quad (6)$$

Theorem :- An almost pseudo-symmetric energy momentum spacetime will be an almost Ricci-symmetric spacetime such that the scalar curvature vanishes.

Proof. From the equation (4), we can write that

$$(\nabla^*_X S_{\text{ric}})(Y,Z) - \frac{1}{2}dr^*(X)g(Y,Z) = \kappa^*(\nabla^*_X T_{\text{energy}})(Y,Z)\,. \quad (7)$$

Combining equations (4), (6) and (7), we get that

$$(\nabla^*_X S_{\text{ric}})(Y,Z) = \{A(X)+B(X)\}S_{\text{ric}}(Y,Z) + A(Y)S_{\text{ric}}(X,Z) + A(Z)S_{\text{ric}}(Y,X) - \frac{r^*}{2}\{\{A(X)+B(X)\}g(Y,Z) + A(Y)g(X,Z) + A(Z)g(Y,X)\} + \frac{1}{2}dr^*(X)g(Y,Z)\,. \quad (8)$$

If $r^* = 0$, then we see that the theorem is proven. □

Definition:- A non-flat spacetime is defined to be a quasi Einstein spacetime [14] if its Ricci tensor (S_{ric}) is not identically zero and satisfies

$$S_{\text{ric}}(X,Y) = ag(X,Y) + bA(X)A(Y),$$

where a and b are scalars.

Theorem:- An almost pseudo symmetric energy momentum spacetime satisfying the 4-dimensional matter distribution of the perfect fluid will be a quasi Einstein spacetime.

Proof. When we combine two equations (4) and (5), it follows that

$$S_{\text{ric}}(Y,Z) = (\frac{r^*}{2} + \kappa^* p^*)g(Y,Z) + \kappa^*(\sigma^* + p^*)A(Y)A(Z). \quad (9)$$

Putting $Z = u_1$ in (9), we can write that

$$S_{\text{ric}}(Y,u_1) = [(\frac{r^*}{2} + \kappa^* p^*) + \kappa^*(\sigma^* + p^*)A(u_1)]A(Y) = JA(Y), \quad (10)$$

where $J = (\frac{r^*}{2} + \kappa^* p^*) + \kappa^*(\sigma^* + p^*)A(u_1)$.
Since matter tensor and Einstein's field equation are divergence-free [15], that means,

$$(\nabla^*_X T_{\text{energy}})(Y, Z) = 0. \tag{11}$$

With the help of equations (4), (6) and (11), we obtain that

$$\begin{aligned}[A(X) + B(X)][S_{\text{ric}}(Y, Z) - \frac{1}{2}r^* g(Y, Z)] + A(Y)[S_{\text{ric}}(X, Z) \\ -\frac{1}{2}r^* g(X, Z)] + A(Z)[S_{\text{ric}}(Y, X) - \frac{1}{2}r^* g(Y, X)] = 0.\end{aligned} \tag{12}$$

Putting $X = u_1$ in (12) and using the equation (10), we have

$$S_{\text{ric}}(Y, Z) = \frac{r^*}{2} g(Y, Z) + [-2J + r^*]A(Y)A(Z).$$

That means, the almost pseudo symmetric manifold on the 4-dimensional matter distribution of perfect fluid is a quasi Einstein manifold. The proof of the theorem is completed. □

3 Codazzi Type Tensor on Almost Pseudo Symmetric Energy Momentum Spacetimes

Definition:- A energy momentum tensor (T_{energy}) on a semi-Riemannian spacetime is said to be Codazzi type [16] energy momentum tensor if it satisfies the condition

$$(\nabla^*_X T_{\text{energy}})(Y, Z) = (\nabla^*_Y T_{\text{energy}})(X, Z). \tag{13}$$

Definition:- A non-flat semi-Riemannian spacetime is called a nearly quasi-Einstein spacetime [17] if the Ricci tensor (S_{ric}) of type $(0, 2)$ is not identically zero and satisfies the condition

$$S_{\text{ric}}(X, Y) = a_1 g(X, Y) + b_1 E(X, Y),$$

where a_1 and b_1 are non-zero scalars and $E(X, Y)$ is a non-zero symmetric tensor of type $(0, 2)$.

Theorem:- An almost pseudo-symmetric spacetime satisfying Codazzi type of perfect fluid energy-momentum tensor is a nearly quasi-Einstein spacetime.

Proof. Using equations (6) and (13), we can conclude that

$$\begin{aligned}B(X)S_{\text{ric}}(Y, Z) - B(Y)S_{\text{ric}}(X, Z) - \frac{1}{2}r^*[B(X)g(Y, Z) \\ -B(Y)g(X, Z)] = 0.\end{aligned} \tag{14}$$

Putting $X = u_1$ in (14) and using (10), it follows that

$$S_{\text{ric}}(Y, Z) = \frac{1}{2}r^* g(Y, Z) + \frac{2J - r^*}{2B(u_1)} B(Y)A(Z). \tag{15}$$

Interchanging Y and Z by Z and Y in equation (15), we obtain that

$$S_{\text{ric}}(Y, Z) = \frac{1}{2}r^* g(Y, Z) + \frac{2J - r^*}{2B(u_1)} B(Z)A(Y). \tag{16}$$

Adding equations (15) and (16), we can write that

$$S_{\text{ric}}(Y, Z) = \frac{1}{2}r^* g(Y, Z) + \frac{2J - r^*}{4B(u_1)} [B(Y)A(Z) + B(Z)A(Y)]. \tag{17}$$

Therefore, the second part on the right-hand side of Equation (17) is symmetric. That means, an almost pseudo-symmetric spacetime with a Codazzi type energy-momentum tensor is a nearly quasi-Einstein spacetime. The proof of this theorem ends here.

