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Landau Levels and Newton-Hooke Dualities

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Abstract. We review the Landau levels of an electron moving in a constant magnetic field transversal to a plane. In the spirit of the work of Gerald Dunne the Landau levels are lifted on a sphere via a stereographic projection and one recovers the Haldane spherical geometry of radial magnetic field originating at a Dirac monopole with a magnetic charge g. The Hilbert space of states living on the Haldane sphere are expressed with the monopole harmonics and in the limit of large magnetic charge one recovers the planar Landau levels. We push the analogy between the 2D harmonic oscillator and the planar Landau problem further and propose the 4D harmonic oscillator as a lift of the Haldane spherical geometry. We employ the Newton-Hooke duality between Coulomb problem and harmonic oscillator to introduce the MICZ-Kepler problem, *i.e.,* the quantum Coulomb-Kepler problem for a motion of an electron in the background field of a dyon (a particle having both electric and magnetic charge). We conclude that MICZ-Kepler model is a natural generalization of the Haldane spherical geometry for the Landau levels.

KEY WORDS: Landau levels, Magnetic monopoles, Monopole harmonics.

1 Introduction

The duality between the motion of a particle in a Newtonian potential $-1/r$ and the harmonic oscillator with Hooke's potential r^2 can be traced back to Principia [4]. We will refer to this duality as Newton-Hooke duality. R.W. Hamilton found out that the hodograph (*i.e.,* the trajectory in the space of velocities) of a Keplerian elliptic orbit is a perfect circle [13]. The hodographs turn out to be great circles of spheres S^3 in the momentum space. The classical geodesic motion on the sphere is amenable to the harmonic oscillator in $\mathbb{R}^4$ which is a manifestation of the Newton-Hooke duality. Vladimir Fock considered the quantum Kepler problem (*i.e.,* hydrogen atom) [10] and compactified the 3D momentum space into a three dimensional sphere S^3. Thus he implicitly explored a quantum version of a Newton-Hooke duality by transforming the Schrödinger equation with Coulomb potential to a Laplace equation on S^3. The classical cyclotronic orbits of an electron in a magnetic field can be thought as great circles on a sphere S^2 or alternatively as hodographs of regularized Coulomb problem [6].

The Hamiltonian for a charged particle in a magnetic field is given by

$$H = \frac{\left(-i\hbar\nabla - \frac{e}{c}\mathbf{A}\right)^2}{2m} , \qquad L_z = -i\hbar \left(\boldsymbol{r} \times \boldsymbol{\nabla}\right) . \tag{1}$$

When we fix the symmetric gauge $\mathbf{A} = B(-y, x)/2$ in the plane, we get an $SO(2)$ invariant Landau Hamiltonian H_L. It differs from the isotropic 2D harmonic oscillator hamiltonian H_{osc} by a term proportional to the angular momentum L_z (spin). The mutually commuting operators H_L, L_z and $H_{osc} = H_L + \Omega L_z$ have common space of eigenstates. We denoted by $\mathbf{Dirac}$ the common Hilbert space of the Landau hamiltonian H_L and the oscillator hamiltonian H_{osc}, having different energy spectra [6] (with Landau levels being infinitely degenerated). We apply the Hooke-Newton duality to the states of the 2D harmonic oscillator and get a fresh insight on the planar Landau problem as a dual of a quantum Coulomb-Kepler problem [5, 6]. The Hilbert space $\mathbf{Dirac}$ is a reducible $\mathfrak{so}(2, 3)$ module for Dirac's remarkable representation of the anti-de Sitter (AdS) group $SO(2, 3)$ [7]. The Bohlin transform incarnates the Newton–Hooke duality [21] between the harmonic oscillator (Landau model) and the quantum Coulomb–Kepler models (with/without magnetic charge):

$$\mathbf{Di} \oplus \mathbf{Rac} \xleftrightarrow{\text{Hooke-Newton}} \{\text{2D dyon-charge system}\} \oplus \{\text{2D hydrogen atom}\} .$$

The spectrum generating group of 2D harmonic oscillator representation $\mathbf{Dirac}$ is the group $Sp(4, \mathbb{R})$ which is the double-covering of the AdS group $SO(2, 3)$. The Landau levels space $\mathbf{Dirac}$ decomposes into two $\mathfrak{so}(2, 3)$-submodules [6]:

(i) $\mathbf{Di} = D(1, 1/2)$, the Hooke-Newton dual quantum system of an electron and a charged magnetic vortex (dyon with helicity $s = \frac{1}{2}$) [21]; (odd spin L_z) ,

(ii) $\mathbf{Rac} = D(1/2, 0)$, the Hooke-Newton dual of 2D hydrogen atom (with helicity $s = 0$) [3]; (even spin L_z) .

The unitary massless conformal $\mathfrak{so}(2, 3)$ represenations $D(E_0, s)$ are classified by the minimal *conformal energy* E_0 and the *helicity* s, with the notations of the seminal work *Massless particles, conformal group, and de Sitter universe* [1]. The conformal energy E turns out to be the eigenvalue of the harmonic oscillator H_{osc}, whereas the helicity s is an intrinsic spin that is related to the minimal value of the angular momentum L_z in the plane.

The present work draws a parallel between the planar Landau problem and the spherical geometry of Haldane [12] via the harmonic oscillators in $\mathbb{R}^4$. The Hooke-Newton dual of the 4D harmonic oscillator has two components:

(i) 3D hydrogen atom (states with integer spin $j = l + s$), (ii) 3D dyogen atom (states with half-integer spin $j = l + s$).

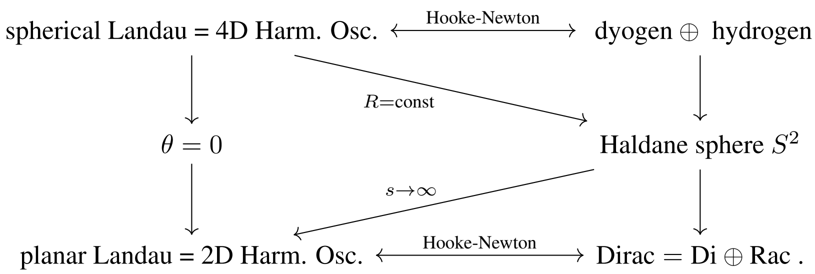
The dyogen atom [25] is a poetic name for the quantum MICZ-Kepler problem: the binary system of a particle with a charge e moving in the field of a dyon (with an electric charge $-e$ and a magnetic charge g) [18,29]. Conformally regularized dyogen atom is then dual to the 4D isotropic harmonic oscillator [25].

