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Numerical Order of Material Change and Variable Energy Density of a Dynamic Quantum Vacuum

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Abstract. A model of time as numerical order of material change inside a fundamental timeless three-dimensional dynamic quantum vacuum defined by a variable energy density is proposed. The emergence of time as a mathematical parameter that does not flow on its own but derives as a consequence of the entanglement between a system and a clock, is explored in a picture where the three-dimensional quantum vacuum is subjected to a dissipative hydrodynamics. Perspectives as regards the treatment of entangled systems of two interacting qubits in the three-dimensional dynamic quantum vacuum subjected to a dissipative hydrodynamics are analysed.

KEY WORDS: three-dimensional dynamic quantum vacuum, variable quantum vacuum energy density, clocks, entanglement between clock and evolving system, numerical order of material change.

1 Introduction

If time has always puzzled philosophers and scientists and the whole development of theoretical physics can be seen as a continuous improvement of the models of space and time, today, in the light of the most significant research on quantum gravity, the notion of time intended as a primary physical reality, as an independent physical dimension which has existence on its own, seems to have no fundamental citizenship. By virtue of the "deformable time" of general relativity and the notion of a timeless quantum vacuum as a fabric of reality introduced by quantum physics, at the beginning of the third millennium a new vision of time is emerging in theoretical physics: particles move in a timeless space background and time as-measured with clocks exists only as an emergent mathematical parameter indicating the numerical order of particles' change [1,2].

In Newtonian physics as well as in conventional quantum mechanics, time is treated as a background idealized parameter which marks the evolution of the system, as a special physical quantity that plays the role of the independent variable of physical evolution. Newtonian mechanics is based on the assumption that such an unobservable, absolute time with a unique topological and metrical structure exists, in terms of which the dynamics is defined. In a similar way, quantum mechanics made time a parameter, external to the theory and not recognizable as an observable. Both in the Schrödinger formulation and in the Heisenberg formulation of standard quantum mechanics, time is not represented as an operator acting on the relevant Hilbert space, but is considered as part of an *a priori* given classical background with a well-defined value, in other words is a Newtonian time, i.e., an absolute global time. In fact, the two main methods of quantization, namely, Dirac's canonical quantization method and Feynman's path integral method are based on classical constraints which become operators annihilating the physical states, and on the sum over all possible classical trajectories, sum over histories, respectively. Both quantization methods rely on the Newtonian global and absolute time [3].

On the other hand, in general relativity, as a consequence of the fully relational dynamics of the gravitational field and of any other dynamical object, of the fact that the space-time metric is no longer fixed, immune to change, Newton's time does not exist but is replaced by different possible internal times, related to specific physical clocks. In general relativity, change is described in terms of a relation among equal footing variables and time is a complicated non-local function of the gravitational field which is associated with a given world line [4].

As regards the nature of time, there is therefore something that strides between the two most important theories of contemporary physics: general relativity does not seem to possess a natural time variable, while quantum theory relies quite heavily on a preferred time, thus generating a situation of impasse when one seeks an unification of gravity with quantum theory or, more precisely, seeks an accommodation of gravity into the quantum framework. As well known, quantization methods, when applied to general relativity, lead to the Wheeler–DeWitt equation [5, 6], a second order functional differential equation, which leads to the so-called frozen formalism time-problem in quantum gravity, according to which, in apparent contradiction with everyday experience, nothing at all seems to happen in the universe. Just as a consequence of the fact that the nature of time in quantum gravity is not yet clear, it emerges the time arbitrariness problem, the so-called "problem of time" of quantum gravity [7–9].

The impossibility to treat time as any other observable in quantum mechanics is incompatible with the tenet of general relativity regarding the introduction of "spacetime". In this way, discussions about the role of time in quantum mechanics have been developed for some decades, leading to different proposals on how to overcome what is considered a weakness of the quantum theory, that mines the capacity of this theory to describe the aspects of the world. An attempt of finding a solution was introduced by D.N. Page and W.K. Wotters in 1983 in the realm of quantum information, which formalizes the idea that the expression “at a certain time t” should be understood as “conditioned to a clock being in a state characterized by a certain value t” [10]. This proposal, which can be called as the “Page and Wootters (PaW) mechanism”, is based on the idea that the actual time t for an evolving system is set by the fact that another system (referred to as the ”clock”) be in a state labelled by the value t, in a picture where system and clock together are assumed to be in an entangled state. The PaW mechanics is physically motivated by the observation that the notion of time can be replaced by the notion of quantum correlations between the clock (C) and the physical system of interest (R), that different states of C correspond different states of R but the whole $U = R + C$ remains in a stationary state. In this sense, in the PaW approach both the dynamics and time emerge from non-local correlations, i.e. the emergence of time and the notion of entanglement can be someway mixed together [11]. The PaW mechanism has been extensively used, and its assumptions scrutinized, in the recent literature, both from the theoretical and the experimental viewpoint [11–25].

Moreover, in recent years it has been shown how the PaW mechanism can be applied to operationally define local time reference frames associated with several quantum clocks, able to investigate time dilation, gravitational interaction and the causal structure of processes in general relativity [25, 26]. The relationship of the PaW formalism with other approaches to quantum gravity and, more generally, to quantum mechanics [27] and to the definition of (time-) reference frames, quantum observables and their classical limit has also been explored [28]. Within the framework of the PaW mechanism, both the quantum Schrodinger equation and the classical Hamilton equations of motion are derived in a manner that consistently identifies the physical quantity that plays the role of time in both equations [29]. Moreover, in [30] a new definition of time has been considered, where the evolving system Γ and its clock C are described as non-interacting, entangled systems, with a discrete spectrum, and it has been explored how the quantum dynamics transforms into a classical-like behaviour when conditions related with macroscopicity are met by the clock alone, or by both the clock and the evolving system, and how the clock C can describe the standard dynamics of Γ regardless of whether the latter is quantum or classical. While the implementation of the PaW constraint leads to strict conditions on the parameters setting the energy scales of the two systems and on the allowed states appearing in the global entangled state, the possibility to describe the full dynamics of Γ can be recovered, when the macroscopic (classical limit) of the clock-system is taken in order to reconcile the quantum definition of time by the PaW mechanism with the usual classical ”time” variable appearing in the Schrodinger equation. This new definition of time implies thus that what we once thought was a fundamental element of our physical reality could indeed be an illusion created by quantum entanglement, that the only reason that an object appears to change over time is because it is entangled with a clock.

In this paper, in affinity with these current research, our aim is to go a step ahead, by introducing a model of clocks in the view of the three-dimensional dynamic quantum vacuum (3D DQV), defined by a variable energy density, which has been developed by the authors of this paper. The structure of the paper is the following. In Section 2, after reviewing the basic features of the model of the 3D DQV, we explore how, by starting from the variable energy density of the 3D DQV, one can build a concept of clock which measures frequency, speed and numerical order of material changes. In Section 3 we analyse the link between numerical order of material change with quantum entanglement in systems of two interacting qubits. In Section 4 we summarize the main results of the paper.

2 Clocks as Measuring Systems of the Numerical Order of Material Change in the Three-Dimensional Dynamic Quantum Vacuum

The model of the 3D DQV introduces the perspective that ordinary matter, dark matter and dark energy are emergent structures which assume the form of specific excited states of the quantum vacuum corresponding to specific fluctuations of a fundamental quantum vacuum energy density. In this model, the background space of physical processes is a 3D quantum vacuum where each elementary particle is determined by elementary reduction-state (RS) processes of creation/annihilation of quanta (more precisely, of virtual pairs particles-antiparticles) corresponding to opportune changes of energy density of the vacuum [31, 32]. The virtual particles-antiparticles corresponding to the RS processes of creation/annihilation of the 3D quantum vacuum give rise to a total zero spin, thus constituting an organized Bose ensemble, analogous to the superfluid helium [33]. By defining the “ground state” of the fundamental quantum vacuum as the physical state characterized by the maximum value of the quantum vacuum energy density namely the Planck energy density

$$\rho_{pE} = \frac{M_{Pl}c^2}{l_p^3} = 4.641266 \times 10^{113} J/m^3, \tag{1}$$

where M_{Pl} is Planck’s mass, c is the light speed and l_p is Planck’s length. In this model the appearance of ordinary matter of the Standard Model corresponds to opportune excited states of the quantum vacuum which are characterized by suitable diminutions of the quantum vacuum energy density corresponding to elementary reduction-state (RS) processes of creation/annihilation of virtual particle/antiparticle pairs, given by relation

$$\Delta\rho_{qvE} \equiv \rho_{pE} - \rho = \frac{mc^2}{V} \tag{2}$$

with respect to the ground state, depending on the amount of mass m of the particle and the volume V of the region where the actualization of the particle occurs. In a similar way, dark energy cannot be considered as a primary physical reality but its action is an emerging process which is generated by opportune quantum vacuum energy density fluctuations $\Delta\rho_{qvE}^{DE}$ on the basis of relation

$$\rho_{DE} \cong \frac{35Gc^2}{2\pi\hbar^4 V}\left(\frac{V}{c^2}\Delta\rho_{qvE}^{DE}\right)^6, \tag{3}$$

where $\hbar$ is Planck's reduced constant and G is Newton's gravitational constant. As shown in [31, 32, 34–36], the dark energy density (3) can be associated to the action of a cosmological constant and is responsible of the curvature of space-time. Here, ordinary matter, dark matter and dark energy can receive somewhat an unifying description in the sense that can be seen as opportune states of the fundamental quantum vacuum intended as the primary matrix of the world. This model of quantum vacuum has the merit to suggest interesting perspectives of unification of gravity and quantum theory and of a completion of the Standard Model where the action of the Higgs boson does not exist as an irreducible reality but emerges from something more fundamental, namely as the interplay of opportune fluctuations of the quantum vacuum energy density (see, for example, [2, 31, 32, 34–36]).

