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Notes on Geometry of Helicoidal Modeling

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Abstract. The use of rototranslations based on rank-2 tensor fields of synectic dual functions for the description of motion is presented. The properties of curvature are derived.

KEY WORDS: Dual functions, Helicoidal modeling, Materials with microstructure, Particle oriento-position, Rototranslation.

1 Introduction

Dual numbers arise naturally in General and Mathematical Physics, where they provide an algebraic representation of first-order infinitesimal structures. They play an essential role in the geometric formulation of physical theories, particularly in the description of tangent spaces, Lie algebras, and infinitesimal transformations. As such, dual numbers constitute a fundamental tool in the mathematical language underlying classical mechanics and related geometric frameworks.

We can also mention a nice application of dual numbers in Continuum Mechanics, where a modeling materials with microstructure often necessitates the incorporation of continua with additional degrees of freedom beyond the traditional three degrees associated with the displacement field. These extra degrees of freedom are linked to additional fields, such as independent rotational fields. The inclusion of such fields in the continuum description results in the formulation of a Cosserat continuum. This approach is particularly useful for materials with layered structures and soft interfaces, or in cases where slip between layers is possible. When these layers undergo bending due to applied loads, the resulting deformation may occur independently of macroscopic displacement [1].

An excellent, and arguably fundamental, reference for an introduction to dual numbers, dual vectors, and dual-valued functions is the paper by Angeles [2]. However, it should be noted that even in this paper not all aspects are explained in full detail. We therefore consider it necessary to first address the derivative of

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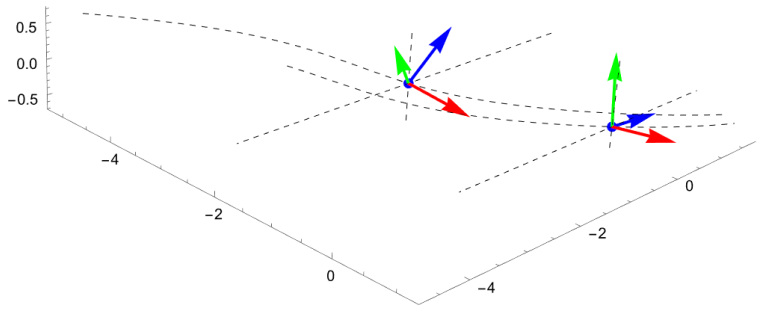

Figure 1. *Modeling materials with microstructure. Material lines.*

a dual-valued function. The conditions of differentiability may then be reflected in a certain visual representation, which we propose.

For modeling materials with internal structure (such as Cosserat media), we use dual number-based helicoidal modeling here. This paper presents several new results that refine the geometric description in particular. These results are formulated in the form of three propositions.

2 Dual Numbers, Dual Functions and the Dual Differentiaton

2.1 Synectic functions on $\mathbb{D}$

Dual numbers are defined as numbers of a form $\widehat{a} = a_0 + a_1\varepsilon$, where a_0, a_1 are real numbers and where the adjoint "infinitesimal" element ε possesses the essential property $\varepsilon^2 = 0$. Dual numbers with $a_0 \neq 0$ are called *regular dual numbers* and dual numbers with $a_0 = 0$ are called *nilpotent dual numbers*. The set of dual numbers is denoted by $\mathbb{D}$ and it forms a two-dimensional commutative unital associative $\mathbb{R}$-algebra. A dual-valued function $\widehat{\varphi}\colon \mathbb{D} \to \mathbb{D}$ of a dual variable $\widehat{x} = x_0 + x_1\varepsilon$ can be represented in the form $\widehat{\varphi}(x_0 + x_1\varepsilon) = \varphi_0(x_0, x_1) + \varphi_1(x_0, x_1)\varepsilon$. We will now deal with the differentiability of dual functions. The conditions below have been known for a long time, but for the autonomy of our text we will state the derivation here. Let us first note that $\mathbb{D}$ as a set is nothing but $\mathbb{R}^2$ (in which a certain multiplication is defined). The usual Euclidean topology is considered in $\mathbb{R}^2$. We say that $\widehat{\varphi}(\widehat{x})$ is *(dual) differentiable at* $\widehat{a}$, if the limit $\lim\limits_{\widehat{h}\to 0} \dfrac{\widehat{\varphi}(\widehat{a}+\widehat{h}) - \widehat{\varphi}(\widehat{a})}{\widehat{h}}$ exists. The value of this limit is called the *(dual) derivative of* $\widehat{\varphi}$ *at* $\widehat{a}$. The increment $\widehat{h}$ must always be a regular dual number, otherwise division by it is not defined.

Proposition 1. *Let $\widehat{\varphi}\colon \mathbb{D} \to \mathbb{D}$ be a dual-valued function of a dual variable $\widehat{x} = x_0 + x_1\varepsilon$. Then $\widehat{\varphi}$ is (dual) differentiable at $\widehat{a} = a_0 + a_1\varepsilon$ if and only if*

$$\frac{\partial \varphi_0}{\partial x_0}(a_0, a_1) = \frac{\partial \varphi_1}{\partial x_1}(a_0, a_1) \qquad \textit{and} \qquad \frac{\partial \varphi_0}{\partial x_1}(a_0, a_1) = 0. \tag{1}$$

