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Study in Bianchi-IX Space-Time with Linear Dependencies on R & T with $f(R,T)$ Gravity

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Abstract. This work investigate an anisotropic cosmological model in the framework of $f(R,T)$ gravity by considering the functional form $f(R,T) = \nu R + \nu T$ within a Bianchi type-IX spacetime. To describe the dynamical evolution of the universe, a fractional linear varying deceleration parameter (FLVDP) of the form $q = (1 - \alpha t)/(1 + \beta t)$ is employed, which effectively captures the transition from an early decelerating phase to the currently observed accelerating phase of the universe. Exact solutions of the modified field equations are obtained, and the behavior of key cosmological parameters such as the Hubble parameter, scale factor, shear scalar, and expansion scalar is analyzed to understand the dynamical features of the model. The results indicate that the universe evolves from an initially anisotropic state toward isotropy as cosmic time progresses. Furthermore, the evolution of physical parameters such as energy density, pressure, and the cosmological constant is examined to explore the role of dark energy in driving the accelerated expansion. The graphical analysis supports the consistency of the model with the expanding universe scenario and shows that the cosmological constant tends toward a very small positive value at late times. These findings contribute to a better understanding of anisotropic cosmological dynamics in modified gravity and provide a useful framework for studying the large-scale evolution of the universe.

KEY WORDS: $f(R,T)$ gravity theory, Bianchi type-IX spacetime,Deceleration parameter, Cosmological models.

1 Introduction

The prominence of Bianchi-IX spacetime in the field of general relativity (GR) can be traced back to its introduction by the Italian mathematician Luigi Bianchi [1]. His early 20$^{\text{th}}$ century contributions to differential geometry and the classification of homogeneous and anisotropic cosmological spacetimes, including the

Bianchi-IX model, have significantly shaped our understanding of the universe's structure and dynamics. These foundational insights later became important in the development of Einstein's GR, which predicted a dynamic and evolving universe, challenging the static cosmological models prevailing at that time. Expanding cosmological solutions were subsequently introduced independently by Georges Lemaître and Alexander Friedmann [2–4], and later supported by Hubble's observations of galactic redshifts, which provided strong evidence for the expansion of the universe.

The cosmological constant (Λ) was first introduced by Einstein in 1917 as a universal repulsive force to maintain a static universe, consistent with the cosmological perspective of that era. However, the nature of Λ remains one of the most significant unresolved problems in modern cosmology. The extremely small observed value of the cosmological constant, $\Lambda_0 \leq 10^{-56}\,\text{cm}^{-2}$, poses a serious theoretical challenge, as it differs drastically from predictions of particle physics. For instance, the Glashow–Salam–Weinberg model [5] predicts values up to 10^{50} times larger, while estimates from the Grand Unified Theory (GUT) suggest discrepancies as large as 10^{107} [6]. This discrepancy is widely known as the cosmological constant problem. Furthermore, observational breakthroughs in the late 1990s, particularly from type Ia supernovae observations conducted by the Supernova Cosmology Project and the High-Z Supernova Team [7–9], together with measurements of the cosmic microwave background (CMB), confirmed that the universe is currently undergoing an accelerated phase of expansion. These discoveries highlight the crucial role of dark energy (DE) or the cosmological constant in describing the large-scale dynamics of the universe.

During the late 1960s and early 1970s, research in GR experienced significant developments regarding the nature of spacetime singularities. In this period, researchers such as Stephen Hawking and Roger Penrose [10] established the singularity theorems, which provided deeper insights into the formation and properties of spacetime singularities. These developments also stimulated efforts to explore extensions of GR through alternative theories of gravity. Motivated by both theoretical considerations and observational challenges such as dark matter (DM) and DE, several modified gravity theories have been proposed, including $f(R)$ [11,12], $f(R,T)$ [13], Brans–Dicke [14], $f(T)$ [15–18], $f(Q)$ [17, 19, 20], and $f(Q,T)$ [21, 22] theories. These frameworks modify Einstein's field equations by introducing additional geometric or matter-coupled terms, thereby providing alternative explanations for cosmic acceleration and other large-scale phenomena. In particular, several recent investigations have developed cosmological models within the framework of $f(R,T)$ gravity to examine the dynamical behavior of anisotropic universes [23–35]. These studies primarily focus on the evolution of the scale factor, matter-energy content, and geometric parameters in Bianchi spacetimes, offering deeper insights into the cosmic evolution of anisotropic universes.

Motivated by these developments, the present work investigates the Bianchi-IX cosmological model within the framework of $f(R,T)$ gravity by considering the functional form $f(R,T) = f(R) + f(T)$ with linear dependencies on the Ricci scalar R and the trace of the energy-momentum tensor T. To describe the transition of the universe from an early decelerating phase to the present accelerating phase, we adopt a generalized deceleration parameter in the form of a fractional linear varying deceleration parameter (FLVDP) given by $q = \frac{1-\alpha t}{1+\beta t}$. This parametrization provides a flexible framework to study the dynamical evolution of the universe. Several recent studies have successfully employed FLVDP-based approaches in different modified theories of gravity to construct viable cosmological models and test them against observational datasets [36–39]. In this paper, we obtain exact solutions of the field equations and analyze the dynamical behavior of important cosmological parameters such as the scale factor, Hubble parameter, energy density, pressure, and cosmological constant. Furthermore, the physical viability of the model is examined through the analysis of energy conditions, providing insights into the evolutionary dynamics of the universe in the context of modified gravity.

This study is divided into seven sections. In Section 2, we discuss the $f(R,T)$ gravity theory briefly. Section 3 provides the metric and field equations governing the cosmological model. Sections 4 and 5 provide the solutions to the field equations and describe the physical and geometric characteristics of the model, respectively. In Section 6, we discuss the energy conditions of the model. In the last section, we summarize our study with concluding remarks.