A crucial perspective suggested by the view of the 3D DQV is that the duration of physical phenomena is not a primary physical reality but depends on the motion of matter and requires the measurement of an observer. In other words, evolution can be described in terms of a mathematical parameter measuring only the ordering of events, with clocks we measure frequency, speed and numerical order of material changes.

The concept of physical clock/time as a parameter measuring the order of the dynamics of processes inside a fundamental timeless background is suggested by various current research. In particular, in [37–39] Elze developed a model of time which implies that "time passes" when there is an observable change, which is localized with the observer, that physical time is an emergent discrete quantity related to the increasing number of incidents, i.e. observable unit changes, and thus only coincidences of points of the trajectory of the system with appropriate detectors exist as physical reality. In analogous way, in [40] Caticha considered an "entropic" notion of time in which time emerges as a device to keep track of the accumulation of small changes and it is only the correlations among the particles of interest and the clock that are observable and not their "absolute" order. Finally, in [41] Prati showed that, in a timeless Hamiltonian framework, a physical system S, if complex enough, can be separated in a subsystem S2 whose dynamics is described, and another cyclic subsystem S1 which behaves as a clock where, as a consequence of the gauge invariance the complex system S can be separated in many ways in a part which constitutes the clock and the rest. This means, from the physical point of view, that the ticking of each subsystem which acts as a clock provides a different reference system in order to describe the dynamics, that as a consequence of the gauge invariance, by changing the separation inside S we obtain different measuring systems of the dynamics taking place in S and therefore that time as an idealized quantity that flows on its own does not exist. On the basis of Prati's model, the numerical order of material changes is defined in a precise mathematical way in the following way. For a given $\bar{\sigma}$, a state ψ of the system S consists of the tensor product of the state $\psi_1(\bar{\sigma}) \in H_1$ and the state $\psi_2(\bar{\sigma}) \in H_2$ where H_1 and H_2 are the Hilbert spaces of the subsystems S1 and S2 respectively. Given the interval (σ_A, σ_B), by introducing the set:

$$\Omega(\sigma_A, \sigma_B) = \{\psi_2(\sigma) \in H_2 / \sigma \in (\sigma_A, \sigma_B)\}, \tag{4}$$

which is a particular set of states associated with the interval (σ_A, σ_B), $\Omega(\sigma_A, \sigma_B) \subset H_2$, physical time T^{S_1} of the interval (σ_A, σ_B), as numerical order of the material motion of the system S2 can be defined mathematically as a counter function k_{AB} that provides the number of states $\psi_2(\sigma) \in \Omega$ of the subsystem S2 whose dynamics is studied that satisfies an appropriate initial condition (namely the origin of measurement) $\psi_1(\sigma) = \bar{\psi}_1$ of the subsystem acting as a clock.

In summary, according to Elze's approach of time, Caticha's approach of entropic time and Prati's model of physical clock time, material changes do not run in a time intended as a primary, fundamental physical reality, but instead time is a mathematical quantity which emerges from a more fundamental timeless background. Now, the model of the 3D DQV allows us to go beyond by showing in an elegant and simple way in what sense clocks represent measuring systems of the numerical order of material changes in the context of an a-temporal description of motion in physics. In this paper, we want to show how the concept of time as sequential numerical order of material change, as a counter function of the number of states regarding the dynamics of the system under consideration, can be implemented in the 3D DQV, as a phenomenon that derives from the fundamental variable quantum vacuum energy density. In this regard, we will make two steps as regards the description of the entanglement between the evolving system under study and the clock: in the first, we will consider that the clock belongs to the family of spin systems of the 3D DQV and the state of the composite system is projected upon generalized coherent states; in the other, we will explore how the energy fluctuations of the vacuum which determine the entanglement between clock and evolving system correspond physically to a dissipative hydrodynamics and act at a fundamental scale where Compton wavelength and Schwarzschild radius are unified.

2.1 Numerical order of material change in clocks modelled as spin-systems

In Quantum Mechanics any physical system is described by a theory defined by some representation of a Lie-algebra $\mathscr{g}$, i.e. by a Hilbert space $\mathscr{H}$ and the commutation relations $[\cdot,\cdot]$ between the operators acting on $\mathscr{H}$ that describe the observables of the system. The specific algebra is identified by requiring that the Hamiltonian $\hat{H}$ ruling the dynamical evolution of the system belongs to $\mathcal{g}$. In our model of 3D DQV, each physical system derives from opportune aggregates of modes of the fundamental 3D DQV characterized by specific quantum vacuum energy density fluctuations, so we can consider that the two systems of interest, namely the clock C and the evolving system Γ, are opportune states of the 3D DQV. By invoking a hydrodynamic picture, Γ is associated with modes of the 3D DQV with mass

$$M = \frac{\Delta\rho_{qvE} V}{c^2} \tag{5}$$

and frequency

$$\omega = c^2/D\,, \tag{6}$$

where n is the number of the RS processes of virtual sub-particles characterizing the vacuum medium and

$$D = \frac{\hbar c^2 n}{2\Delta\rho_{qvE} V} \tag{7}$$

is the diffusion coefficient associated with the scattering of the virtual sub-particles characterizing the vacuum medium in a given volume V. By substituting (7) into (6), the frequency of the elementary modes of the vacuum are therefore

$$\omega = \frac{2\Delta\rho_{qvE} V}{\hbar n}\,. \tag{8}$$

By introducing the normalized bosonic operators $\frac{\hat{a}^{(\dagger)}}{\frac{1}{c}\sqrt{\Delta\rho_{qvE} V}}$ such that

$$\left[\hat{a},\, \hat{a}^{\dagger}\right] = \frac{c^2 I}{\Delta\rho_{qvE} V}\,, \tag{9}$$

the Hamiltonian of Γ may be defined as

$$\hat{H}_{\Gamma} = \frac{\hat{p}^2}{2M} + \frac{1}{2}M\omega^2\hat{q}^2 = \frac{4{\Delta\rho_{qvE}}^3 V^3}{\hbar^2 n^2 c^2}\left(\hat{n} + \frac{c^2 I}{2\Delta\rho_{qvE} V}\right) \tag{10}$$

with $\hat{n} = \hat{a}^{\dagger}a$, $\hat{q} = \sqrt{\frac{\hbar n}{4\Delta\rho_{qvE} V}}\left(\hat{a}^{\dagger} + a\right)$ and $\hat{p} = \frac{\Delta\rho_{qvE} V}{c^2}\sqrt{\frac{\Delta\rho_{qvE} V}{\hbar n}}\left(\hat{a}^{\dagger} - a\right)$. Then, we consider that the clock C belongs to the family of spin systems of the 3D DQV, described by the Lie algebra $su(2)$, whose representation on some $(2J{+}1)$-dimensional Hilbert space is spanned by the operators $\{\hat{J}_0, \hat{J}_1, \hat{J}_2\}$ such that $[\hat{J}_i, \hat{J}_j] = i\epsilon_{ijk}\hat{J}_k$ with $i(j,k) = 0,\, 1,\, 2$, and the Casimir $\hat{J}_0^2 + \hat{J}_1^2 + \hat{J}_2^2 = J(J+1)I$ that provides the spin J of the system. More precisely, C is a spin-J system if it is immerged in an external magnetic field pointing in a direction labelled by the above index . Moreover, we use the normalized spin operators $\hat{j}_i = \hat{J}_i/J$, such that

$$\left[\hat{j}_i, \hat{j}_j\right] = \frac{i}{J}\hat{j}_k\epsilon_{ijk} \tag{11}$$