The energy spectrum of hydrogen and dyogen atom have accidental degeneracy explained by the hidden symmetry $SO(4)$ [2]. The eigenstates have in common the radial part $R_{Nj}(\rho)$ written in terms of the associated Laguerre polynomials

$$\Psi_{N,j,m,s} \propto e^{-\rho} \rho^j L^{2j+1}_{N-j-1}(2\rho)\ {}^sY^m_j(\varphi, \theta, \gamma) \, . \tag{2}$$

The angular part is given by monopole harmonics ${}^sY^m_j$ [8, 28]. For $s = 0$ the monopole harmonics coincide with the spherical harmonics, ${}^0Y^m_j = Y^m_j$ and we obtain the eigenstates of the hydrogen atom.

In his study of the Fractional Quantum Hall effect Haldane [12] proposed a compact spherical geometry for the electron gas by compactifying the plane to a sphere. We show that the states on the Haldane sphere S^2 can be deduced from the 4D harmonic oscillator by suppressing its radial variable via the Hopf fibration $SU(2)/U(1)$. Haldane's spherical model allows for a planar limit when the magnetic flux grows to infinity. Our construction is summarized by the table



The 4D isotropic oscillator has the spectrum generating group $U(2, 2)$ [17]. It interpolates between the states of the conformal regularized MICZ-Kepler models. In view of the isomorphism $SU(2, 2)/\mathbb{Z}_2 \cong SO(2, 4)$ the 4D oscillator Hilbert space is a reducible $SO(2, 4)$-module with two orbits; hydrogen atom (even spins with vacuum $s = 0$) and dyogen atom (odd spins with ground state $s = \frac{1}{2}$). The angular momentum $\mathcal{L}_i$ generate the isometries of the sphere $SU(2) \cong S^3$; the rotations are the counterparts of the magnetic translation operator $b^{\pm}$ in the plane. In the same vein, as explained by Greiter [11], one can lift the energy creation and annihilation operators $a^{\pm}$ to the helicity lowering and raising operators $\mathcal{S}_{\mp}$. Equivalently, Hasebe [14] understood that $\mathcal{S}_{\mp}$ are the Edth operators in disguise, changing the helicity number s of the monopole harmonics.

We finally propose to employ the full-fledged 4D isotropic oscillator with no radial reduction as a model for the Landau levels. Instead of one Haldane sphere in the coordinate space we propose a tower of spheres in the momentum space parametrized by the helicity s.

2 Landau Levels in the Plane

An electron in a constant uniform magnetic field $\boldsymbol{B}$ is propagating in a helix having as axis the direction of the field. In the center of mass frame the classical motion of an electron is on a circle in a plane perpendicular to the magnetic field $\boldsymbol{B}$ with a cyclotronic frequency $\omega_c = \frac{eB}{mc}$. Under the symmetric gauge $\mathbf{A} = \frac{B}{2}(-y, x)$, we get a constant magnetic field along the z-axis $\mathbf{B} = (0, 0, B)$ transversal to the plane. Due to the rotational symmetry the angular momentum $\boldsymbol{L} = \boldsymbol{r} \times \boldsymbol{p}$ is transversal to the plane of motion and it is a constant of motion $\boldsymbol{L} = (0, 0, L_z)$. With the help of the magnetic length $\ell = \sqrt{\frac{\hbar c}{eB}}$ we introduce dimensionless holomorphic coordinates in the plane

$$z = \frac{x + iy}{2\ell}, \qquad \partial = \frac{\partial}{\partial z} = \ell \left(\frac{\partial}{\partial x} - i \frac{\partial}{\partial y} \right) = \frac{\ell}{\hbar} \left(p_y + i p_x \right) \ .$$

The Landau Hamiltonian, eq. (1) in symmetric gauge and the angular momentum operator in the new coordinates take the form

$$H_L = \frac{\hbar \omega_c}{2} \left(z\bar{z} - \partial\bar{\partial} - z\partial + \bar{z}\bar{\partial} \right) \ , \quad L_z = \hbar \left(z\partial - \bar{z}\bar{\partial} \right) \ . \tag{3}$$

The operators H and L_z commute, hence can be simultaneously diagonalized

$$H\psi_{n,m} = \hbar\omega_c \left(n + \frac{1}{2} \right) \psi_{n,m} \ , \quad L_z \psi_{n,m} = \hbar m \psi_{n,m} \ , \quad n \geq 0, \quad m \geq -n \ . \tag{4}$$

States with the same quantum number n have the same energy and form the n-th Landau level. Each Landau level is infinitely degenerated.

It is worth noting that the harmonic oscillator Hamiltonian H_{osc} differs from H_L by a factor of L_z and has finite degeneracy

$$H_{osc} = H_L + \frac{\omega_c}{2} L_z \ , \qquad H_{osc}\psi_{n,m} = \hbar\omega_c \left(n + \frac{m}{2} \right) \psi_{n,m} \ .$$

The 2D harmonic oscillator is also rotationally invariant, $[H_{osc}, L_z] = 0$. Hence it shares a common Hilbert space with the Landau problem and the difference resides in the energy eigenvalues.

We define energy and magnetic translation creation and annihilation operators

$$\begin{aligned} a^- &= \frac{1}{\sqrt{2}}(z + \bar{\partial}) \, ; \qquad & b^- &= \frac{1}{\sqrt{2}}(\bar{z} + \partial) \, , \\ a^+ &= \frac{1}{\sqrt{2}}(\bar{z} - \partial) \, ; \qquad & b^+ &= \frac{1}{\sqrt{2}}(z - \bar{\partial}) \, . \end{aligned} \tag{5}$$

In symmetric gauge $a^{\pm}$ and $b^{\pm}$ generate two mutually commuting Heisenberg algebras

$$[a^-, a^+] = \mathbb{1} \;, \qquad [b^-, b^+] = \mathbb{1} \;.$$

Using our complex operators defined above, we can rewrite the operators as

$$H_L = \hbar\omega_c \left(a^+ a^- + \frac{1}{2}\right) \;, \qquad L_z \quad = \hbar \left(b^+ b^- - a^+ a^-\right) \;. \tag{6}$$

In view of the commutation relations, $a^{\pm}$ are energy creation/annihilation operators, whereas $b^{\pm}$ are raising/lowering operators for the angular momentum.

$$[H_L, a^{\pm}] = \pm\hbar\omega_c a^{\pm} \;, \qquad [L_z, b^{\pm}] = \pm\hbar b^{\pm} \;.$$

The operators $b^{\pm}$ are zero modes of the Hamiltonian

$$[H_L, b_{\pm}] = 0 \;,$$

hence these are referred to as magnetic translations.