2 Brief about $f(R,T)$ Theory of Gravity

The $f(R,T)$ gravity theory, introduced by Harko et al. in 2011 [13], presents a substantial modification of Einstein's general theory of relativity (GTR). The fundamental equation for this $f(R,T)$ theory can be written as follows:

$$\mathscr{A} = \int \left(\frac{f(R,T)}{2k} + \mathscr{L}_m \right) dV, \tag{1}$$

where the term $dV = \sqrt{-g}d^4x$ signifies the volume element, with g being the determinant of the metric tensor and d^4x representing the infinitesimal volume in spacetime. The inclusion of dV is crucial for handling very small volumes in the context of spacetime. Furthermore, $\mathscr{L}_m$ denotes the Lagrangian density of matter. The stress-energy tensor T_{kl} for matter is defined as follows:

$$T_{kl} = -\frac{2\delta(\sqrt{-g}\mathscr{L}_m)}{\sqrt{-g}\delta g^{kl}}, \tag{2}$$

the term $T = g^{kl}T_{kl}$ is the trace of the above equation. Here $\mathscr{L}_m$ depends only on the metric tensor components g_{kl} and not on its derivatives, therefore we obtain

$$T_{kl} = \mathscr{L}_m g_{kl} - 2\frac{\partial \mathscr{L}_m}{\partial g^{kl}}, \tag{3}$$

by varying the action $\mathscr{A}$ given in Eq. (1) w.r.t. the metric tensor components g_{kl}, we find the field equations for $f(R,T)$ gravity

$$F(R,T)R_{kl} - \frac{1}{2}f(R,T)g_{kl} + (g_{kl}\Box - \nabla_k\nabla_l)F(R,T) = kT_{kl} - \mathscr{F}(R,T)T_{kl} - \mathscr{F}(R,T)\Theta_{kl}, \quad (4)$$

where

$$\Theta_{kl} = g_{kl}\mathscr{L}_m - 2\left(T_{kl} + g^{kl}\frac{\partial^2\mathscr{L}_m}{\partial g^{kl}\partial g^{ij}}\right), \quad (5)$$

$F(R,T) = \frac{\partial f(R,T)}{\partial R}, \mathscr{F}(R,T) = \frac{\partial f(R,T)}{\partial T}$, $\Box = g^{kl}\nabla_l\nabla_k = \nabla^k\nabla_k$ and ∇_k stands for the covariant derivative. $k = \frac{8\pi G}{c^4}$, here c is the speeed of light and G is the gravitational constant.

The standard energy momentum tensor for perfect fluid is

$$T_{kl} = (p+\rho)u_k u_l - pg_{kl}, \quad (6)$$

where p and ρ represent the energy density and pressure of the fluid, respectively. We also have a 4-velocity vector denoted as $u^k = (0,0,0,1)$ in co-moving coordinates. This vector satisfies two crucial conditions: first, $u^k u_k = 1$ and second, $u^k\nabla_l u_k = 0$. we consider the Lagrangian density of matter as $\mathscr{L}_m = -p$ which leads to the following outcome

$$\theta_{kl} = -2T_{kl} - pg_{kl}, \quad (7)$$

The field equations in the $f(R,T)$ gravity theory are impacted by the characteristics of the matter field denoted by θ_{kl}. Within the framework of $f(R,T)$ gravity, several cosmological models can be developed, and Harko et al.(2011) [13] have introduced a classification system that categorizes these models into three specific groups. These categories can be summarized under three cases:

- Case(i):
  When $f(R,T) = R + 2f(T)$,
  Here $f(T)$ is an arbirtary function of trace T of stress energy tensor.
- Case (ii):
  when $f(R,T) = f(R) + f(T)$,
  Here $f(R)$ and $f(T)$ are arbitrary functions of Ricci scalar R and T respectively.
- Case (iii):
  when $f(R,T) = f(R) + f(R)f(T)$,
  Here $f(R), f(R)$ are arbitrary functions of R and $f(T)$ is an arbitrary function of T.

In this communication, our focus is on the second case, where the expression for the modified gravity theory is given by $f(R,T) = f(R) + f(T)$ in the context of a Bianchi-IX type universe. Here, $f(R)$ represents how the geometry of spacetime depends on the Ricci scalar R, which encodes information about the curvature or metric aspects of spacetime in the modified theory of gravity. The term $f(T)$ is a function of the trace T of the EMT, which relates to the matter content or energy distribution in spacetime. This term describes how gravity is influenced by the properties and distribution of matter in the theory.

Here we use $G = c = 1$, now using Eq. (7) in Eq. (2) the field Eqs. (4) takes the form

$$f'(R)R_{kl} - \frac{1}{2}f(R)g_{kl} + (g_{kl}\Box - \nabla_k\nabla_l)f^{'}(R) = 8\pi T_{kl} + f'(T)T_{kl} + \left\{f'(T)p + \frac{1}{2}f(T)\right\}g_{kl}, \quad (8)$$

Now considering the linear choices of

$$f(R) = \nu R \quad \text{and} \quad f(T) = \nu T, \quad (9)$$

where ν is an any constant, Eq. (8) takes the form

$$\nu R_{kl} - \frac{1}{2}\nu R g_{kl} + (g_{kl}\Box - \nabla_k\nabla_l)\nu = 8\pi T_{kl} + \nu T_{kl} + \nu\left\{g_{kl}p + \frac{1}{2}Tg_{kl}\right\}, \quad (10)$$

since $g_{kl}\Box - \nabla_k\nabla_l = 0$, we get

$$R_{kl} - \frac{1}{2}Rg_{kl} = \left(\frac{8\pi + \nu}{\nu}\right)T_{kl} + \left(p + \frac{T}{2}\right)g_{kl}. \quad (11)$$

Now from GR the Einstein tensor $G_{kl} \equiv R_{kl} - \frac{1}{2}g_{kl}R$. Therefore above equation becomes

$$G_{kl} - \left(p + \frac{T}{2}\right)g_{kl} = \left(\frac{8\pi + \nu}{\nu}\right)T_{kl}. \quad (12)$$

The cosmological constant Λ was introduced by Einstein in 1917 [40] and In GR, the EFE with cosmological constant Λ is written as

$$G_{kl} - \Lambda g_{kl} = 8\pi T_{kl}. \quad (13)$$

Now on comparing Eqs. (12) and (13) we get

$$G_{kl} - \Lambda g_{kl} = 8\pi T_{kl}, \quad (14)$$

and hence

$$\Lambda = p + \frac{T}{2}, \quad (15)$$

where we choose the Λ is a DE source.