and thus the Hamiltonian of the clock reads

$$\hat{H}_C = \epsilon J \hat{j}_0 + FI \tag{12}$$

with ϵJ and F positive energies. If we refer to the PaW mechanism, we assume that Γ and C do not interact namely that the total Hamiltonian of the composite system can be expressed as

$$\hat{H} = \hat{H}_C \otimes \hat{I}_\Gamma + \hat{I}_C \otimes \hat{H}_\Gamma \tag{13}$$

and that the state Ψ of the composite system is isolated, i.e. is a pure state of the total Hamiltonian (13) such that $\hat{H}|\Psi\rangle = 0$ and is entangled. By considering the mathematics developed in [30], in the Hilbert spaces H_C and H_Γ of the clock and the evolving system, spanned respectively by $\{|J, m\rangle, \, m = -J, -J+1, \ldots, J\}$ and $\{|n\rangle, n \in N\}$, any state of the composite system Ψ can be expressed as

$$|\Psi\rangle = \sum_{n=0}^{\infty} \sum_{m=-J}^{J} c_{nm} |J, m\rangle_C |n\rangle_\Gamma \quad \text{with} \quad \sum_{n,m} |c_{nm}|^2 = 1 \tag{14}$$

and therefore one has the constraint

$$\sum_{n,m} c_{nm} \left[\epsilon(m+J) - \frac{\Delta\rho_{qvE} V}{c^2} \left(n + \frac{1}{2} \right) \right] |J, m\rangle_C |n\rangle_\Gamma = 0 \tag{15}$$

implying

$$c_{nm} = c_m \delta \left[\epsilon(m+J) - \frac{2\Delta\rho_{qvE} V}{\hbar n} \left(n + \frac{1}{2} \right) \right]. \tag{16}$$

This means that the coefficients c_{nm} turn out to be different from zero if the quantum numbers n and m satisfy the following constraint:

$$n + \frac{1}{2} = \kappa r \, (m + J) \, , \tag{17}$$

where

$$\kappa = \frac{\epsilon J \hbar n c^2}{2 \Delta\rho_{qvE}{}^2 V^2} \quad \text{and} \quad r = \frac{\Delta\rho_{qvE} V}{c^2 J} \, . \tag{18}$$

On the basis of equations (17) and (18), in order to satisfy the constraints regarding the state of the composite system, once κ and r are fixed, only some states $|J, m\rangle_c |n\rangle_\Gamma$ can appear in the decomposition (14). Moreover, since the state Ψ is entangled, at least two pairs $(m, \, n)$ exist that satisfy equation (17). In the most general case, there exist pairs $(m, \, n)$ if

$$\kappa r = \frac{2 i_n + 1}{2 i_m}, \quad i_n i_m \in \mathbb{N} \, , \tag{19}$$

$$m = i_m \, (2l + 1) - J, \quad n = i_n \, (2l + 1) + l \tag{20}$$

$$\text{with} \quad l = 0, 1, 2, \ldots, \left\lfloor \frac{J}{i_m} - \frac{1}{2} \right\rfloor \tag{21}$$

$$\text{and} \quad J \geq 3i_m/2\,, \tag{22}$$

where the last inequality ensures that there are at least two allowed (m, n) pairs, i.e. that $|\Psi\rangle$ is entangled. When equation (17) holds, the coefficients in (14) depend on one index only, say m, and one can write

$$|\Psi\rangle = \sum_{m\in\mathscr{A}} c_m |J, m\rangle_C |n\rangle_\Gamma \quad \text{with} \quad n_m = \kappa r\,(m+J) - \frac{1}{2} \tag{23}$$

with $\mathscr{A}$ the set of integers m consistent with equations (19)-(22). Following [29, 30], one can consider Generalized Coherent States (GCS) for the clock, i.e. the $su(2) - CS$, also known as Spin Coherent States (SCS), defined as

$$|\Omega_C\rangle = e^{\Omega \hat{J}_+ - \Omega^* \hat{J}_-} |J, -J\rangle_C\,, \tag{24}$$

where $\hat{J}_\pm = \hat{J}_1 \pm i\hat{J}_2$ and $\Omega(\theta,\varphi) = (\theta/2)e^{-i\varphi}$. Here, if one partially project the state of the composite system upon SCS of C, one obtains a normalized state $\phi_\theta(\varphi)$ which satisfies a Schrödinger equation

$$i\hbar\epsilon \frac{d}{d\varphi} |\phi_\theta(\varphi)\rangle_\Gamma = \hat{H}_\Gamma |\phi_\theta(\varphi)\rangle_\Gamma \tag{25}$$

with $\hbar\varphi/\epsilon$ as time and the following mathematical expression for the wave function of the composite system:

$$|\phi_\theta(\varphi)\rangle_\Gamma = \frac{\sum\limits_{m\in\mathscr{A}} c_m \binom{2J}{m+J}^{1/2} \left(\cos\frac{\theta}{2}\right)^{J-m} \left(\sin\frac{\theta}{2}\right)^{J+m} e^{-i\varphi(J+m)} |n_m\rangle_\Gamma}{\sum\limits_{m\in\mathscr{A}} |c_m|^2 \binom{2J}{m+J} \left(\cos\frac{\theta}{2}\right)^{2(J-m)} \left(\sin\frac{\theta}{2}\right)^{2(J+m)}}\,. \tag{26}$$

On the basis of the approach based on equations (10)-(26), once the parameters of the Hamiltonians (10) and (12) are fixed, one finds that the system C can mark the time for the evolution of Γ via the PaW mechanism only if its dynamics is limited to a subspace of $\hat{H}_\Gamma$, defined by the variable quantum vacuum energy density $\Delta\rho_{qvE}$, the frequency of the modes of the vacuum ω depending of the fluctuations of the quantum vacuum energy density themselves, the parameter ϵ, and the Casimir. The fact that time measured by the clock C emerges only if the dynamics of the system under study is associated with a peculiar subspace of the Hamiltonian $\hat{H}_\Gamma$ means that time does not label an external, absolute, global, idealized temporal dimension that flows on its own in the universe but exists only as a mathematical parameter measuring the sequential numerical order of material change that is ultimately determined by the variable quantum vacuum energy density and the frequency modes of the vacuum. The entanglement between the evolving system Γ and the clock C can be associated to more fundamental fluctuations of the energy density of the 3D DQV, and these fluctuations generate the numerical order of material change of the system Γ that we experience as time. In other words, one can say that the 3D DQV can be considered as the ultimate origin of the entanglement between the system under study and the clock and thus of the speed and numerical order which describes the material change of our system.

In this picture, the ground state of the 3D DQV, corresponding to the Planck energy density (1), is ultimately a timeless background. Time has origin in the form of sequential numerical order of material change when the 3D DQV overcomes a transition towards excited states that are characterized by opportune fluctuations of the quantum vacuum energy density and are associated with opportune modes of the vacuum with certain frequencies which make that only some states of the form $|J, m\rangle_c |n\rangle_\Gamma$ appear in the decomposition (14), that are fixed by specific values of the quantities κ and r. As a consequence of these fundamental processes regarding the 3D DQV, the dynamics of the evolving system Γ that is ruled by the Schrödinger equation (25) can be considered a direct result of the coupling between the clock and the system in a picture where the quantity $\hbar\varphi/\epsilon$ representing time as numerical order of change is given by the ratio between two parameters associated with the spin coherent states and the energy of the clock, which ultimately derive from the fluctuations of the quantum vacuum energy density themselves.

In summary, we can say that time measured by clocks emerges as a mathematical quantity which is generated by the frequency modes of the 3D DQV associated with the variations of the quantum vacuum energy density with respect to its ground state. In other words, the entanglement between the evolving system and the clock generates a counter function labelled by the hands of the clock which is linked with the frequency modes of the 3D DQV, namely with the variations of the quantum vacuum energy density with respect to the ground state of the vacuum. Therefore, it is the frequency modes of the 3D DQV, i.e. the changes of the quantum vacuum energy density which occur in correspondence to the entanglement between the evolving system and the clock, that justify in what sense the research mentioned above regarding clock/time imply a concept of time as an emergent mathematical quantity that measures the numerical order of the dynamics. Elze's definition of time as an emergent discrete quantity related to the observable unit changes, corresponding to the coincidences of points of the trajectory of the system with appropriate detectors, Caticha's "entropic" notion of time as a device to keep track of the accumulation of small changes and Prati's counter function k_{AB} that provides the number of states of the subsystem whose dynamics is studied that satisfies an appropriate initial condition (namely the origin of measurement) of the subsystem acting as a clock, all these models imply that time is a measuring system of the numerical order of material changes as a consequence of the fluctuations of the energy density of the fundamental timeless 3D DQV. Now, the next step regards the search for an appropriate mathematical formulation of time intended as numerical order of material changes associated with the elementary fluctuations of the energy density of the 3D DQV, which can be considered somewhat universal.