The scalar product on the Hilbert space is chosen to be

$$(\psi, \chi) = \int dx \wedge dy \, \psi^*(x,y)\chi(x,y) = 4\ell^2 \int dz \wedge d\bar{z} \, \psi^*(z,\bar{z})\chi(z,\bar{z}) \;.$$

We denote the Hilbert space [6] as

$$\mathbf{Dirac} = \bigoplus_{n\geq 0} \mathbf{Dirac}_n = \bigoplus_{n\geq 0} \; \bigoplus_{m\geq -n} \psi_{n,m}(z,\bar{z}) \;. \tag{7}$$

The ground state $\psi_{0,0}(z,\bar{z})$ in the lowest Landau level is a Gaussian function with the distribution $|\psi_{0,0}|^2$ and a standard deviation equal to the magnetic length ℓ:

$$\psi_{0,0}(z,\bar{z}) = \frac{1}{\ell\sqrt{2\pi}} e^{-z\bar{z}} = \frac{1}{\ell\sqrt{2\pi}} e^{-\frac{x^2+y^2}{4\ell^2}} \;. \tag{8}$$

Any state in the lowest Landau level $n = 0$ (LLL) is represented by an arbitrary holomorphic function multiplying the ground state $f(z)e^{-z\bar{z}}$. The basis of the LLL, or $\mathbf{Dirac}_0$, is provided by the holomorphic monomials:

$$\psi_{0,0}(z,\bar{z}) = (m!)^{-\frac{1}{2}}(b^+)^m \psi_{0,0}(z,\bar{z}) = z^m e^{-z\bar{z}}/\sqrt{2^m m! 2\pi\ell^2} \;,$$

since the application of the "zero modes" operators b^+ does not change the energy level n. The lowest Landau level is a Bargmann space. The higher Landau level $\mathbf{Dirac}_n$ is another Bargmann space , built on a higher vacuum state

$$\psi_{n,-n} = \frac{1}{\sqrt{2\pi\ell^2 2^n n!}} \bar{z}^n e^{-z\bar{z}} \qquad \mathbf{Dirac}_n = \bigoplus_{m\geq 0} \mathbb{C}(b^+)^m \psi_{n,-n} \;.$$

The higher vacuum state of the Landau level $\mathbf{Dirac}_n$ is created by the vacuum state in the LLL via the n-fold application of the energy creation operator

$$\psi_{n,-n} = (a^+)^n \psi_{0,0}/\sqrt{n!}\,. \tag{9}$$

Any state in the Hilbert space can be generated from the ground state via the creation operators [6]

$$\psi_{n,m}(z,\bar{z}) = \frac{(a^+)^n}{\sqrt{n!}} \frac{(b^+)^{n+m}}{\sqrt{(n+m)!}} \psi_{0,0}$$

$$\psi_{n,m}(z,\bar{z}) = c_{n,m} e^{-|z|^2} \begin{cases} z^m L_n^m(2|z|^2), & m \geq 0 \\ \bar{z}^{|m|} L_{n-|m|}^{|m|}(2|z|^2), & -n \leq m < 0 \end{cases}$$

where L_n^m denotes associated Laguerre polynomials, n is the Landau energy level index, and m is the angular momentum eigenvalue.

3 Haldane Sphere

In his seminal work [12] Duncan Haldane proposed to rewrite the planar Landau problem on the surface of a two-sphere with transversal magnetic field to explain some features of the quantum Hall effect [12]. He assumed a constant radial magnetic field

$$\mathbf{B} = g\frac{\hat{\mathbf{r}}}{r^2} = \nabla \times \boldsymbol{\mathcal{A}}\,, \qquad \boldsymbol{\mathcal{A}}(x,y,z) = \frac{g}{r(r+z)}(-y,x,0)\,, \tag{10}$$

produced by a Dirac magnetic monopole with magnetic charge g, placed at the center of the coordinate system. To better understand the origin of Haldane's spherical geometry one can follow the approach of Gerald Dunne [9] who starts with Pauli hamiltonian $\mathcal{H}$ for with the quantum motion of an electron in the background field of a Dirac monopole. The magnetic flux through a sphere of radius R is obtained by dividing the total flux by the quant of a magnetic flux

$$\frac{\Phi}{\Phi_0} = \frac{4\pi R^2 B}{2\pi\hbar c/e} = \frac{2R^2}{\ell^2} = 2eg/\hbar c = 2s \in \mathbb{Z}\,.$$

The quantization of the magnetic flux then follows from Dirac's quantization condition. We introduce the dimensionless variable $\rho = R/\ell$ and the flux condition yields the equation of a sphere

$$\rho^2 = s\,, \qquad 2s \in \mathbb{Z}\,.$$

Dunne then restricts the motion to a sphere of radius $R = \sqrt{s}\ell$

$$\mathcal{H}_s = \frac{1}{2m}\left(-i\hbar\nabla - \frac{e}{c}\boldsymbol{\mathcal{A}}\right)^2\Bigg|_{r=R}\,. \tag{11}$$

The restriction to a sphere yields the centrifugal Hamiltonian proposed by Haldane in ad hoc manner [12] through the dynamical angular momentum

$$\mathcal{H}_s = \frac{\Lambda^2}{2\mu R^2} = \frac{\omega_c}{2}\frac{\Lambda^2}{\hbar s}\,, \qquad \Lambda = \boldsymbol{r} \times (-i\hbar\nabla - \frac{e}{c}\boldsymbol{\mathcal{A}})\,. \tag{12}$$

We can use a stereographic projection to transfer wave functions from the plane to the sphere of radius $\rho = \sqrt{s}$

$$z = \rho e^{i\varphi} \tan\frac{\theta}{2}\,, \qquad \rho = \frac{R}{\ell}. \tag{13}$$

The Haldane centrifugal Hamiltonian (12) in the new coordinates reads

$$\mathcal{H}_s = \frac{\hbar\omega_c}{2}\left[z\bar{z} - \left(1 + \frac{|z|^2}{s}\right)^2 \partial\bar{\partial} - \left(1 + \frac{|z|^2}{s}\right)(z\partial - \bar{z}\bar{\partial})\right]\,. \tag{14}$$