□

4 Quadratic Killing on an Almost Pseudo Symmetric Energy-Momentum Spacetime

Definition:- An almost pseudo symmetric spacetime whose energy-momentum tensor is quadratic Killing if it satisfies the condition

$$(\nabla^*_X T_{\text{energy}})(Y, Z) + (\nabla^*_Y T_{\text{energy}})(X, Z) + (\nabla^*_Z T_{\text{energy}})(X, Y) = 0. \tag{18}$$

Theorem:- If an almost pseudo-symmetric spacetime whose energy-momentum tensor is quadratic killing, then the spacetime is a nearly quasi-Einstein spacetime.

Proof. From equations (6), (4) and (18), we get

$$\begin{aligned} &\{3A(X) + B(X)\}\{S_{\text{ric}}(Y, Z) - \frac{r^*}{2}g(Y, Z)\} + \{3A(Y) + B(Y)\} \\ &\quad \{S_{\text{ric}}(X, Z) - \frac{r^*}{2}g(X, Z)\} + \{3A(Z) + B(Z)\}\{S_{\text{ric}}(X, Y) \\ &\quad - \frac{r^*}{2}g(X, Y)\} = 0\,. \end{aligned} \tag{19}$$

Putting $X = u_1$ in (19), we declared that

$$\begin{aligned} &\{3A(u_1) + B(u_1)\}\{S_{\text{ric}}(Y, Z) - \frac{r^*}{2}g(Y, Z)\} + \{3A(Y) + B(Y)\} \\ &\quad \{S_{\text{ric}}(u_1, Z) - \frac{r^*}{2}g(u_1, Z)\} + \{3A(Z) + B(Z)\}\{S_{\text{ric}}(u_1, Y) \\ &\quad - \frac{r^*}{2}g(u_1, Y)\} = 0. \end{aligned} \tag{20}$$

Equations (10) and (20) can be written by arranging that

$$S_{\text{ric}}(Y,Z) = \frac{r^*}{2\{3A(u_1)+B(u_1)\}}g(Y,Z) + \frac{r^*-2J}{\{3A(u_1)+B(u_1)\}}A(Y)A(Z) + \frac{r^*-2J}{2\{3A(u_1)+B(u_1)\}}[A(Y)B(Z)+A(Z)B(Y)]\,. \quad (21)$$

simplify equation (21), we get that

$$S_{\text{ric}}(Y,Z) = \frac{r^*}{2\{3A(u_1)+B(u_1)\}}g(Y,Z) + \frac{r^*-2J}{2\{3A(u_1)+B(u_1)\}}E_1(Y,Z)\,,$$

where $E_1(Y,Z) = 2A(Y)A(Z) + [A(Y)B(Z) + A(Z)B(Y)]$ and $E_1(Y,Z)$ is symmetric. That means, an almost pseudo-symmetric spacetime with a quadratic killing energy-momentum tensor is a nearly quasi-Einstein spacetime. Therefore, the theorem is proven.

□

5 Recurrent Energy Momentum Tensor on an Almost Pseudo-Symmetric Energy Momentum Spacetime

Definition:- A spacetime of dimension n is called an energy momentum-recurrent spacetime if

$$(\nabla^*_X T_{\text{energy}})(Y,Z) = A(X)T_{\text{energy}}(Y,Z)\,. \quad (22)$$

Theorem:- If an almost pseudo-symmetric energy momentum spacetime with perfect fluid satisfying recurrent energy-momentum tensor, then spacetime is a quasi-Einstein spacetime.

Proof. From equations (6) and (22), we get

$$B(X)T_{\text{energy}}(Y,Z) + A(Y)T_{\text{energy}}(X,Z) + A(Z)T_{\text{energy}}(Y,X) = 0\,. \quad (23)$$

Combining equations (4) and (23), we obtain that

$$B(X)[S_{\text{ric}}(Y,Z) - \frac{r^*}{2}g(Y,Z)] + A(Y)[S_{\text{ric}}(X,Z) - \frac{r^*}{2}g(X,Z)] + A(Z)[S_{\text{ric}}(X,Y) - \frac{r^*}{2}g(X,Y)] = 0\,. \quad (24)$$

Putting $X = u_1$ in (24) and using equation (10), we have

$$S_{\text{ric}}(Y,Z) = \frac{r^*}{2}g(Y,Z) + \frac{r^*-2J}{2B(u_1)}A(Y)A(Z). \quad (25)$$

That means equation (25) is a quasi-Einstein spacetime. The proof of the theorem is completed. □

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