2.2 Numerical order of material change as a consequence of the dissipative hydrodynamics of the vacuum at a fundamental unifying scale

In Section 2.1 we have analysed how the entanglement between the clock and the evolving system can be mathematically described in a picture where the clock is modelled as a spin-system, explaining in what sense time emerges as numerical order of material change in the case of spin coherent states of the clock. Now our aim is to go a step ahead, to search for a fundamental mathematical definition of time as numerical order, namely to find the mathematical relation between time and the energy density fluctuations of the 3D DQV. In this regard, we start by the hypothesis that, as a consequence of the entanglement between systems and clocks, the states of the clock are ultimately determined by the bath of elementary quantum vacuum energy density fluctuations subjected to a dissipative hydrodynamics.

By following the treatment made in [42], we consider that the Universe is a bipartite quantum system constituted by the clock C and the evolving system Γ where a pure state of the universe is expressed in terms of the orthonormal basis $\{|x\rangle_\Gamma \otimes |t\rangle_C = |x\rangle_\Gamma |t\rangle_C\}$ where $\{|x\rangle_\Gamma\}$ and $\{|t\rangle_C\}$ are orthonormal basis of the Hamiltonian of the system $\hat{H}_\Gamma$ and the Hamiltonian of the clock $\hat{H}_C$ respectively. On the basis of the results obtained in [43], the evolving system Γ and the clock C are ultimately associated with states of the 3D DQV whose geometry is characterized by the existence of a scale which provides an unification of Compton wavelength and Schwarzschild radius, namely the generalized Compton wavelength given by relation

$$R'_C = R'_S = \sqrt{\left(\frac{\beta\hbar c}{\Delta\rho_{qvE}V}\right)^2 + \left(\beta l_p^2 \frac{\Delta\rho_{qvE}V}{\hbar c}\right)^2}\,, \tag{27}$$

where the parameter β is a fluctuating quantity which expresses the fact that here space-time fluctuations fix the minimal length scale only on average and can be associated with the Planck scale. In the light of the scale (27), microscopic systems and macroscopic regime can be seen as emergent physical structures which derive from more elementary objects, which can be called “sub-Planckian black holes of the variable energy density of the 3D DQV”, i.e. mechanical objects which simultaneously have the properties of elementary particles and of black holes, which can be seen as the collective behaviour of the more fundamental variable quantum vacuum energy density [43]. We observe that, in the regime $\Delta\rho_{qvE}V/c^2 \ll M_{Pl}$ (where M_{Pl} is the Planck mass), the generalized Compton wavelength (27) tends to the standard value of the Compton wavelength of elementary particles which corresponds to the approximation of (27) with relation

$$R'_C \approx \frac{\beta\hbar c}{\Delta\rho_{qvE}V}\left[1+\frac{2}{\beta^2}\left(\frac{\Delta\rho_{qvE}V}{c^2 M_{Pl}}\right)^4\right], \tag{28}$$

while for $\Delta\rho_{qvE}V/c^2 \gg M_{Pl}$ the generalized Compton wavelength (4) tends to the standard value of the Schwarzschild radius which corresponds to the approximation of (27) with relation

$$R'_S \approx \beta l_p^2 \frac{\Delta\rho_{qvE}V}{\hbar c}\left[1+\frac{\beta^2}{8}\left(\frac{M_{Pl}c^2}{\Delta\rho_{qvE}V}\right)^4\right] \tag{29}$$

in agreement with the results obtained in [44–49].

Now, in the background ruled by the generalized Compton wavelength (27), we consider that the entanglement between the clock and the evolving system can be associated to a dissipative hydrodynamics of the vacuum, namely ultimately corresponds to deviations of the vacuum from the superfluid features near the Planck scale that can be expressed by an opportune dispersion relation of the form

$$\omega^2 = c^2k^2 - \frac{c^4\hbar^2}{\Delta\rho_{qvE}{}^2V^2}k^4, \tag{30}$$

where the wave number k is assimilated just to the inverse of the generalized Compton wavelength (27), yielding [50]

$$\omega^2 = \frac{c^2}{\left(\frac{\beta\hbar c}{\Delta\rho_{qvE}V}\right)^2 + \left(\beta l_p^2\frac{\Delta\rho_{qvE}V}{\hbar c}\right)^2} - \frac{c^4\hbar^2}{\Delta\rho_{qvE}{}^2V^2\left[\left(\frac{\beta\hbar c}{\Delta\rho_{qvE}V}\right)^2 + \left(\beta l_p^2\frac{\Delta\rho_{qvE}V}{\hbar c}\right)^2\right]^2}. \tag{31}$$

In the light of the dissipative features of the 3D DQV described by the dispersion relation (31), we can say that the states of the clock are ruled by a non-linear Schrödinger equation, a new peculiar version of the Gross–Pitaevskij equation of the form

$$i\hbar\frac{\partial\psi}{\partial t} = \frac{-\hbar^2}{2m}\nabla^2\psi + \frac{a^2}{2\pi}\left[\frac{c^2}{\left(\frac{\beta\hbar c}{\Delta\rho_{qvE}V}\right)^2 + \left(\beta l_p^2\frac{\Delta\rho_{qvE}V}{\hbar c}\right)^2} - \frac{c^4\hbar^2}{\Delta\rho_{qvE}{}^2V^2\left[\left(\frac{\beta\hbar c}{\Delta\rho_{qvE}V}\right)^2 + \left(\beta l_p^2\frac{\Delta\rho_{qvE}V}{\hbar c}\right)^2\right]^2}\right]^{1/2} m|\psi|^2\psi + U\psi, \tag{32}$$

where m is the mass of each virtual particle (namely sub-planckian black hole) of the physical vacuum, U is the potential energy relating to the single virtual particle and ν is a viscosity coefficient having the dimensions $\frac{\text{length}^2}{\text{time}}$ which can be expressed as $\nu = a^2\omega/(2\pi)$ where a is the scattering length between the virtual particles. As a consequence of the dissipative hydrodynamics described by the Gross-Pitaevskij equation (32), the time marked by the clock is a continuous label which has the physical meaning of numerical order of material change associated with the dynamics of the system under study, that can be associated to the inverse of the frequency defined by the dispersion relation (31), namely

$$t = \left[\frac{c^2}{\left(\frac{\beta\hbar c}{\Delta\rho_{qvE}V}\right)^2 + \left(\beta l_p^2\frac{\Delta\rho_{qvE}V}{\hbar c}\right)^2} - \frac{c^4\hbar^2}{\Delta\rho_{qvE}{}^2V^2\left[\left(\frac{\beta\hbar c}{\Delta\rho_{qvE}V}\right)^2 + \left(\beta l_p^2\frac{\Delta\rho_{qvE}V}{\hbar c}\right)^2\right]^2}\right]^{-\frac{1}{2}} \tag{33}$$

and can be associated to a quantum observable $\hat{\mathscr{T}}$ as follows:

$$\hat{\mathscr{T}}|t\rangle_C = \left[\frac{c^2}{\left(\frac{\beta\hbar c}{\Delta\rho_{qvE}V}\right)^2 + \left(\beta l_p^2\frac{\Delta\rho_{qvE}V}{\hbar c}\right)^2} - \frac{c^4\hbar^2}{\Delta\rho_{qvE}{}^2V^2\left[\left(\frac{\beta\hbar c}{\Delta\rho_{qvE}V}\right)^2 + \left(\beta l_p^2\frac{\Delta\rho_{qvE}V}{\hbar c}\right)^2\right]^2}\right]^{-\frac{1}{2}} |t\rangle_c\,. \tag{34}$$