We can extract a measure factor from the wavefunctions of the Haldane Hamiltonian $\mathcal{H}_s$ written in complex coordinates (14)

$$\Psi(z,\bar{z}) = \frac{1}{\left(1 + \frac{|z|^2}{s}\right)^s}\,\mathring{\Psi}(z,\bar{z})\,. \tag{15}$$

The reduced wavefunctions are given with Jacobi polynomials [9]

$${}^s\mathring{\Psi}_l^m = {}^g c_n^m\, z^m P_l^{(m,2s-m)}\left(\frac{1 - \frac{|z|^2}{s}}{1 + \frac{|z|^2}{s}}\right)\,. \tag{16}$$

Algebraic approach. The dynamical angular momentum $\boldsymbol{\Lambda}$ of a charged particle propagating in the background field of a magnetic monopole is not a conserved quantity. Moreover, the commutation relations of the components of $\boldsymbol{\Lambda}$ do not close an algebra

$$[\Lambda_a, \Lambda_b] = i\hbar\epsilon_{abc}(\Lambda_c + \hbar s\, x_c/r).$$

The conserved quantity is a corrected angular momentum $\boldsymbol{L}$ as Poincaré showed in 1896 [24]

$$\boldsymbol{L} = \boldsymbol{\Lambda} - \hbar s\hat{\boldsymbol{r}}\,, \qquad \hat{\boldsymbol{r}} := \frac{\boldsymbol{r}}{r}\,.$$

The correction is the contribution to the angular momentum of the electromagnetic field of the magnetic monopole via the Poynting vector. The now conserved angular momentum generates an $\mathfrak{su}(2)$ algebra

$$[L_a, L_b] = i\hbar\epsilon_{abc}L_c\,, \qquad \boldsymbol{L}^2 = \boldsymbol{\Lambda}^2 + \hbar^2 s^2\,,$$

whose Casimir operator $\boldsymbol{L}^2$ is simply related to the square of the dynamical angular momentum Λ^2 in view of the condition $\boldsymbol{r}\cdot\boldsymbol{\Lambda}=0$. It is convenient to introduce raising and lowering angular momentum operators $L_\pm = L_1 \pm iL_2$

$$[L_+, L_-] = 2\hbar L_3, \qquad [L_3, L_\pm] = \pm\hbar L_\pm \ .$$

In the complex coordinates the generators of the $\mathfrak{su}(2)$ algebra take the form

$$\begin{aligned}
\frac{L_+}{\hbar} &= \frac{1}{\sqrt{s}} z^2\partial + \sqrt{s}\bar{\partial} - \sqrt{s} z = \frac{1}{\sqrt{s}} z^2\partial - \sqrt{2s} b^+ \\
\frac{L_-}{\hbar} &= -\frac{1}{\sqrt{s}} \bar{z}^2\bar{\partial} - \sqrt{s}\partial - \sqrt{s}\bar{z} = -\frac{1}{\sqrt{s}} \bar{z}^2\bar{\partial} - \sqrt{2s} b^- \\
\frac{L_3}{\hbar} &= z\partial - \bar{z}\bar{\partial} - s = \frac{L_z}{\hbar} - s \ .
\end{aligned} \tag{17}$$

Finally the Haldane centrifugal potential (12) reads

$$\mathcal{H}_s = \frac{\omega_c}{2}\frac{\Lambda^2}{\hbar s} = \frac{\omega_c}{2}\frac{\boldsymbol{L}^2 - \hbar^2 s^2}{\hbar s} \ .$$

On an irreducible $\mathfrak{su}(2)$-representation of spin $j = l + s$ the Casimir operator takes the constant value $\boldsymbol{L}^2 = \hbar^2 j(j+1)$ hence

$$\mathcal{H}_s = \frac{\hbar\omega_c}{2}\frac{\hbar^2 j(j+1) - \hbar^2 s^2}{\hbar^2 s} = \hbar\omega_c \left\{ \left(l + \frac{1}{2}\right) + \frac{l(l+1)}{2s} \right\}$$

where the orbital quantum number l indexes the Landau levels on the sphere. In the limit of large flux $2s$ grows indefinitely, suppressing the second term, and we get the equidistant levels of the harmonic oscillator.

The Hilbert space of the wavefunctions on the sphere $\rho^2 = s$ should carry a structure of a representation of the $\mathfrak{su}(2)$-algebra, eq. (17) and can be decomposed to irreducible components. Any irreducible $\mathfrak{su}(2)$-representation is the span

$$V_l^{(s)} = \bigoplus_{m=-l}^{l+2s} \mathbb{C}\ {}^s\Psi_l^m \ , \qquad \dim V_l^{(s)} = 2(l+s)+1 = 2j+1 \ ,$$

where the indices correspond to the Casimir operator and the plane angular momentum projection $L_z = L_3 + \hbar s$

$$\begin{aligned}
\boldsymbol{L}^2\ {}^s\Psi_l^m &= \hbar^2(l+s)(l+s+1)\ {}^s\Psi_l^m \ , & l &\geq 0 \ , \\
L_z\ {}^s\Psi_l^m &= \hbar m\ {}^s\Psi_l^m \ , & -l &\leq m \leq l+2s \ .
\end{aligned}$$

We can then traverse the momentum ladder by applying $L_\pm$ starting from a lowest or a highest angular momentum state

$$L_-\ {}^s\Psi_l^{m=-l} = 0 \ , \qquad L_+\ {}^s\Psi_l^{m=l+2s} = 0 \ .$$

These states are given by

$$ {}^{s}\Psi_{l}^{m=-l} \propto \frac{\bar{z}^{l}}{\left(1+\frac{|z|^2}{s}\right)^{l+s}} , \qquad {}^{s}\Psi_{l}^{m=l+2s} \propto \frac{z^{l+2s}}{\left(1+\frac{|z|^2}{s}\right)^{l+s}} . \tag{18} $$

Comparing the spectrum of the angluar momentum eigenfunctions ${}^{s}\Psi_{l}^{m}$ with the defining relations of the monopole harmonics [8, 28]

$$ \begin{aligned} \boldsymbol{L}^2 \ {}^{s}Y_{j}^{m'} &= \hbar^2 j(j+1) \ {}^{s}Y_{j}^{m'} , & j = l+s \\ L_3 \ {}^{s}Y_{j}^{m'} &= \hbar m' \ {}^{s}Y_{j}^{m'} , & -j \le m' \le j \\ L_{\pm} {}^{s}Y_{j}^{m'} &= \hbar\sqrt{(j\mp m')(j+1\pm m')} \ {}^{s}Y_{j}^{m'\pm 1} , & \end{aligned} \tag{19} $$

we conclude that via a shift $m' = m - s$ the wavefunctions on the Haldane sphere can be related to the monopole harmonics by

$$ {}^{-s}Y_{j}^{m'}(\theta,\phi) \propto {}^{s}\Psi_{l=j-s}^{m'+s}(z,\bar{z}) . \tag{20} $$