3 Metric and Field Equations

At the core of this study lies the spatially homogeneous and anisotropic Bianchi-IX spacetime, which provides a versatile framework for examining anisotropic cosmological models. This spacetime allows for distinct expansion rates along different spatial directions while preserving homogeneity on spatial hypersurfaces, making it ideal for investigating the dynamics of the early universe and the effects of anisotropy.

The Bianchi-IX metric is expressed as:

$$ds^2 = -dt^2 + A^2 dx^2 + B^2 dy^2 + (B^2 \sin^2 y + A^2 \cos^2 y) dz^2 - 2A^2 \cos y \, dx dz, \quad (16)$$

where A, B are functions of cosmic time t. Now using Eqs. (6) and (16) the field Eq. (14) becomes

$$\frac{\ddot{B}}{B} + \frac{\ddot{A}}{A} + \frac{\dot{A}\dot{B}}{AB} + \frac{1}{4}A^2 B^{-4} = -\left(\frac{8\pi + \nu}{\nu}\right) p + \Lambda, \quad (17)$$

$$2\frac{\ddot{B}}{B} + \left(\frac{\dot{B}}{B}\right)^2 + \frac{1}{B^2} - \frac{3}{4}A^2 B^{-4} = -\left(\frac{8\pi + \nu}{\nu}\right) p + \Lambda, \quad (18)$$

$$2\frac{\dot{A}\dot{B}}{AB} + \left(\frac{\dot{B}}{B}\right)^2 + \frac{1}{B^2} - \frac{1}{4}A^2 B^{-4} = \left(\frac{8\pi + \nu}{\nu}\right) \rho + \Lambda. \quad (19)$$

Before solving the solutions of these field equations, we establish specific physical parameters. These parameters serve as crucial instruments for quantifying and characterizing the universe's expansion, contraction, and overall development on a cosmic scale. They provide valuable insights into the universe's structure and dynamics, in accordance with cosmological theories.

The average scale factor $a(t)$ in cosmology is the mean expansion factor of the universe over time. It quantifies how the universe's size changes as it evolves, essential for understanding cosmic expansion and is defined as

$$a(t) = \sqrt[3]{AB^2}. \quad (20)$$

The equation

$$H = \frac{1}{3}\left(H_1 + 2H_2\right), \quad (21)$$

defines the mean Hubble parameter, denoted as (H). It provides crucial information about the universe's expansion rate, offering insights into its age and past, present, and future dynamics. where $H_1 = \dot{A}/A$ and $H_2 = \dot{B}/B$ are directional Hubble parameters.

Spatial volume (V) refers to the 3D-size of a region in the universe. It plays a crucial role in understanding the universe's large-scale structure and expansion as described by the Big Bang theory and is defined as

$$V(t) = AB^2. \tag{22}$$

The expansion scalar (θ) measures changes in distances between nearby cosmic objects. A positive θ implies cosmic expansion, a negative θ suggests contraction, and $\theta = 0$ indicates constant expansion and is defined as

$$\theta = 3H = \sum_{i=1}^{3} H_i = H_1 + 2H_2. \tag{23}$$

Mean Anisotropic Parameter (A_m) quantifies spatial inhomogeneities, helping us account for variations in different directions and better understand the universe's isotropy or anisotropy.

$$A_m = \frac{1}{3}\left[\left(\frac{H_1}{H} - 1\right)^2 + 2\left(\frac{H_2}{H} - 1\right)^2\right]. \tag{24}$$

Shear Scalar (σ) is essential for analyzing the deformation and stretching of cosmic structures, which plays a role in the large-scale behavior of the universe.

$$\sigma^2 = \frac{1}{2}\sigma_{kl}\sigma^{kl} = \frac{1}{2}\sum_{i=1}^{3} H_i^2 - \frac{1}{6}\theta^2, \qquad i = 1, 2, 3. \tag{25}$$

where $\sigma_{kl} = \frac{1}{2}\left(u_{j;i} + u_{i;j}\right) h_k^i h_l^j - \frac{1}{3}\theta h_{kl}$ defines the shear tensor, and $h_{kl} = g_{kl} + u_k u_l$ is referred to as the projection tensor.

The deceleration parameter (q) in cosmology measures the rate of change in the universe's expansion. When positive, it indicates deceleration due to mutual gravitational attraction. A negative value signifies accelerating expansion, likely driven by DE. $q = 0$ implies a constant expansion rate. It offers insights into cosmic dynamics and is defined as

$$q(t) = -\frac{\ddot{a}a}{\dot{a}^2} = -\frac{\dot{H}}{H^2} - 1. \tag{26}$$

We can also write it as

$$H(t) = \frac{1}{\int (q(t) + 1)dt + c_1}, \tag{27}$$

where c_1 is the constant of integration.

4 Solutions of the Field Equations

The field Eqs. (17)–(19) represent a set of highly nonlinear field equations. There are five unknown parameters, namely $(A, B, p, \rho,$ and $\Lambda)$. Hence, in order to derive explicit solutions for these equations, it becomes essential to establish an extra relationship among these parameters. Since we know the relation between the component σ_1^1 of the shear scalar and the expansion scalar θ, i.e., $\sigma_1^1 \propto \theta$, leads to the expression

$$A = B^n, \tag{28}$$

where n is positive constant. This relationship was originally proposed by Throne [41].