The physical meaning of equation (34) is just that the entanglement between the evolving system and the clock generates a counter function labelled by the hands of the clock which is linked with the frequency modes of the 3D DQV, namely with the variations of the quantum vacuum energy density, that are ultimately determined by the dissipative features of the vacuum. In particular, in the regime $\Delta\rho_{qvE}V/c^2 \ll M_{Pl}$, which corresponds to the quantum regime, the generalized Compton wavelength (27) can be approximated with relation (28) and therefore for the numerical order of material change (33) one can use the approximation

$$t = \left[\frac{c^2}{\left[\frac{\beta\hbar c}{\Delta\rho_{qvE}V}\left[1 + \frac{2}{\beta^2}\left(\frac{\Delta\rho_{qvE}V}{c^2M_{Pl}}\right)^4\right]\right]^2} - \frac{c^4\hbar^2}{\Delta\rho_{qvE}{}^2V^2\left(\frac{\beta\hbar c}{\Delta\rho_{qvE}V}\left[1 + \frac{2}{\beta^2}\left(\frac{\Delta\rho_{qvE}V}{c^2M_{Pl}}\right)^4\right]\right)^4}\right]^{-\frac{1}{2}}, \tag{35}$$

while in the regime $\Delta\rho_{qvE}V/c^2 \gg M_{Pl}$, which corresponds to the regime of classical general relativity, the generalized Compton wavelength (27) tends to relation (29) and, therefore, for the numerical order of material change (33) one can use the approximation

$$t = \left[\frac{c^2}{\left(\beta l_p^2 \frac{\Delta\rho_{qvE}V}{\hbar c}\left[1+\frac{\beta^2}{8}\left(\frac{M_{Pl}c^2}{\Delta\rho_{qvE}V}\right)^4\right]\right)^2} - \frac{c^4\hbar^2}{{\Delta\rho_{qvE}}^2 V^2\left(\beta l_p^2 \frac{\Delta\rho_{qvE}V}{\hbar c}\left[1+\frac{\beta^2}{8}\left(\frac{M_{Pl}c^2}{\Delta\rho_{qvE}V}\right)^4\right]\right)^4} \right]^{-\frac{1}{2}} . \quad (36)$$

Moreover, always taking account the results obtained in [42], we assume that $\hat{H}_C$ and the clock's observable $\hat{\mathscr{T}}$ obey the commutation relation

$$\left[\hat{\mathscr{T}}, \hat{H}_C\right] = i\hbar\,, \quad (37)$$

which physically means that $\hat{H}_C$ is the generator of displacements in the position of the clock's hands, i.e. that for an infinitesimal displacement dt it follows that $e^{-i\hat{H}_C dt/\epsilon\hbar}|t\rangle_C = |t+dt\rangle_C$. Now, the state of Γ when the clock reads t, i.e. when the clock is in the state $|t\rangle_C$, is obtained by projecting $|\Psi\rangle$ onto $|t\rangle_C$ and can be denoted as $|\phi(t)\rangle_\Gamma = \langle t|\Psi\rangle$. In this way we obtain

$$\frac{\partial}{\partial\varphi}|\phi(t)\rangle_\Gamma = \frac{i}{\hbar}\langle t|\hat{H}_C|\Psi\rangle\,. \quad (38)$$

Taking account, on the basis of (13), that $\hat{H}_C \otimes \hat{\mathbb{I}}_\Gamma = \hat{H} - \hat{\mathbb{I}}_C \otimes \hat{H}_\Gamma$ and that $\hat{H}|\Psi\rangle = 0$, one obtains

$$\frac{\partial}{\partial t}|\phi(t)\rangle_\Gamma = \frac{i}{\hbar}\langle t|\hat{H} - \hat{\mathbb{I}}_C \otimes \hat{H}_\Gamma|\Psi\rangle = \frac{-i}{\hbar}\langle t|\hat{\mathbb{I}}_C \otimes \hat{H}_\Gamma|\Psi\rangle = \frac{-i}{\hbar}\hat{H}_\Gamma\langle t|\Psi\rangle$$
$$= \frac{-i}{\hbar}\hat{H}_\Gamma|\phi(t)\rangle_\Gamma \quad (39)$$

and therefore

$$i\hbar\frac{d}{dt}|\phi(t)\rangle_\Gamma = \hat{H}_\Gamma|\phi(t)\rangle_\Gamma\,, \quad (40)$$

which is the usual Schrödinger equation for the state of the evolving system.

On the basis of the approach based on equations (27)-(40), one can draw the following crucial results. The dynamical scenario satisfied by the relative state of the rest of the Universe, namely the system under consideration (and expressed by the time-dependent Schrödinger equation) emerges from the static image of the non-evolving state of the universe as a consequence of the entanglement between the system Γ and the clock. In this picture, time marked by the clock is an emergent mathematical quantity determined by the variations of the quantum vacuum energy density and therefore the counter function invoked by Prati can be associated with the quantity (33):

$$k_{AB} = \left[\frac{c^2}{\left(\frac{\beta\hbar c}{\Delta\rho_{qvE}V}\right)^2 + \left(\beta l_p^2 \frac{\Delta\rho_{qvE}V}{\hbar c}\right)^2} - \frac{c^4\hbar^2}{\Delta\rho_{qvE}{}^2 V^2 \left[\left(\frac{\beta\hbar c}{\Delta\rho_{qvE}V}\right)^2 + \left(\beta l_p^2 \frac{\Delta\rho_{qvE}V}{\hbar c}\right)^2\right]^2} \right]^{-\frac{1}{2}}, \quad (41)$$

which can be approximated with relation

$$k_{AB} = \left[\frac{c^2}{\left[\frac{\beta\hbar c}{\Delta\rho_{qvE}V}\left[1 + \frac{2}{\beta^2}\left(\frac{\Delta\rho_{qvE}V}{c^2 M_{Pl}}\right)^4\right]\right]^2} - \frac{c^4\hbar^2}{\Delta\rho_{qvE}{}^2 V^2 \left(\frac{\beta\hbar c}{\Delta\rho_{qvE}V}\left[1 + \frac{2}{\beta^2}\left(\frac{\Delta\rho_{qvE}V}{c^2 M_{Pl}}\right)^4\right]\right)^4} \right]^{-\frac{1}{2}} \quad (42)$$

in the regime $\Delta\rho_{qvE}V/c^2 \ll M_{Pl}$, and with relation

$$k_{AB} = \left[\frac{c^2}{\left(\beta l_p^2 \frac{\Delta\rho_{qvE}V}{\hbar c}\left[1 + \frac{\beta^2}{8}\left(\frac{M_{Pl}c^2}{\Delta\rho_{qvE}V}\right)^4\right]\right)^2} - \frac{c^4\hbar^2}{\Delta\rho_{qvE}{}^2 V^2 \left(\beta l_p^2 \frac{\Delta\rho_{qvE}V}{\hbar c}\left[1 + \frac{\beta^2}{8}\left(\frac{M_{Pl}c^2}{\Delta\rho_{qvE}V}\right)^4\right]\right)^4} \right]^{-\frac{1}{2}} \quad (43)$$

in the regime $\Delta\rho_{qvE}V/c^2 \gg M_{Pl}$. Thus, in the model of the 3D DQV where time is a measure of the numerical order of material change associated with a dissipative hydrodynamics, by substituting (33) inside equation (40), the Schrödinger equation (40) reads

$$i\hbar \frac{d}{d\left(\left[\frac{c^2}{\left(\frac{\beta\hbar c}{\Delta\rho_{qvE}V}\right)^2 + \left(\beta l_p^2 \frac{\Delta\rho_{qvE}V}{\hbar c}\right)^2} - \frac{c^4\hbar^2}{\Delta\rho_{qvE}{}^2 V^2 \left[\left(\frac{\beta\hbar c}{\Delta\rho_{qvE}V}\right)^2 + \left(\beta l_p^2 \frac{\Delta\rho_{qvE}V}{\hbar c}\right)^2\right]^2} \right]^{-\frac{1}{2}}\right)} |\phi(t)\rangle_\Gamma = \hat{H}_\Gamma |\phi(t)\rangle_\Gamma\,, \quad (44)$$

which, since in the quantum regime one can use the approximation (35) – namely (42) – for the numerical order of material change, may be also written as

$$i\hbar \frac{d}{d\left(\left[\frac{c^2}{\left[\frac{\beta\hbar c}{\Delta\rho_{qvE}V}\left[1+\frac{2}{\beta^2}\left(\frac{\Delta\rho_{qvE}V}{c^2 M_{Pl}}\right)^4\right]\right]^2} - \frac{c^4\hbar^2}{\Delta\rho_{qvE}{}^2V^2\left(\frac{\beta\hbar c}{\Delta\rho_{qvE}V}\left[1+\frac{2}{\beta^2}\left(\frac{\Delta\rho_{qvE}V}{c^2 M_{Pl}}\right)^4\right]\right)^4}\right]^{-\frac{1}{2}}\right)}|\phi(t)\rangle_\Gamma = \hat{H}_\Gamma|\phi(t)\rangle_\Gamma \,. \tag{45}$$

Equation (44), and the equivalent equation (45), can be considered as the "timeless" version of the Schrödinger equation in the 3D DQV.