4 Planar Limit of the Haldane Sphere

We can recover the planar wavefunctions as a limit $R \to \infty$ from the wavefunctions on the Haldane sphere. The spherical surface near the North pole will be indistinguishable from a plane. Since the magnetic field $B = g/R^2$ (10) should be maintained constant in the planar Landau problem, we have to require that the magnetic flux $g \to \infty$ along with the radius $R \to \infty$ such that $g/R^2 = const$. Under this limit $s \to \infty$ the Haldane wavefunctions reduce to their planar counterparts

$$ \lim_{s\to\infty} \frac{1}{\left(1+\frac{|z|}{s}^2\right)^{s}} = e^{-|z|^2} , $$

$$ \lim_{s\to\infty} z^m P_n^{(m,2s-m)}\left(\frac{1-\frac{|z|}{s}^2}{1+\frac{|z|}{s}^2}\right) = z^m L_n^m(2|z|^2) , $$

where we have used the limiting relationship that connects Jacobi and Laguerre polynomials [9],

$$ \lim_{\beta\to\infty} P_n^{(\alpha,\beta)}\left(1-\frac{2x}{\beta}\right) = L_n^{\alpha}(x) . $$

Therefore in the limit of infinite magnetic flux the Hilbert space of Haldane wavefunctions, eq. (15) with given angular momentum quantum number $l = n$

approximate n-th Landau level in the Hilbert space of the planar Landau level

$$V_n^{(s)} = \bigoplus_{m=-n}^{n+2s} \mathbb{C}\ {}^s\Psi_n^m(z,\bar{z}) \xrightarrow[s\to\infty]{} \mathbf{Dirac}_n = \bigoplus_{m\geq -n} \psi_{n,m}(z,\bar{z}) ,$$

where the energy spectra on both sides are also identified in the limit

$$\lim_{s\to\infty} \hbar\omega_c \left(n + \frac{1}{2} + \frac{n(n+1)}{2s} \right) = \hbar\omega_c \left(n + \frac{1}{2} \right) .$$

On an algebraic level in the limit $s \to \infty$ we have an example a Wigner-Inonu contraction of the $\mathfrak{su}(2)$-algebra to the Heisenberg algebra. Indeed we can define rescaled angular momentum operators

$$\tilde{L}_+ = -\frac{L_+}{\sqrt{2s}}, \quad \tilde{L}_- = -\frac{L_-}{\sqrt{2s}}, \quad \tilde{L}_3 = \frac{1}{s} L_3 , \tag{21}$$

which satisfy commutation relations

$$[\tilde{L}_+, \tilde{L}_-] = \tilde{L}_3, \qquad [\tilde{L}_3, \tilde{L}_\pm] = \pm\frac{1}{s}\tilde{L}_\pm . \tag{22}$$

We can now safely take the limit and obtain that the $\mathfrak{su}(2)$-algebra of the Haldane sphere rotations is contracted to the Heisenberg algebra of magnetic translations in the plane

$$\tilde{L}_\pm \xrightarrow[s\to\infty]{} b^\pm , \qquad \tilde{L}_3 \xrightarrow[s\to\infty]{} -\mathbb{1} .$$

The contraction of the Haldane hamiltonian in terms of the rescaled operators becomes

$$\begin{aligned} \mathcal{H}_s &= \frac{\hbar\omega_c}{2} \left(\frac{L_+L_- + L_3(L_3-1) - s^2}{s} \right) \\ &= \hbar\omega_c \left(\tilde{L}_+\tilde{L}_- + \frac{1}{2}\tilde{L}_3(s\tilde{L}_3 - 1) - \frac{s}{2} \right) \\ &= \hbar\omega_c \left(\tilde{L}_+\tilde{L}_- - L_z + \frac{1}{2} + \frac{1}{2s} L_z(L_z - 1) \right) , \end{aligned}$$

where we used $\tilde{L}_3 = \frac{L_z}{s} - \mathbf{1}$. Hence in the planar limit

$$\lim_{s\to\infty} \mathcal{H}_s = \hbar\omega_c \left(b^+b^- - L_z + \frac{1}{2} \right) = \hbar\omega_c \left(a^+a^- + \frac{1}{2} \right)$$

which is exactly the planar Landau Hamiltonian H_L.

5 4D Harmonic Oscillator and Haldane Sphere

We will now show that the eigenfunctions of the Haldane hamiltonian $\mathcal{H}_s$ can be obtained by reduction of the eigenfunctions of the 4D harmonic oscillator

$$\tilde{\mathcal{H}}_{osc} = \frac{1}{2\mu}(p_1^2+p_2^2+p_3^2+p_4^2)+\frac{\mu\Omega^2}{2}(u_1^2+u_2^2+u_3^2+u_4^2)\,, \qquad \Omega = \frac{\omega_c}{2}\,, \quad (23)$$

with real coordinates $(u_1, u_2, u_3, u_4) \in \mathbb{R}^4$. We make use of the spinor coordinates $\boldsymbol{z} = (z_1, z_2) \in \mathbb{C}^2$ instead of the real coordinates of $\mathbb{R}^4 \cong \mathbb{R}_+ \times S^3$ where we separated the radial and the angular part.