As we know that, the variable deceleration parameter is a valuable tool in cosmology for characterizing the universe's expansion, identifying the dominant cosmic components, testing theories, and probing the nature of DE. It plays a central role in shaping our understanding of the cosmos and its evolution. So in this communication we take fractional linear varying deceleration parameter (FLVDP) as proposed by Mishra & Chand (2016) [42].

$$q(t) = \frac{1-\alpha t}{1+\beta t}, \tag{29}$$

where α and β are positive constants. From the above equation, it can be readily verified that the deceleration parameter (q) is positive for $\alpha t < 1$, indicating a decelerating phase during this epoch. Conversely, for $\alpha t > 1$, it transitions to negative values, signifying an accelerating phase of the universe. This behavior highlights a transition in the deceleration parameter, shifting from an early decelerated phase to the currently observed accelerating phase, consistent with recent observations of SNe Ia [7–9]. Furthermore, as $t \to \infty$, the deceleration parameter approaches $q \to -\alpha/\beta$.

By considering the observed value of the DP at the present time ($t_0 = 13.8$ billion years) as -0.73, as documented in [43], we can establish a relationship between the constants α and β.

$$10.074\beta - 13.8\alpha + 1.73 = 0. \tag{30}$$

Substituting the value of the DP (q) from Eq. (29) into Eq. (27), we can derive the expression for the Hubble parameter (H).

$$H(t) = \frac{1}{\left(1-\frac{\alpha}{\beta}\right)t + \left(\frac{\alpha+\beta}{\beta^2}\right)\log(1+\beta t) + c_2}, \tag{31}$$

here c_2 is constant. Because the expansion rate was extremely rapid during the inflationary period, we set c_2 to zero. Hence the above equation becomes

$$H(t) = \frac{1}{\left(1-\frac{\alpha}{\beta}\right)t + \left(\frac{\alpha+\beta}{\beta^2}\right)\log(1+\beta t)}. \tag{32}$$

Its expanded form is

$$H = \frac{\dot{a}}{a} = \mu_0 \frac{1}{t} + \mu_1 + \mu_2 t + \mu_3 t^2 + \mu_4 t^3 + \mu_5 t^4 + \mu_6 t^5 + \mathcal{O}(t^6). \quad (33)$$

where

$$\mu_0 = \frac{1}{2}, \qquad \mu_1 = \frac{\alpha+\beta}{8}, \qquad \mu_2 = \frac{1}{96}\left(3\alpha^2 - 2\alpha\beta - 5\beta^2\right),$$
$$\mu_3 = \frac{1}{384}\left(3\alpha^3 - 7\alpha^2\beta + \alpha\beta^2 + 11\beta^3\right),$$
$$\mu_4 = \frac{1}{23040}\left(45\alpha^4 - 180\alpha^3\beta + 230\alpha^2\beta^2 + 28\alpha\beta^3 - 427\beta^4\right),$$
$$\mu_5 = \frac{1}{30720}\left(15\alpha^5 - 85\alpha^4\beta + 190\alpha^3\beta^2 - 178\alpha^2\beta^3 - 61\alpha\beta^4 + 407\beta^5\right).$$

Now on integrating Eq. (33), we get the value of scale factor as

$$a(t) = a_0\sqrt{t}e^{Q(t)}, \quad (34)$$

where $Q(t) = \mu_1 t + \frac{\mu_2}{2}t^2 + \frac{\mu_3}{3}t^3 + \frac{\mu_4}{4}t^4 + \frac{\mu_5}{5}t^5 + \frac{\mu_6}{6}t^6 + \mathcal{O}(t^7)$.

The graphical behavior of the DP shown in Figure 1, along with the Hubble parameter in Figure 2 and the scale factor in Figure 3, illustrates the expanding

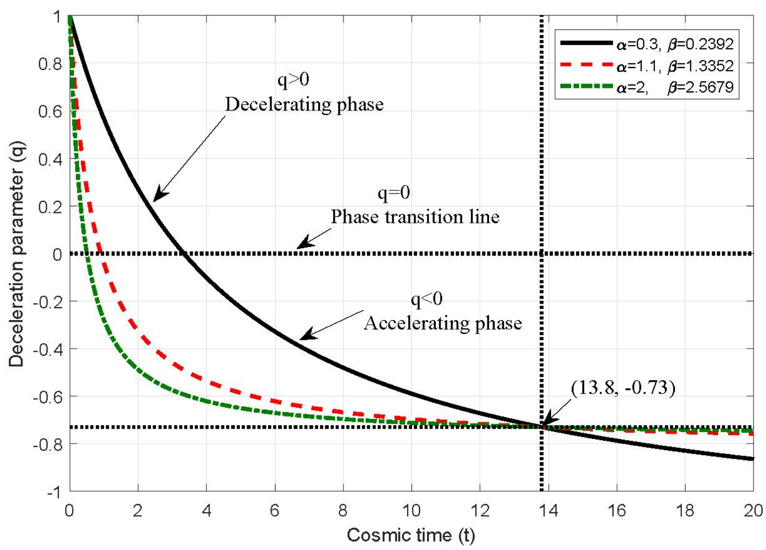

Figure 1. Time evolution of the DP, showing a shift from decelerated expansion in the past to accelerated expansion in the present era. The plot is based on selected values of the constants α and β, capturing the universe's transition from deceleration to acceleration.

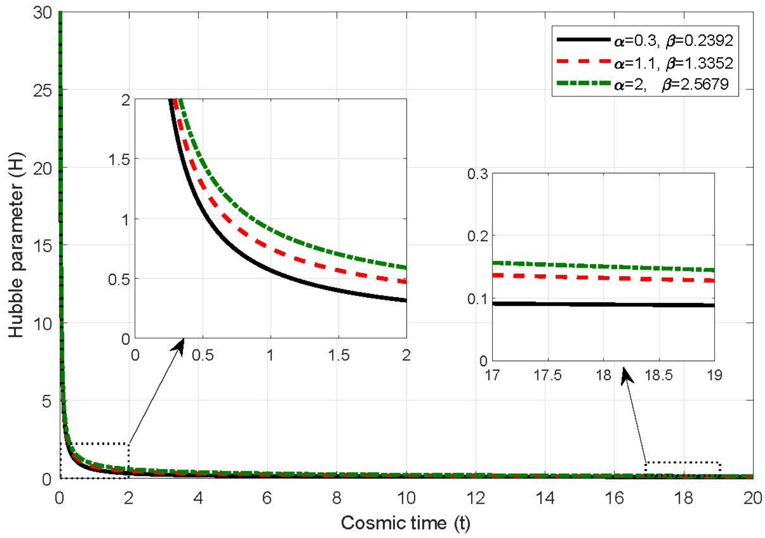

Figure 2. Evolution of the Hubble parameter H as a function of cosmic time t. Starting high at $t = 0$, H steadily decreases, approaching zero as $t \to \infty$. The plot is based on selected values of α and β.