Now, we must emphasize that, when the counter function provided by the hands of the clock has values in the range

$$\left[0, \frac{1}{\sqrt{\frac{c^2}{\left(\frac{\beta\hbar c}{\Delta\rho_{qvE}V}\right)^2+\left(\beta l_p^2\frac{\Delta\rho_{qvE}V}{\hbar c}\right)^2} - \frac{c^4\hbar^2}{\Delta\rho_{qvE}{}^2V^2\left[\left(\frac{\beta\hbar c}{\Delta\rho_{qvE}V}\right)^2+\left(\beta l_p^2\frac{\Delta\rho_{qvE}V}{\hbar c}\right)^2\right]^2}}}\right],$$

the global state of the universe may be expressed as

$$|\Psi\rangle = \left[\frac{c^2}{\left(\frac{\beta\hbar c}{\Delta\rho_{qvE}V}\right)^2+\left(\beta l_p^2\frac{\Delta\rho_{qvE}V}{\hbar c}\right)^2} - \frac{c^4\hbar^2}{\Delta\rho_{qvE}{}^2V^2\left[\left(\frac{\beta\hbar c}{\Delta\rho_{qvE}V}\right)^2+\left(\beta l_p^2\frac{\Delta\rho_{qvE}V}{\hbar c}\right)^2\right]^2}\right]^{\frac{1}{4}} \int \Psi(x,t)|x\rangle_C \, dx \, dt \tag{46}$$

and thus the relative state of the evolving system is

$$\begin{aligned} |\phi(t)\rangle_\Gamma &= \left[\frac{c^2}{\left(\frac{\beta\hbar c}{\Delta\rho_{qvE}V}\right)^2+\left(\beta l_p^2\frac{\Delta\rho_{qvE}V}{\hbar c}\right)^2} - \frac{c^4\hbar^2}{\Delta\rho_{qvE}{}^2V^2\left[\left(\frac{\beta\hbar c}{\Delta\rho_{qvE}V}\right)^2+\left(\beta l_p^2\frac{\Delta\rho_{qvE}V}{\hbar c}\right)^2\right]^2}\right]^{\frac{1}{4}} \int \Psi(x,t)|x\rangle_\Gamma \, dx \\ &= \left[\frac{c^2}{\left(\frac{\beta\hbar c}{\Delta\rho_{qvE}V}\right)^2+\left(\beta l_p^2\frac{\Delta\rho_{qvE}V}{\hbar c}\right)^2} - \frac{c^4\hbar^2}{\Delta\rho_{qvE}{}^2V^2\left[\left(\frac{\beta\hbar c}{\Delta\rho_{qvE}V}\right)^2+\left(\beta l_p^2\frac{\Delta\rho_{qvE}V}{\hbar c}\right)^2\right]^2}\right]^{\frac{1}{4}} |\tilde{\phi}(t)\rangle_\Gamma \,, \end{aligned} \tag{47}$$

where we have defined

$$|\tilde{\phi}(t)\rangle_\Gamma = \int \Psi(x,t)|x\rangle_\Gamma \, dx \,. \tag{48}$$

As a consequence, we can say that considering values of the numerical order t within a finite interval

$$\left[0, \frac{1}{\sqrt{\frac{c^2}{\left(\frac{\beta\hbar c}{\Delta\rho_{qvE}V}\right)^2+\left(\beta l_p^2\frac{\Delta\rho_{qvE}V}{\hbar c}\right)^2} - \frac{c^4\hbar^2}{\Delta\rho_{qvE}{}^2V^2\left[\left(\frac{\beta\hbar c}{\Delta\rho_{qvE}V}\right)^2+\left(\beta l_p^2\frac{\Delta\rho_{qvE}V}{\hbar c}\right)^2\right]^2}}}\right]$$

corresponds to studying a part of the history of the Universe that, from the standard time-based perspective, is perceived as having a finite duration

$$\frac{1}{\sqrt{\frac{c^2}{\left(\frac{\beta\hbar c}{\Delta\rho_{qvE}V}\right)^2+\left(\beta l_p^2\frac{\Delta\rho_{qvE}V}{\hbar c}\right)^2} - \frac{c^4\hbar^2}{\Delta\rho_{qvE}{}^2V^2\left[\left(\frac{\beta\hbar c}{\Delta\rho_{qvE}V}\right)^2+\left(\beta l_p^2\frac{\Delta\rho_{qvE}V}{\hbar c}\right)^2\right]^2}}}\,.$$

In other words, a finite range of values

$$\left[0, \frac{1}{\sqrt{\frac{c^2}{\left(\frac{\beta\hbar c}{\Delta\rho_{qvE}V}\right)^2+\left(\beta l_p^2\frac{\Delta\rho_{qvE}V}{\hbar c}\right)^2} - \frac{c^4\hbar^2}{\Delta\rho_{qvE}{}^2V^2\left[\left(\frac{\beta\hbar c}{\Delta\rho_{qvE}V}\right)^2+\left(\beta l_p^2\frac{\Delta\rho_{qvE}V}{\hbar c}\right)^2\right]^2}}}\right]$$

for the clock's position observable, and a consequent normalized wave function within that finite interval, can be regarded as the result of having measured the observable (projector)

$$\mathscr{P} = \int_0^{\frac{1}{\sqrt{\frac{c^2}{\left(\frac{\beta\hbar c}{\Delta\rho_{qvE}V}\right)^2+\left(\beta l_p^2\frac{\Delta\rho_{qvE}V}{\hbar c}\right)^2} - \frac{c^4\hbar^2}{\Delta\rho_{qvE}{}^2V^2\left[\left(\frac{\beta\hbar c}{\Delta\rho_{qvE}V}\right)^2+\left(\beta l_p^2\frac{\Delta\rho_{qvE}V}{\hbar c}\right)^2\right]^2}}}} |t\rangle\langle t| dt$$

post-selecting the measurement result 1, namely are the result of projecting the state of the Universe onto the subspace spanned by eigenstates of $\hat{\mathscr{T}}$ with eigenvalues

$$t \in \left[0, \left(\frac{c^2}{\left(\frac{\beta\hbar c}{\Delta\rho_{qvE}V}\right)^2+\left(\beta l_p^2\frac{\Delta\rho_{qvE}V}{\hbar c}\right)^2} - \frac{c^4\hbar^2}{\Delta\rho_{qvE}{}^2V^2\left[\left(\frac{\beta\hbar c}{\Delta\rho_{qvE}V}\right)^2+\left(\beta l_p^2\frac{\Delta\rho_{qvE}V}{\hbar c}\right)^2\right]^2}\right)^{-\frac{1}{2}}\right].$$

In terms of the wave function of the composite system $\Psi(x,t)$ equation (40) reads

$$i\hbar\frac{d}{dt}\Psi(x,t)=\hat{H}_\Gamma\Psi(x,t)\,, \tag{49}$$

which implies that, if Γ and C are non-entangled, then $\phi(t)_\Gamma$ is an eigenstate of $\hat{H}_\Gamma$, i.e. $\phi(t)_\Gamma$ is a stationary state and Γ does not evolve. In other words, $\phi(t)_\Gamma$ evolves only if there is an entanglement between the clock and the rest of the universe, and this confirms the fact that in the 3D DQV model time is not a physical dimension which has a primary physical existence and flows on its own, but is an emergent mathematical quantity that manifests itself as a numerical order of the dynamics of a given system when there is the entanglement between this system and the clock. At a fundamental level, when there is no entanglement between the clock and the rest of the universe, one deals with a quantum timeless picture.

Moreover, the degree of entanglement between clock and the rest of the universe and the corresponding numerical orders associated with the system under study Γ can be quantified by considering the degree of mixedness of the density matrix of Γ given by

$$\rho_\Gamma=\int_0^{\sqrt{\frac{1}{\frac{c^2}{\left(\frac{\beta\hbar c}{\Delta\rho_{qvE}V}\right)^2+\left(\beta l_p^2\frac{\Delta\rho_{qvE}V}{\hbar c}\right)^2}-\frac{c^4\hbar^2}{\Delta\rho_{qvE}{}^2V^2\left[\left(\frac{\beta\hbar c}{\Delta\rho_{qvE}V}\right)^2+\left(\beta l_p^2\frac{\Delta\rho_{qvE}V}{\hbar c}\right)^2\right]^2}}}} \times|\phi(t)_\Gamma\rangle\langle\phi(t)_\Gamma|dt\,, \tag{50}$$