Spinors. The sphere S^3 is the subspace of $\mathbb{R}^4 \cong \mathbb{C}^2$ defined by the quadric

$$S^3_\rho := \left\{ \boldsymbol{Z} := \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} \in \mathbb{C}^2 : \boldsymbol{Z}^\dagger \boldsymbol{Z} = \bar{z}_1 z_1 + \bar{z}_2 z_2 = \rho \right\} .$$

The spinor parametrization of the sphere S^3_ρ by three Euler angles $\theta \in [0, \pi]$, $\varphi \in [0, 2\pi)$ and $\gamma \in [0, 4\pi)$ shows the isomorphism $SU(2) \cong S^3$

$$z_1 = u_1+iu_2 = \sqrt{\rho}\sin\frac{\theta}{2}e^{\frac{i}{2}(\gamma+\varphi)}\,, \qquad z_2 = u_3+iu_4 = -\sqrt{\rho}\cos\frac{\theta}{2}e^{\frac{i}{2}(\gamma-\varphi)}\,.$$

If we set $\rho = 1$ then the elements of the Lie group $SU(2)$ have parametrization

$$g = \begin{pmatrix} z_1 & -\bar{z}_2 \\ z_2 & \bar{z}_1 \end{pmatrix} = \begin{pmatrix} \sin\frac{\theta}{2}e^{\frac{i}{2}(\gamma+\varphi)} & \cos\frac{\theta}{2}e^{-\frac{i}{2}(\gamma-\varphi)} \\ -\cos\frac{\theta}{2}e^{\frac{i}{2}(\gamma-\varphi)} & \sin\frac{\theta}{2}e^{-\frac{i}{2}(\gamma+\varphi)} \end{pmatrix} \in SU(2)\,.$$

In terms of the spinor variables the hamiltonian of the 4D isotropic harmonic oscillator takes the concise form

$$\tilde{\mathcal{H}}_{osc} = \hbar\Omega(\bar{\boldsymbol{Z}}\boldsymbol{Z} - \bar{\boldsymbol{D}}\boldsymbol{D}) := \hbar\Omega(\bar{z}_1 z_1 + \bar{z}_2 z_2 + \bar{w}_1 w_1 + \bar{w}_2 w_2)\,, \qquad (24)$$

where we have defined the two spinors

$$\boldsymbol{D} = \begin{pmatrix} w_1 \\ w_2 \end{pmatrix} = \begin{pmatrix} \frac{\partial}{\partial z_1} \\ \frac{\partial}{\partial z_2} \end{pmatrix} .$$

The connection between the Landau plane $z \in \mathbb{C}$ and the homogeneous coordinates $\boldsymbol{Z} \in \mathbb{C}^2$

$$\frac{z}{\rho} = -\frac{z_1}{z_2} = e^{i\varphi}\tan\frac{\theta}{2}\,, \qquad \boldsymbol{Z} := \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} .$$

The hamiltonian $\tilde{\mathcal{H}}_{osc}$ of the 4D harmonic oscillator is a lift of the 2D Hamiltonian $H_{osc} = \hbar\Omega(\bar{z}z - \bar{\partial}\partial)$.

Hopf fibration. The points on S^3_ρ are projected onto the two dimensional sphere S^2_ρ by

$$x_1 + ix_2 = \rho \sin\theta e^{i\varphi} = -2z_1\bar{z}_2 \,, \qquad x_3 = \rho\cos\theta = z_2\bar{z}_2 - z_1\bar{z}_1 \,.$$

The kernel of the projection is $(e^{i\gamma}z_1, e^{i\gamma}z_2) = U(1)\boldsymbol{z}$. The projection π : $(z_1, z_2) \mapsto (x_1, x_2, x_3)$ becomes one-to-one if we add the angle $\gamma \in [0, 4\pi)$

$$-x_k = \boldsymbol{z}\sigma_k\bar{\boldsymbol{z}} \,, \qquad \gamma = \arg z_1 + \arg z_2 \,. \tag{25}$$

The points of S^2_ρ are S^1-rays in S^3 and we end up with the Hopf fibration with base the unit sphere $S^2_{\rho=1} = SU(2)/U(1)$.

The 4D isotropic oscillator is invariant under rotations, that is, the Hamiltonian $\tilde{\mathcal{H}}_{osc}$ commutes with the $SU(2)$-generators $\mathcal{L}_i$ defined in the Appendix. The set of mutually commuting generators that we choose to diagonalize on the bound states of (regularized) MICZ-Kepler consists of the Hamiltonian $\tilde{\mathcal{H}}_{osc}$, the square of the angular momentum $\mathcal{L}^2$, its projection $\mathcal{L}_3$ and the helicity $\mathcal{S}_3$

$$(\mathcal{H}_{osc} - N)\Psi_{N,j,m,s} = 0\,, \qquad (\mathcal{S}_3 - 2s)\Psi_{N,j,m,s} = 0\,,$$
$$\left(\mathcal{L}^2 - j(j+1)\right)\Psi_{N,j,m,s} = 0\,, \qquad (\mathcal{L}_3 - m)\Psi_{N,j,m,s} = 0\,.$$

The spectrum of the operator S_3 accounts for the helicity $s = eg/\hbar c$ of the electro-magnetic field carrying the interaction between the dyon and the electron. The total angular momentum operator $\mathcal{L} - \mathcal{S}$ transforms in an $SU(2)$-representation of spin $j = l+s$. The component $\mathcal{L}_3$ gives the magnetic quantum number m. The spectrum of the hamiltonian $\mathcal{H}_{osc}$ gives the principal quantum number $N = n_r + l + s + 1$ where n_r is the radial quantum number.

Mack and Todorov [17] have shown that the Hilbert space of a 4D isotropic oscillator carries a representation of the conformal group $U(2,2)$. The center of $U(2,2)$ is generated by the helicity operator $\mathcal{S}_3$ (see in the Appendix. The eigenstates $\Psi_{N,j,m,s}$ span a massless representation $SU(2,2)$ of helicity s.

The general solution of the 4D harmonic oscillator was obtained by by separation of variables of $\mathbb{R}^4$ by $(\rho, \theta, \varphi, \gamma) \in \mathbb{R}_+ \times S^3$

$$\Psi_{N,j,m,s} \propto R_{Nj}(\rho)\mathcal{D}^{(j)}_{sm}(\varphi, \theta, \gamma) \,, \tag{26}$$

where $\mathcal{D}^{(j)}_{sm}(\varphi, \theta, \gamma)$ stands for the Wigner $\mathcal{D}$-matrix and the radial part is given by the associated Laguerre polynomials

$$R_{Nj}(\rho) = e^{-\rho}\rho^j L^{2j+1}_{N-j-1}(2\rho) \,.$$

The 4D oscillator radial reduction of by $\rho = const$ does not affect the angular degrees of freedom. We have already matched the wavefunction on the Haldane sphere to the monopole harmonics in eq. (20). In view of the specialization of the Wigner function to monopole harmonics living on the two dimensional sphere

$$ {}^{s}Y_{j}^{m}(\phi,\theta) = \sqrt{\frac{2j+1}{4\pi}} D^{(j)}_{-sm}(\varphi,\theta,\gamma=0) \in SU(2)/U(1) \,, $$

we get the correspondence between the Haldane wavefunctions and Wigner matrices

$$ {}^{s}\Psi_{l}^{m} \leftrightarrow \mathcal{D}^{j}_{s\ m'} \,, \qquad m' = m - s \,. \tag{27} $$

We conclude that the Landau level quantization in a spherical geometry can be deduced from the reduction of the conformal invariant harmonic oscillator in $\mathbb{R}^4$.