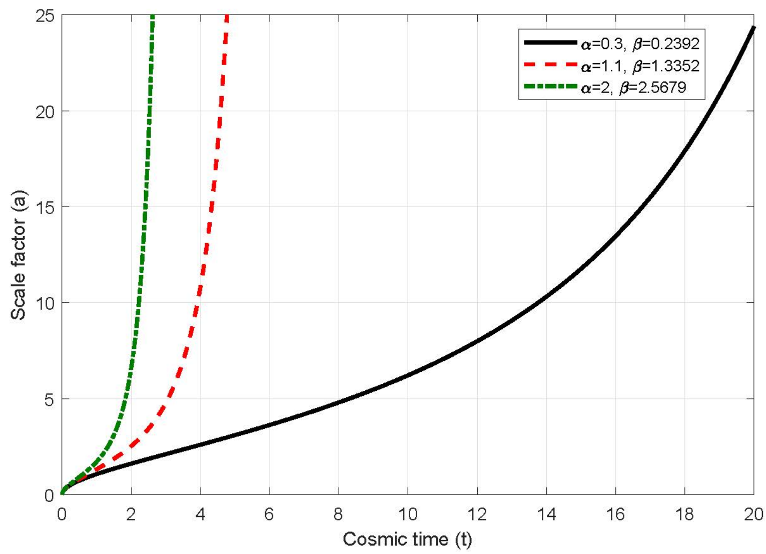

Figure 3. Time evolution of the scale factor $a(t)$. Starting at zero when $t = 0$, the scale factor increases indefinitely, approaching infinity as $t \to \infty$. The plot is based on Eq. (34), considering the first three terms of $Q(t)$ for selected values of α and β and $a_0 = 1$.

nature of the universe. The FLVDP curve in Figure 1 exhibits a phase transition from positive to negative values, indicating that the universe evolves from an early decelerating phase to the present accelerating phase. The vertical line at $t = 13.8$ Gyr intersects the curve at $q = -0.73$, representing the current value of the DP, which is consistent with the observational estimates reported by Cunha (2009) [43]. Furthermore, as cosmic time increases, the value of q continues to decrease, suggesting a more accelerated expansion in the future, and asymptotically approaches $-\alpha/\beta$ as $t \to \infty$. The Hubble parameter H remains non-negative throughout the cosmic evolution, while the scale factor increases monotonically with time, both confirming the continuous expansion of the universe. At the initial epoch ($t \to 0$), H diverges to infinity and the scale factor approaches zero, indicating the presence of a Big-Bang type singularity, which is consistent with earlier findings [44–50].

Now from Eqs. (22), (28), and (34) we get

$$A = \left\{a_0\sqrt{t}e^{Q(t)}\right\}^{\frac{3n}{n+2}}, \textit{and} \ \ B = \left\{a_0\sqrt{t}e^{Q(t)}\right\}^{\frac{3}{n+2}}. \tag{35}$$

Thus the metric Eq. (16) becomes

$$\begin{aligned} ds^2 = & - dt^2 + \left\{a_0\sqrt{t}e^{Q(t)}\right\}^{\frac{6n}{n+2}} dx^2 + \left\{a_0\sqrt{t}e^{Q(t)}\right\}^{\frac{6}{n+2}} dy^2 \\ & + \left[\left\{a_0\sqrt{t}e^{Q(t)}\right\}^{\frac{6}{n+2}} \sin^2 y + \left\{a_0\sqrt{t}e^{Q(t)}\right\}^{\frac{6n}{n+2}} \cos^2 y\right] dz^2 \\ & - 2\left\{a_0\sqrt{t}e^{Q(t)}\right\}^{\frac{6n}{n+2}} \cos y dx dz. \end{aligned} \tag{36}$$

In the upcoming section, we will understand the universe's evolution by studying the physical and geometric aspects of the cosmological model.

5 Physical and Geometric Characteristics of the Cosmological Model

In this section, we focus on the physical and geometric properties of the model defined in Eq. (36). We recognize that comprehending these properties is essential for gaining a deeper understanding of the universe's fundamental characteristics, structure, and behavior. This examination allows us to make predictions and provide explanations for observable phenomena, such as the cosmic microwave background radiation and the distribution of galaxies. Here we will discuss the physical parameters outlined in Eqs. (22)–(25), which are integral to the analysis of cosmological models. Hence we determined as

$$V(t) = \left[a_0\sqrt{t}e^{Q(t)}\right]^3, \tag{37}$$

$$\theta(t) = 3\left\{\frac{1}{2t} + \dot{Q}(t)\right\}. \tag{38}$$

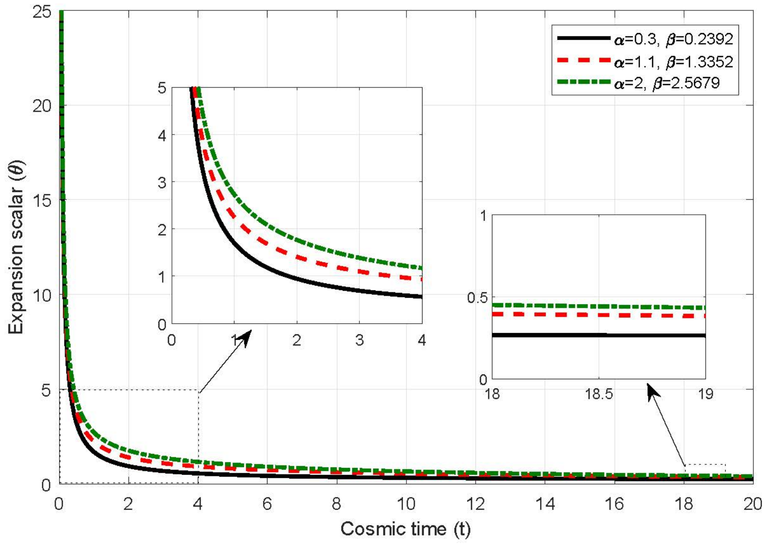

Figure 4. Evolution of the expansion scalar θ over cosmic time t. The scalar starts at infinity when $t = 0$ and decreases to zero as t approaches infinity for selected values of α and β with $n = 0.9$.