which leads to define the entanglement measure

$$\varepsilon=1-\mathrm{Tr}\,\rho_\Gamma^2=1-\left[\frac{c^2}{\left(\frac{\beta\hbar c}{\Delta\rho_{qvE}V}\right)^2+\left(\beta l_p^2\frac{\Delta\rho_{qvE}V}{\hbar c}\right)^2}-\frac{c^4\hbar^2}{\Delta\rho_{qvE}{}^2V^2\left[\left(\frac{\beta\hbar c}{\Delta\rho_{qvE}V}\right)^2+\left(\beta l_p^2\frac{\Delta\rho_{qvE}V}{\hbar c}\right)^2\right]^2}\right]$$
$$\times\int_0^{\sqrt{\frac{1}{\frac{c^2}{\left(\frac{\beta\hbar c}{\Delta\rho_{qvE}V}\right)^2+\left(\beta l_p^2\frac{\Delta\rho_{qvE}V}{\hbar c}\right)^2}-\frac{c^4\hbar^2}{\Delta\rho_{qvE}{}^2V^2\left[\left(\frac{\beta\hbar c}{\Delta\rho_{qvE}V}\right)^2+\left(\beta l_p^2\frac{\Delta\rho_{qvE}V}{\hbar c}\right)^2\right]^2}}}}$$
$$\times\int_0^{\sqrt{\frac{1}{\frac{c^2}{\left(\frac{\beta\hbar c}{\Delta\rho_{qvE}V}\right)^2+\left(\beta l_p^2\frac{\Delta\rho_{qvE}V}{\hbar c}\right)^2}-\frac{c^4\hbar^2}{\Delta\rho_{qvE}{}^2V^2\left[\left(\frac{\beta\hbar c}{\Delta\rho_{qvE}V}\right)^2+\left(\beta l_p^2\frac{\Delta\rho_{qvE}V}{\hbar c}\right)^2\right]^2}}}}$$
$$\times|\langle\tilde{\phi}(t)_\Gamma||\tilde{\phi}(t)_\Gamma\rangle|^2dt\,dt'\,. \tag{51}$$

The quantity (51) can be interpreted as a measure of the average entanglement between clock and the rest of the universe that generates a numerical order of material change belonging to the interval

$$\left[0, \frac{1}{\sqrt{\frac{c^2}{\left(\frac{\beta\hbar c}{\Delta\rho_{qvE}V}\right)^2+\left(\beta l_p^2\frac{\Delta\rho_{qvE}V}{\hbar c}\right)^2}-\frac{c^4\hbar^2}{\Delta\rho_{qvE}{}^2V^2\left[\left(\frac{\beta\hbar c}{\Delta\rho_{qvE}V}\right)^2+\left(\beta l_p^2\frac{\Delta\rho_{qvE}V}{\hbar c}\right)^2\right]^2}}}\right].$$

In other words, it allows us to evaluate in a quantitative way the connection between the evolution of the system under study and the dissipative processes of the 3D DQV at the fundamental scale represented by the generalized Compton wavelength, when there is an entanglement between the system and the clock. When there is no entanglement between system and clock one obtains $\varepsilon = 0$ and therefore there is no time.

In summary, on the basis of equations (27)-(51), we can conclude that, at a fundamental level, time emerges as a mathematical quantity measuring the numerical order of material change by following the connection between the evolution of the system under study and the dissipative processes of the 3D DQV at the fundamental scale represented by the generalized Compton wavelength, and this connection can be therefore considered the fundamental reason which explains in what sense Elze's definition of time, Caticha's notion of time and Prati's counter function k_{AB} imply that time is a measuring system of the numerical order of material changes.

3 Time, Three-Dimensional Quantum Vacuum and Quantum Entanglement

Although entanglement remains today a mysterious feature of the quantum world, some recent research have tried to extract en emergent spacetime geometry from the entanglement properties of the quantum state [51–55]. In the light of these research, it becomes natural to explore the connection between time and quantum entanglement. In this regard, some works have shown that an evolving quantum system is entangled with quantum time [10, 56, 57]. Another recent study has explored the correlation between the external time-system entanglement and the conventional internal entanglement of a system that contains two entangled qubits, i.e. the entanglement of the system within itself, finding that the interacting system can evolve faster than the non-interacting system if the interaction is sufficiently strong [58]. In the view of time as an emergent mathematical parameter which measures the sequential numerical order of material motion in space, which is fixed by the variable quantum vacuum energy density subjected to dissipative features, quantum entanglement characterizing EPR-type experiments does not require existence of "hidden variables" in the sense that the 3D DQV itself is an immediate information medium between particles which does not require time (duration) [59–61].

As regards the relation between time and entanglement, in [62], Vedral underlines that the view of time as entanglement practically reflects the fact that we never observe time directly: this can be considered the real reason of the link between time and entanglement. We usually observe the position (of the hand of the clock, the sun or the stars) or some other observable of a periodically evolving system. Therefore, when we are measuring the evolution of the system under consideration, we are always talking about the system's states with respect to the state of the clock. Moreover, in Vedral's treatment, by assuming that the clock states are constructed in such a way that

$$e^{-iH_c d\tau/\hbar}|\psi_c\rangle = |\psi_c(\tau + d\tau)\rangle \,, \tag{52}$$

i.e. that the clock Hamiltonian generates shifts between one clock time and the immediate next clock time (where H_c is the Hamiltonian of the clock), the evolution of the system relative to the states of the clock is of the form

$$i\hbar \frac{d}{d\tau}|\psi_s\rangle = H_s|\psi_s\rangle \,, \tag{53}$$

(H_s being the Hamiltonian of the system) and thus the system undergoes the Schrödinger-type evolution relative to the ticking of the clock. On the basis of Vedral's approach, time therefore arises internally without the need for any global time, thus proving a potentially important argument in cosmology where there are presumably no clocks to measure time outside the universe. According to the point of view of the authors of this paper, the fact that time emerges internally without the need for any global absolute, idealized temporal dimension (which cannot be observed directly) means, in other words, that time measured by clocks does not exist as a primary physical reality but is only an emergent mathematical quantity measuring the numerical order of material changes.

Now, in order to explore in detail the link between entanglement and numerical order of material change induced by the dissipative features of the 3D DQV from the mathematical point of view, we consider the entanglement for a system containing two interacting qubits A and B whose basis vectors are eigenstates of the correspondent local Hamiltonians:

$$\hat{H}_A|0\rangle_A = 0 \qquad \hat{H}_A|1\rangle_A = \varepsilon|1\rangle_A \,, \tag{54}$$

$$\hat{H}_B|0\rangle_B = 0 \qquad \hat{H}_B|1\rangle_B = \varepsilon|1\rangle_B \,. \tag{55}$$

The total Hamiltonian can be expressed as

$$\hat{H}_{\text{total}} = \hat{H}_A \otimes \hat{\mathbb{1}}_B + \hat{\mathbb{1}}_A \otimes \hat{H}_B + \hat{H}_{\text{int}} \,, \tag{56}$$

where the interaction Hamiltonian $\hat{H}_{\text{int}}$ able to generating the interaction between the two qubits is linked on the dissipative features of the 3D DQV and is given by relation

$$\hat{H}_{\text{int}} = \frac{2\lambda\Delta\rho_{qvE}V}{n}\left(|1\rangle\langle 0| \otimes |1\rangle\langle 0| + |0\rangle\langle 1| \otimes |0\rangle\langle 1|\right) \,, \tag{57}$$

where λ is an adimensional parameter indicating the strength of the interaction.

By solving the Schrödinger equation (44), if when there is no material change $|\psi(0)\rangle = |00\rangle$ is the state of the system of the two qubits, the entangled state of the two qubits can be associated to the numerical order of material change induced by the 3D DQV on the basis of relation

$$|\psi(t)\rangle = \cos \frac{2\lambda\Delta\rho_{qvE}V}{n\hbar\left(\frac{c^2}{\left(\frac{\beta\hbar c}{\Delta\rho_{qvE}V}\right)^2+\left(\beta l_p^2\frac{\Delta\rho_{qvE}V}{\hbar c}\right)^2} - \frac{c^4\hbar^2}{\Delta\rho_{qvE}{}^2V^2\left[\left(\frac{\beta\hbar c}{\Delta\rho_{qvE}V}\right)^2+\left(\beta l_p^2\frac{\Delta\rho_{qvE}V}{\hbar c}\right)^2\right]^2}\right)^{\frac{1}{2}}}|00\rangle$$

$$+ i\sin \frac{2\lambda\Delta\rho_{qvE}V}{n\hbar\left(\frac{c^2}{\left(\frac{\beta\hbar c}{\Delta\rho_{qvE}V}\right)^2+\left(\beta l_p^2\frac{\Delta\rho_{qvE}V}{\hbar c}\right)^2} - \frac{c^4\hbar^2}{\Delta\rho_{qvE}{}^2V^2\left[\left(\frac{\beta\hbar c}{\Delta\rho_{qvE}V}\right)^2+\left(\beta l_p^2\frac{\Delta\rho_{qvE}V}{\hbar c}\right)^2\right]^2}\right)^{\frac{1}{2}}}|11\rangle\,. \quad (58)$$