6 4D Harmoninc Oscillator versus MICZ-Kepler Problem

The Wigner functions $\mathcal{D}$ span a basis of the space of square-integrable functions on S^3, $\mathcal{L}^2(S^3)$. The Hopf fibration gives an isomorphism between the space of functions $\mathcal{L}^2(S^3)$ and the space of square-integrable sections of complex line bundles over S^2. In practice the Wigner functions $\mathcal{D}$ can be written in terms of the monopole harmonics [8], (see Appendix).

A one-to-one correspondence $\Psi \leftrightarrow \tilde{\Psi}$ between smooth functions on S^3 and smooth section of the line bundle $\pi : S^3 \to S^2$ is given by

$$ \Psi_{N,j,m,s}(\rho,\theta,\phi,\gamma) = {}^{s}\tilde{\Psi}_{N,j,m}(\rho,\theta,\phi) e^{is(\phi+\gamma)} \,. \tag{28} $$

With this $U(1)$-reduction [16, 20] from the 4D harmonic oscillator one obtains the MICZ-Kepler hamiltonian [18, 29]

$$ \tilde{\mathcal{H}}_{s}^{MICZ} = \frac{1}{2\mu}\left(\boldsymbol{p} - \frac{e}{c}\boldsymbol{\mathcal{A}}\right)^2 - \frac{e^2}{r} + \frac{\hbar^2 s^2}{2\mu r^2} \quad , \quad s = \frac{eg}{\hbar c} \,, \tag{29} $$

describing the classical motion of a charged particle in the background of dyon with electric charge e and magnetic charge g given by the potential $\mathcal{A}$, eq. (14). The geometric quantization scheme has been applied by Mladenov and Tzanov [19] to the orbit compact manifolds. A $U(1)$-reduction gave rise to a quantum MIC-Kepler problems indexed by the helicity quantum number s. The system of dyon-dyon interaction has been also obtained in [22] via $U(1)$-reduction. However, with a global gauge transformation the dyon-dyon Hamiltonian can be reduced to the previous charge-dyon case. Barut and his collaborators (see [2] and the references therein) studied in details the conformal spectrum generating algebra $\mathfrak{so}(2,4)$ of Kepler-type problems (hydrogen atom $s = 0$ and its magnetized version $s \neq 0$) and its 4D oscillator representation.

The massless $SU(2,2)$-representations (minimal representations in the sense of Joseph) have been classified in the influential work of Mack and Todorov [17]. The ones of helicity zero $s = 0$ provided a group-theoretic regularization of the hydrogen atom revealing the hidden conformal symmetry of its spectrum. However, the $SU(2,2)$-representations of helicity $s \neq 0$ have been left without physical interpretation. Iwai found out later [15] that a massless $SU(2,2)$-representation of helicity s provides the Hilbert space of the quantum MICZ-Kepler $\tilde{\mathcal{H}}_s$.

After the $U(1)$-reduction, eq. (28), the eigenfunctions of the MICZ-Kepler have an energy spectrum

$$\tilde{\mathcal{H}}_s\ {}^s\tilde{\Psi}_{N,j,m}(\rho,\theta,\phi) = \varepsilon_N\, {}^s\tilde{\Psi}_{N,j,m}(\rho,\theta,\varphi)\ , \qquad \varepsilon_N = -\frac{\mu c^2\alpha^2}{2N^2}\ ,$$

depending only on the principal quantum number $N = n_r + l + |s| + 1$ with half-integer values $N \geq |s| + 1$. The spin $j = l + |s|$ ranges on integers or half-integers $|s| \leq j \leq N - 1$ and m ranges $-j \leq m \leq j$. Hence the degeneracy of the MICZ-Kepler bound state in an orbit with principal quantum number N is

$$\varepsilon_N = -\frac{\mu c^2\alpha^2}{2(n_r + l + |s| + 1)^2}\ , \qquad deg(\varepsilon_N) = N^2 - s^2 = \sum_{j=|s|}^{N-1}(2j+1).$$

We are now ready to give a new interpretation: The radial reduction $\rho = const$ to a sphere of the MICZ-Kepler hamiltonian coincides (up to a constant) with the Haldane centrufugal hamiltonian $\mathcal{H}_s$, eq. (11). Obviously the radial reduction of the 4D oscillator wavefunctions (26) associated via Hooke-Newton duality with MICZ-Kepler model recovers the wavefunctions on the Haldane spherical model for Landau levels, see eq. (20) and eq. (27). It is easy to see $\mathcal{H}_s$ as the angular part of the MICZ-Kepler Hamiltonian $\tilde{\mathcal{H}}_s$. The advantage of our new interpretation is that now the bound states of the MICZ-Kepler can be seen as representatives of the electron states in the Haldane spherical geometry even without imposing the radial reduction.

7 Conclusions

The Pauli hamiltonian, eq. (11) (whose restriction yields the Haldane hamiltonian) has one flaw: there are no bound states of an electron in the field of a magnetic monopole. The way out is to consider the dyon background field instead, that is, the monopole is also electrically charged, and model the spherical Landau problem with the bound states of the MICZ-Kepler problem. We have identified the Haldane centrifugal Hamiltonian as the angular part of the MICZ-Kepler hamiltonian, eq. (29). It is then natural to remove the radial constraint $\rho = const$ and consider the full-fledged MICZ-Kepler hamiltonian as a spherical model on a stack of concentric spheres in the momentum space. The ground state of the MICZ-Kepler model with half-integer helicity parameter s is then associated with a momentum sphere S^3 of radius $\rho = \sqrt{s}$.

The main advantage of the proposed MICZ-Kepler extension of the Haldane model is its built-in symmetry $SO(2,4)$ which is the maximal conformal symmetry of the Maxwell equations in Minkowski space $\mathbb{R}^{1,3}$. It has a natural Majorana reduction [6, 27] to the planar Landau problem enjoying the $SO(2,3)$-symmetry of the Maxwell equations in Minkowski space $\mathbb{R}^{1,2}$.