In Figure 4, the expansion scalar θ depicts a rapid, high-value expansion at the universe's inception ($t = 0$). As time progresses towards infinity, θ approaches zero, indicating a gradual slowing of the universe's expansion, potentially due to the increasing influence of DE, marking different cosmic phases.

$$\sigma^2 = \frac{3(n-1)^2}{(n+2)^2} \left\{ \frac{1}{2t} + \dot{Q}(t) \right\}^2 . \tag{39}$$

Figure 5 depicts the shear scalar σ^2 in cosmology. At $t = 0$, it shows a remarkably high value, symbolizing an early phase of rapid cosmic expansion, potentially linked to cosmic inflation. As time progresses towards infinity, σ^2 gradually tends to zero, indicating an acceleration in the universe's expansion, likely due to the increasing influence of DE, marking a shift from rapid expansion to a more gradual pace in cosmic evolution.

$$A_m = 2 \left(\frac{n-1}{n+2} \right)^2 . \tag{40}$$

In our model the anisotropic parameter (40) quantifies the deviation from isotropy. For $n = 1$, the parameter becomes zero, indicating perfect isotropy where the universe expands uniformly in all directions. As n deviates from 1, the value of A_m increases, reflecting growing anisotropy. For large n, the anisotropic parameter approaches 2, indicating significant anisotropy. This parameter provides a

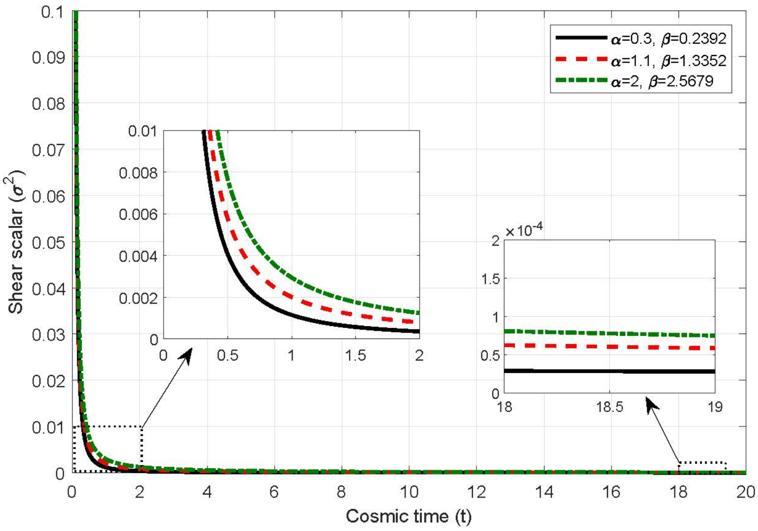

Figure 5. Evolution of the shear scalar σ^2 over cosmic time t. The scalar begins at infinity at $t = 0$ and tends to zero as $t \to \infty$ for selected values of α and β with $n = 0.9$.

measure of directional expansion differences in the universe. And the directional Hubble parameters of the model becomes

$$H_1 = \frac{3n}{n+2}\left\{\frac{1}{2t} + \dot{Q}(t)\right\} \quad \text{and} \quad H_2 = \frac{3}{n+2}\left\{\frac{1}{2t} + \dot{Q}(t)\right\}. \tag{41}$$

where $\dot{Q}(t)$ represents the first order derivative of $Q(t)$ w.r.t cosmic time t.
The isotropic condition σ^2/θ^2 is obtained as

$$\frac{\sigma^2}{\theta^2} = \frac{(n-1)^2}{3(n+2)^2}. \tag{42}$$

Equation 37 indicates that at $t = 0$, the universe had no spatial volume, representing a singularity. However, with time, it undergoes exponential growth, reflecting the rapid expansion from an infinitesimal state, a fundamental aspect of the Big Bang theory. The other physical parameters such as expansion scalar $\theta(t)$, shear scalar σ^2 and directional Hubble parameters H_i show that the universe had a highly dynamic and divergent state at the moment of the Big Bang $(t = 0)$, with rapid expansion and significant distortion. However, as time moves forward, the universe's expansion becomes more isotropic and less distorted, eventually leading to a near-zero expansion rate as it approaches infinity. This reflects the evolution of the universe from its initial singularity to a more uniform and less distorted state. While the Eq. (42) represents the isotropic condition for the universe. It asserts that when $n = 1$, the ratio σ^2/θ^2 equals zero, signifying complete isotropy. In an isotropic universe, expansion is uniform in all directions, adhering to the cosmological principle. However, when n is not equal to 1 ($n \neq 1$) the ratio σ^2/θ^2 remains a non-zero constant. This implies that for values $n \neq 1$, the universe exhibits anisotropy, with a constant, non-zero value of σ^2/θ^2. In an anisotropic universe, expansion rates vary in different spatial directions, indicating a lack of isotropy and a preferred direction of expansion.

So, Eq. (42) essentially tells us that for $n = 1$, the universe is perfectly isotropic, but for $n \neq 1$, it is anisotropic, and the degree of anisotropy is represented by the constant value of σ^2/θ^2.