The global state is thus

$$|\Psi\rangle = \frac{1}{\sqrt{N}}\sum_{t'=0}^{N-1}|t'\rangle\otimes|\psi(t')\rangle\,, \quad (59)$$

where

$$|\psi(t')\rangle = \cos\left(\frac{2\lambda\Delta\rho_{qvE}V}{n\hbar\left(\frac{c^2}{\left(\frac{\beta\hbar c}{\Delta\rho_{qvE}V}\right)^2+\left(\beta l_p^2\frac{\Delta\rho_{qvE}V}{\hbar c}\right)^2} - \frac{c^4\hbar^2}{\Delta\rho_{qvE}{}^2V^2\left[\left(\frac{\beta\hbar c}{\Delta\rho_{qvE}V}\right)^2+\left(\beta l_p^2\frac{\Delta\rho_{qvE}V}{\hbar c}\right)^2\right]^2}\right)^{\frac{1}{2}}}\frac{t'}{N-1}\right)|00\rangle$$

$$+ i\sin\left(\frac{2\lambda\Delta\rho_{qvE}V}{n\hbar\left(\frac{c^2}{\left(\frac{\beta\hbar c}{\Delta\rho_{qvE}V}\right)^2+\left(\beta l_p^2\frac{\Delta\rho_{qvE}V}{\hbar c}\right)^2} - \frac{c^4\hbar^2}{\Delta\rho_{qvE}{}^2V^2\left[\left(\frac{\beta\hbar c}{\Delta\rho_{qvE}V}\right)^2+\left(\beta l_p^2\frac{\Delta\rho_{qvE}V}{\hbar c}\right)^2\right]^2}\right)^{\frac{1}{2}}}\frac{t'}{N-1}\right)|11\rangle\,. \quad (60)$$

where

$$t' = \left[\frac{c^2}{\left(\frac{\beta\hbar c}{\Delta\rho'_{qvE}V}\right)^2+\left(\beta l_p^2\frac{\Delta\rho'_{qvE}V}{\hbar c}\right)^2} - \frac{c^4\hbar^2}{\Delta\rho'_{qvE}{}^2V^2\left[\left(\frac{\beta\hbar c}{\Delta\rho'_{qvE}V}\right)^2+\left(\beta l_p^2\frac{\Delta\rho'_{qvE}V}{\hbar c}\right)^2\right]^2}\right]^{-\frac{1}{2}}\,. \quad (61)$$

According to equation (59), the evolution from the state $|\psi(0)\rangle = |00\rangle$ is possible in $N-1$ steps. By following the mathematics developed in [58], one obtains that the fidelity measuring the "distance" between the initial and final states in the 3D DQV is

$$\Delta\psi = |\langle\psi(t)|\psi(0)\rangle| = \cos\frac{2\lambda\Delta\rho_{qvE}V}{n\hbar\left(\frac{c^2}{\left(\frac{\beta\hbar c}{\Delta\rho_{qvE}V}\right)^2+\left(\beta l_p^2\frac{\Delta\rho_{qvE}V}{\hbar c}\right)^2} - \frac{c^4\hbar^2}{\Delta\rho_{qvE}{}^2V^2\left[\left(\frac{\beta\hbar c}{\Delta\rho_{qvE}V}\right)^2+\left(\beta l_p^2\frac{\Delta\rho_{qvE}V}{\hbar c}\right)^2\right]^2}\right)^{\frac{1}{2}}}\,. \quad (62)$$

In this way, on finds that the minimum numerical order of material change which corresponds to the evolution of the initial state to the orthogonal final state satisfies the following constraint:

$$\frac{1}{\sqrt{\frac{c^2}{\left(\frac{\beta\hbar c}{\Delta\rho_{qvE}V}\right)^2+\left(\beta l_p^2\frac{\Delta\rho_{qvE}V}{\hbar c}\right)^2} - \frac{c^4\hbar^2}{\Delta\rho_{qvE}{}^2V^2\left[\left(\frac{\beta\hbar c}{\Delta\rho_{qvE}V}\right)^2+\left(\beta l_p^2\frac{\Delta\rho_{qvE}V}{\hbar c}\right)^2\right]^2}}} = \frac{\pi\hbar n}{4\lambda\Delta\rho_{qvE}V} \quad (63)$$

namely

$$\lambda = \frac{\pi\hbar n}{4\Delta\rho_{qvE}V}\left(\frac{c^2}{\left(\frac{\beta\hbar c}{\Delta\rho_{qvE}V}\right)^2+\left(\beta l_p^2\frac{\Delta\rho_{qvE}V}{\hbar c}\right)^2} - \frac{c^4\hbar^2}{\Delta\rho_{qvE}{}^2V^2\left[\left(\frac{\beta\hbar c}{\Delta\rho_{qvE}V}\right)^2+\left(\beta l_p^2\frac{\Delta\rho_{qvE}V}{\hbar c}\right)^2\right]^2}\right)^{\frac{1}{2}}. \quad (64)$$

Relation (64) expresses the connection that is required between the constant indicating the strength of the interaction between the two entangled qubits and the fluctuations of the quantum vacuum energy density in order to generate the direct evolution of the initial state to the final state under consideration inside the 3D DQV. Taking account that, in the light of the results obtained in [58], in the case of two non-interacting qubits, the minimum time corresponding to the minimum fidelity is

$$t_*^{NI} = \frac{\pi\hbar}{2\varepsilon}\,, \quad (65)$$

one derives that the speed of evolution of the two interacting qubits is greater – i.e. corresponds to a lower numerical order of material change induced by the 3D DQV – than that of two non-interacting qubits if the parameter λ characterizing the interaction is sufficiently strong compared to the energy scale of the individual subsystems, namely if it satisfies relation

$$\lambda > \frac{\varepsilon n}{2\Delta\rho_{qvE} V} . \tag{66}$$

On the basis of the approach based on equations (56)-(66), we can conclude that the evolution of an entangled system of two interacting qubits is faster, in the sense that requires a lower material change induced by the fluctuations of the energy density of the 3D DQV, with respect to the non-interacting system, if the interaction is sufficiently strong.

4 Conclusions

Since the original 1883 Ernst Mach quote "It is utterly beyond our power to measure the changes of things by time. Quite the contrary, time is an abstraction, at which we arrive by means of the changes of things" [63], to the "no time" interpretation of the Jacobi-Barbour-Bertotti theory (in which time is replaced with a reparametrization invariant action that is integrated over an unphysical evolution parameter) [64–67], to the Rovelli thermal time hypothesis where time is the expression of our ignorance of the full microstate, a reflex of our incomplete knowledge of the state of the world [68–78], till to arrive to more recent current studies [37–41, 79–83, 83, 85], the idea that physical time cannot be considered as a primary physical reality, but is an emergent quantity, that at a fundamental level the background space of physics is timeless, has attracted a significant attention. The view, developed in this paper, of time as a mathematical parameter which measures the sequential numerical order of material change in a fundamental timeless three-dimensional dynamic quantum vacuum, and that emerges when there is an entanglement between a system and a clock, allows us to go a step ahead, showing that time intended as a counter function of the dynamics of a system is ultimately determined by (and can be associated with) opportune fluctuations of the quantum vacuum energy density. In particular, we have shown in what sense the entanglement between clock and evolving system can be interpreted as the result of the dissipative hydrodynamics characterizing the three-dimensional quantum vacuum at a scale, the generalized Compton wavelength, where Compton wavelength and Schwarzschild radius are unified.

Our model of 3D DQV characterized by a dissipative hydrodynamics allow us to obtain the standard Schrödinger equation as an emergent fact from the fundamental static image of the timeless vacuum, in the sense that the standard Schrödinger equation derives as a result of the entanglement between a system and a clock which ultimately corresponds to opportune fluctuations of the quantum vacuum energy density with dissipative features which act at the generalized Compton wavelength, which emerges therefore as a fundamental universal scale of nature. We have got, with this model, a mathematical scheme which can explain in what sense, according to important approaches of emergent time existing in the literature (such as Elze's approach of time, Caticha's approach of entropic time and Prati's model of physical clock time), material changes do not run in a time intended as a primary, fundamental physical reality, but instead time is a mathematical quantity which emerges from a more fundamental timeless background. Moreover, we have shown that an important connection exists between the degree and speed of evolution of an entangled system containing two interacting qubits, and the numerical order of material change induced by the quantum vacuum energy density fluctuations.

In summary, we can conclude that the considerations and the mathematical formalisms developed in this paper have the potential to open suggestive unifying re-readings of the results of the fundamental approaches of emergent time where duration of material change originates from the entanglement between physical systems. In this regard, they introduce a novel way to express, at a fundamental level, the processualism view considered in [86], where physical processes are mapped in terms of interaction/correlation between physical systems and a process is not something unique that evolves through time but is what happens to a given system while it interacts with other physical systems and, by proposing the generalized Compton wavelength as a fundamental scale of nature, imply new scenarios of treating quantum gravity yet to be explored.

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