Appendix

Wigner functions $\mathcal{D}$

The Wigner $\mathcal{D}$-matrix is an irreducible representation of $SU(2)$ of dimension $(2j+1)^2$ with $j = l + s$

$$\begin{aligned} \boldsymbol{L}^2\ \mathcal{D}^j_{sm'} &= \hbar^2 j(j+1)\ \mathcal{D}^j_{sm'}\,, \qquad && j \geq 0\,, \\ L_3\ \mathcal{D}^j_{sm'} &= \hbar m'\ \mathcal{D}^j_{sm'}\,, && -j \leq m' \leq j\,. \end{aligned}$$

It is convenient to use Euler angles (ϕ, θ, γ) for a complete description of rotations, that is, for an arbitrary rotation along an arbitrary axis. Since $SO(3)$ is a three-parameter Lie group, any rotation can be uniquely specified by 3 independent angles. The rotation operator can be decomposed as

$$R(\phi, \theta, \gamma) = R_z(\phi) R_y(\theta) R_z(\gamma)$$

where $R_z(\phi)$ is a rotation by ϕ about the z-axis. More concretely,

$$R(\phi, \theta, \gamma) = e^{-i\phi L_z/\hbar} e^{-i\theta L_y/\hbar} e^{-i\gamma L_z/\hbar}$$

where L_x, L_y, L_z are the angular momentum operators along the respective axis, the generators of infinitesimal rotations. Then the Wigner $\mathcal{D}-$matrices are the matrix elements of the rotation operator in the basis of angular momentum eigenstates $|j, m'\rangle$,

$$\mathcal{D}^j_{sm'}(\phi, \theta, \gamma) = \langle j, s | R(\phi, \theta, \gamma) | j, m' \rangle$$

where j represents the total angular momentum, while m', s are two different angular momentum eigenvalues. Equivalently,

$$\mathcal{D}^j_{sm'}(\phi, \theta, \gamma) = e^{-is\phi} d^j_{sm'}(\theta) e^{-im'\gamma}\,,$$

where the matrix elements of Wigner's small d-matrix are given as

$$d^j_{sm'}(\theta) = \langle j, s | e^{-i\theta L_y/\hbar} | j, m' \rangle\,.$$

Let $\boldsymbol{Z} = (z_1, z_2) \in \mathbb{C}^2$ be a spinor. We define the angular momentum operators $\mathcal{L}_3, \mathcal{L}_\pm := \mathcal{L}_1 \pm \mathcal{L}_2$ which commute with the helicity operators $S_3, S_\pm$ [6, 11],

$$\begin{aligned} \Lambda_k &= \tfrac{1}{2}(\boldsymbol{Z}\sigma_k \boldsymbol{D} - \bar{\boldsymbol{D}}\sigma_k \bar{\boldsymbol{Z}}) \qquad && \leftrightarrow \quad \mathcal{L}_k = \Lambda_k + \mathcal{S}_k, \\ \mathcal{S}_k &= \tfrac{1}{2}\left(\boldsymbol{Z}^\flat \sigma_k \boldsymbol{D}^\flat - \bar{\boldsymbol{D}}^\flat \sigma_k \bar{\boldsymbol{Z}}^\flat\right) && \qquad [\Lambda_k, \mathcal{S}_l] = 0\,, \end{aligned}$$

where the *half-conjugation* $\flat$ is an involution $\boldsymbol{Z}^\flat := (\bar{z}_1\ z_2)$.

The operators $\mathcal{L}_k$ belong to the spectrum generating algebra $SU(2,2)$ of the 4D isotropic harmonic oscillator [2,17]. The helicity operator S_3 is representing the center of the conformal group $U(2,2)$. Its eigenvalue is the helicity

$$S_3 = -2s\mathbb{1} = z^\alpha \frac{\partial}{\partial z^\alpha} - \bar{z}^\alpha \frac{\partial}{\partial \bar{z}^\alpha} = \boldsymbol{Z}\boldsymbol{D} - \bar{\boldsymbol{Z}}\bar{\boldsymbol{D}} \qquad s = 0, \pm\frac{1}{2}, \pm 1, \dots .$$

In fact the half-integer helicity s labels the zero-mass representations of $U(2,2)$.

We are dealing with two mutually commuting $\mathfrak{su}(2)$ algebras, $[\mathcal{L}_i, \mathcal{S}_j] = 0$,

$$\begin{aligned} [\mathcal{L}_+, \mathcal{L}_-] &= 2\mathcal{L}_z, & [\mathcal{S}_+, \mathcal{S}_-] &= 2\mathcal{S}_z, \\ [\mathcal{L}_z, \mathcal{L}_\pm] &= \pm\mathcal{L}_\pm\,, & [\mathcal{S}_z, \mathcal{S}_\pm] &= \pm\mathcal{S}_\pm\,. \end{aligned} \tag{30}$$

These commuting operators are acting on the Haldane sphere's homogeneous coordinates Z. Their Wigner contractions yield two Heisenberg algebras

$$\mathcal{L}_\pm \xrightarrow[s\to\infty]{} b^\pm\,, \qquad \mathcal{S}_\pm \xrightarrow[s\to\infty]{} a^\mp\,,$$

of creation/annihilation $b^\pm$ ($a^\mp$) angular momentum (energy) operators.

The $SU(2) \times SU(2)$-action of the momentum and helicity operators on the Wigner $\mathcal{D}$-matrices is given by

$$\begin{aligned} &\mathcal{S}_\pm \mathcal{D}^j_{-sm'} = -\sqrt{(s \mp m)(s \pm m + 1)}\mathcal{D}^j_{-s\mp 1, m'}\,, \quad \mathcal{S}_3 \mathcal{D}^j_{sm'} = -s\mathcal{D}^j_{sm'} \\ &\mathcal{L}_\pm \mathcal{D}^j_{sm'} = -\sqrt{(j \mp m)(j \pm m + 1)}\mathcal{D}^j_{s, m'\pm 1}\,, \quad \mathcal{L}_3 \mathcal{D}^j_{sm'} = \hbar m' \mathcal{D}^j_{sm'}\,. \end{aligned} \tag{31}$$

The Casimir operators of the two $\mathfrak{su}(2)$-algebras have a common eigenvalue

$$\mathcal{L}^2 = \mathcal{S}^2 = j(j+1)\,.$$

The hydrogen atom is described by the zero helicity $s = 0$ representation [17]. A helicity $s \neq 0$ representation describes charge-dyon system with magnetic charge $g = \hbar cs/e$, which we will refer to as the dyogen atom [25].

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