By substituting the equation of state (EoS) for DE $p = \xi\rho$ where $\xi = \frac{1}{3}$ into the field equations mentioned in Eqs. (17)–(19), we derive expressions for the energy density $\rho(t)$ pressure $p(t)$ and cosmological constant $\Lambda(t)$ as

$$\begin{aligned} \rho = \frac{3}{2}\left(\frac{\nu}{8\pi+\nu}\right)\Bigg[&-\frac{9n(n-1)}{(n+2)^2}H^2 - \frac{3(n+1)}{n+2}\dot{H} \\ &+ \left\{a_0\sqrt{t}e^{Q(t)}\right\}^{\frac{-6}{n+2}} - \frac{1}{2}\left\{a_0\sqrt{t}e^{Q(t)}\right\}^{\frac{6(n-2)}{n+2}}\Bigg], \end{aligned} \tag{43}$$

$$\begin{aligned} p = -\frac{1}{2}\left(\frac{\nu}{8\pi+\nu}\right)\Bigg[&-\frac{9n(n-1)}{(n+2)^2}H^2 - \frac{3(n+1)}{n+2}\dot{H} \\ &+ \left\{a_0\sqrt{t}e^{Q(t)}\right\}^{\frac{-6}{n+2}} - \frac{1}{2}\left\{a_0\sqrt{t}e^{Q(t)}\right\}^{\frac{6(n-2)}{n+2}}\Bigg], \end{aligned} \tag{44}$$

$$\begin{aligned} \Lambda(t) = \frac{1}{2}\Bigg[&\frac{9n(n-1)+54}{(n+2)^2}H^2 + \frac{3(n+5)}{n+2}\dot{H} \\ &+ \left\{a_0\sqrt{t}e^{Q(t)}\right\}^{\frac{-6}{n+2}} - \frac{1}{2}\left\{a_0\sqrt{t}e^{Q(t)}\right\}^{\frac{6(n-2)}{n+2}}\Bigg]. \end{aligned} \tag{45}$$

Figure 6 illustrates a decrease in the universe's energy density as time progresses towards infinity. This signifies that the universe is becoming less dense as it expands. Ultimately, at infinity, the energy density appears to approach a nearly negligible value, indicating a universe with extremely low energy content. This pattern aligns with the current cosmological understanding of an ever-expanding universe.

Figure 7 graph portrays the pressure within the universe as a function of time. Initially, the pressure is negative, indicating a state of attraction or gravitational collapse. As time advances, the pressure increases, potentially signifying the influence of DE or another repulsive force causing cosmic expansion. Ultimately, as time approaches infinity, the pressure tends to zero, implying a balance between attractive and repulsive forces and the universe settling into a stable state. This behavior aligns with cosmological models involving DE.

The graph 8 of the cosmological constant Λ depicts a positive, diminishing function with time, approaching infinitesimal values. This behavior signifies the role

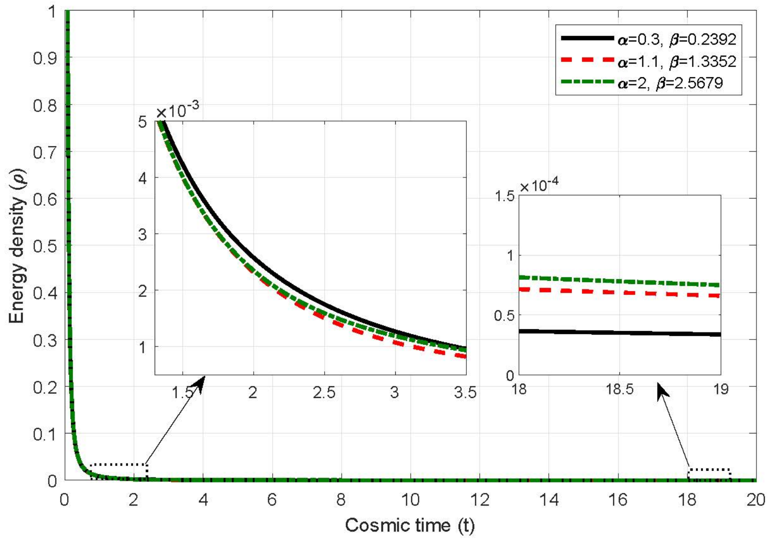

Figure 6. Evolution of energy density (ρ) over cosmic time t. The energy density indicating a dense early universe at $t = 0$, and decreases to zero as $t \to \infty$ approaches infinity for selected values of α and β, with $\nu = 0.1$,$a_0 = 1$, and $n = 0.9$.

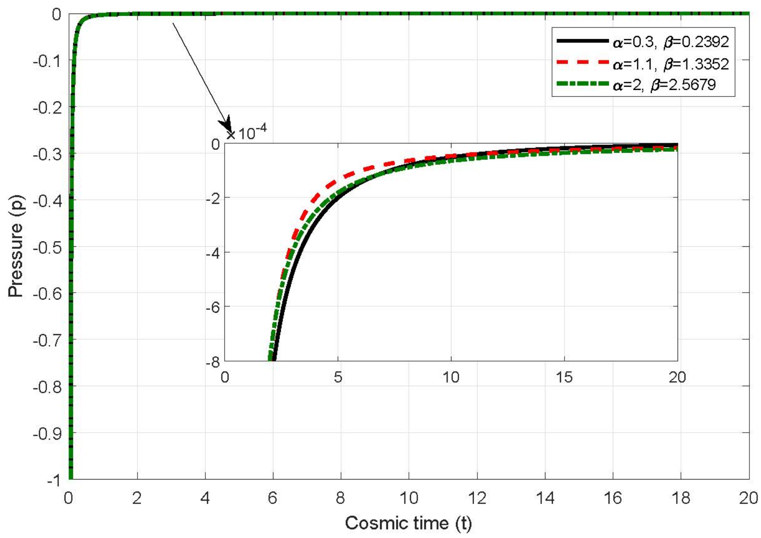

Figure 7. Evolution of pressure p over cosmic time t. The pressure exhibits a highly negative value at $t = 0$ and gradually approaches 0 as $t \to \infty$ for selected values of α and β, with $\nu = 0.1$,$a_0 = 1$, and $n = 0.9$

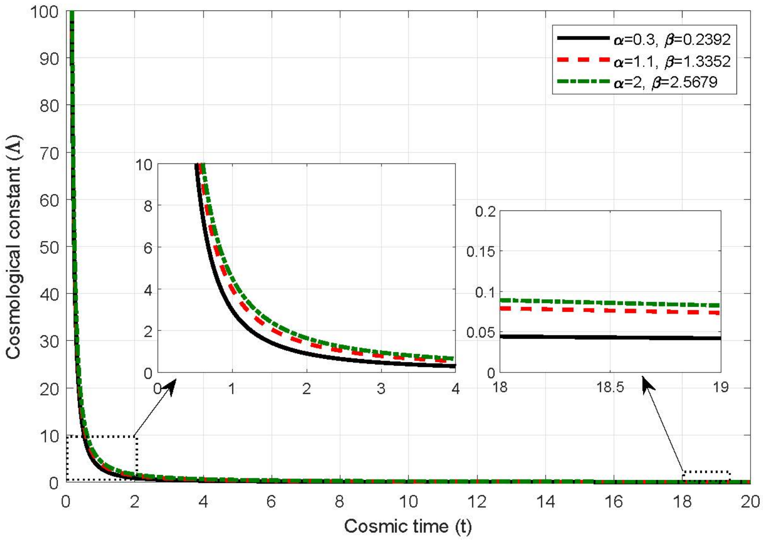

Figure 8. Evolution of the cosmological constant (Λ) over cosmic time t. The value is initially very high at $t = 0$ and decreases to zero as t approaches infinity for selected values of α and β, with $\nu = 0.1$, $a_0 = 1$, and $n = 0.9$.

of DE in the universe. DE associated with a positive Λ, leads to cosmic acceleration. As time progresses, DE's influence grows, and other forms of matter become less significant. In the distant future, DE is expected to dominate, driving exponential cosmic expansion, potentially resulting in a "big rip" scenario.

In the upcoming section, we will assess the model's viability, predictions, theory validity, and the universe's fate by examining the energy conditions of the model.

6 Energy Conditions

Energy conditions play an important role in cosmology and general relativity as they impose physical constraints on the distribution of matter and energy in the universe [51]. These conditions help in understanding how the matter content influences the geometry of spacetime and the dynamics of cosmic expansion. They are also useful for testing the physical viability of cosmological models and studying the large-scale evolution of the universe. In this work, we examine the time-varying behavior of the following energy conditions:

Weak energy condition (WEC): The WEC requires that the energy density satisfies $\rho \geq 0$ and $\rho - p \geq 0$, ensuring that the local energy density remains non-negative.

Null energy condition (NEC): The NEC demands that $\rho + p \geq 0$, which restricts the occurrence of exotic spacetime configurations.

Strong energy condition (SEC): The SEC requires $\rho + 3p \geq 0$ along with the WEC, imposing a stronger constraint on the matter content that is important in gravitational collapse and cosmological dynamics.

In Figure 9, the WEC and NEC start with positive values but eventually tend toward zero. Meanwhile, the SEC consistently remains at zero. This may reflect changing energy distributions in the system, adhering to or deviating from these energy conditions in the context of GR and cosmology.

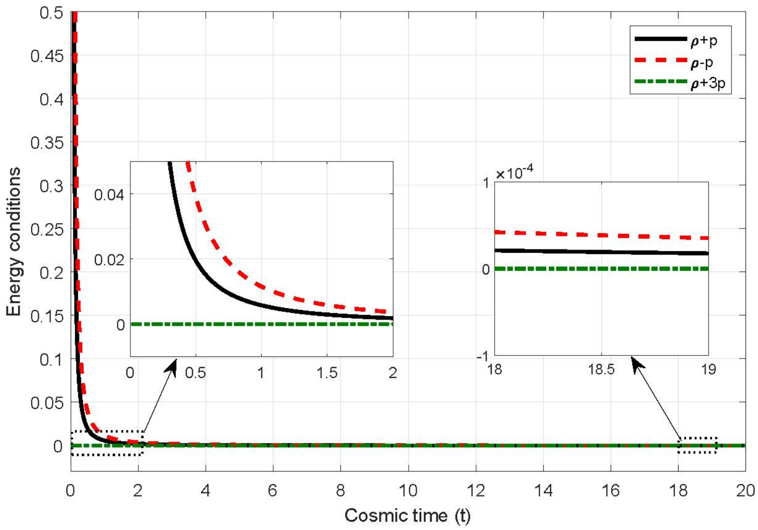

Figure 9. Analysis of energy conditions over cosmic time t. The graph illustrates the behavior of various energy conditions, reflecting the dynamics of the universe as it evolves for $\alpha = 0.3$, $\beta = 0.2392$, $\nu = 0.1$, $a_0 = 1$, and $n = 0.9$.

7 Concluding Remark

The present investigation examines the functional form $f(R,T) = \nu R + \nu T$ within the framework of $f(R,T)$ gravity for an anisotropic Bianchi type-IX spacetime, employing the FLVDP of the form $q = \dfrac{1-\alpha t}{1+\beta t}$. The proposed model provides important insights into the dynamics of cosmic evolution by exploring the interplay between DE, anisotropy, and key cosmological parameters. By deriving exact solutions to the modified field equations, the model successfully describes the transition of the universe from an early decelerating phase to the presently observed accelerating phase, which is essential for understanding the large-scale dynamical behavior of the universe.

The analysis further suggests that the universe gradually evolves toward isotropy, as the initial anisotropies diminish with cosmic expansion. The behavior of fundamental cosmological parameters such as the Hubble parameter (H), scale factor (a), shear scalar (σ^2), and expansion scalar (θ) consistently supports the scenario of an expanding universe. Moreover, the graphical behavior of the energy density (ρ) shows a decreasing trend with cosmic time, while the pressure (p) remains negative, indicating the dominance of DE in driving the accelerated expansion. The cosmological constant (Λ) also evolves toward a very small positive value at late times, as illustrated in Figures 6, 7, and 8, which is consistent with current observational expectations.

Overall, the results obtained in this work contribute to a deeper understanding of the role of DE and anisotropy in cosmic evolution. The analytical framework and graphical analysis presented here provide a useful basis for further investigations into anisotropic cosmological models and their implications for the past, present, and future dynamics of the universe.

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