Proof. $\Rightarrow$: We assume that $\widehat{\varphi}'(\widehat{a}) = \lim\limits_{\widehat{h}\to 0} \dfrac{\widehat{\varphi}(\widehat{a}+\widehat{h}) - \widehat{\varphi}(\widehat{a})}{\widehat{h}}$ exists. That is, we get the same number whether we go with $\widehat{h}$ to zero from any direction $\vec{d}$, say from a direction $\vec{d} = (\cos\alpha, \sin\alpha)$ (but of course with the requirement $\alpha \neq \pi/n + k\pi$, $k \in \mathbb{Z}$, that we do not divide by a nilpotent number). Therefore we express $\widehat{h} = t\cos\alpha + t\varepsilon\sin\alpha$ and then

$$\begin{aligned}
\lim_{\widehat{h}\to 0} \frac{\widehat{\varphi}(\widehat{a}+\widehat{h}) - \widehat{\varphi}(\widehat{a})}{\widehat{h}} &= \lim_{t\to 0} \frac{\widehat{\varphi}(\widehat{a} + t\cos\alpha + t\varepsilon\sin\alpha) - \widehat{\varphi}(\widehat{a})}{t\cos\alpha + t\varepsilon\sin\alpha} \\
&= \frac{1}{\cos\alpha + \varepsilon\sin\alpha}\Big(\lim_{t\to 0} \frac{\varphi_0(a_0 + t\cos\alpha, a_1 + t\sin\alpha) - \varphi_0(a_0, a_1)}{t} \\
&\qquad + \varepsilon \lim_{t\to 0} \frac{\varphi_1(a_0 + t\cos\alpha, a_1 + t\sin\alpha) - \varphi_1(a_0, a_1)}{t}\Big) \\
&= \frac{1}{\cos\alpha + \varepsilon\sin\alpha}\Big({\varphi_0}'_{\vec{d}}(a_0, a_1) + \varepsilon {\varphi_1}'_{\vec{d}}(a_0, a_1)\Big) \\
&= \frac{\cos\alpha - \varepsilon\sin\alpha}{\cos^2\alpha}\Big({\varphi_0}'_{\vec{d}}(a_0, a_1) + \varepsilon {\varphi_1}'_{\vec{d}}(a_0, a_1)\Big) \\
&= \frac{1}{\cos\alpha}{\varphi_0}'_{\vec{d}}(a_0, a_1) + \Big(\frac{1}{\cos\alpha}{\varphi_1}'_{\vec{d}}(a_0, a_1) - \frac{\sin\alpha}{\cos^2\alpha}{\varphi_0}'_{\vec{d}}(a_0, a_1)\Big)\varepsilon.
\end{aligned}$$

Expressing the directional derivatives using the gradient, we get

$$\begin{aligned}
\frac{1}{\cos\alpha}{\varphi_0}'_{\vec{d}}(a_0, a_1) &= \frac{1}{\cos\alpha}\Big(\frac{\partial \varphi_0}{\partial x_0}(a_0, a_1)\cos\alpha + \frac{\partial \varphi_0}{\partial x_1}(a_0, a_1)\sin\alpha\Big) \\
&= \frac{\partial \varphi_0}{\partial x_0}(a_0, a_1) + \frac{\partial \varphi_0}{\partial x_1}(a_0, a_1)\tan\alpha
\end{aligned}$$

for the regular part and

$$\begin{aligned}
\frac{1}{\cos\alpha}{\varphi_1}'_{\vec{d}}(a_0, a_1) &- \frac{\sin\alpha}{\cos^2\alpha}{\varphi_0}'_{\vec{d}}(a_0, a_1) \\
&= \frac{1}{\cos\alpha}\Big(\frac{\partial \varphi_1}{\partial x_0}(a_0, a_1)\cos\alpha + \frac{\partial \varphi_1}{\partial x_1}(a_0, a_1)\sin\alpha\Big) \\
&\quad - \frac{\sin\alpha}{\cos^2\alpha}\Big(\frac{\partial \varphi_0}{\partial x_0}(a_0, a_1)\cos\alpha + \frac{\partial \varphi_0}{\partial x_1}(a_0, a_1)\sin\alpha\Big) \\
&= \frac{\partial \varphi_1}{\partial x_0}(a_0, a_1) + \Big(\frac{\partial \varphi_1}{\partial x_1}(a_0, a_1) - \frac{\partial \varphi_0}{\partial x_0}(a_0, a_1)\Big)\tan\alpha \\
&\quad - \frac{\partial \varphi_0}{\partial x_1}(a_0, a_1)\tan^2\alpha.
\end{aligned}$$

for the nilpotent part. Since neither the regular nor the nilpotent part depends on the angle α, we have

$$\frac{\partial \varphi_0}{\partial x_1}(a_0, a_1) = 0 \qquad \text{and} \qquad \frac{\partial \varphi_0}{\partial x_0}(a_0, a_1) = \frac{\partial \varphi_1}{\partial x_1}(a_0, a_1).$$

$\Leftarrow$: The fulfillment of the conditions (1) reads as $\varphi_0(x_0, x_1) = \varphi_0(x_0)$ and $\varphi_1(x_0, x_1) = \psi(x_0) + \varphi_0'(x_0)x_1$ in the point $\widehat{a}$. Then

$$\begin{aligned}\lim_{\widehat{h}\to 0} \frac{\widehat{\varphi}(\widehat{a}+\widehat{h}) - \widehat{\varphi}(\widehat{a})}{\widehat{h}} = \lim_{[h_0,h_1]\to[0,0]} \Big[&\varphi_0(a_0+h_0) - \varphi_0(a_0) + (\psi(a_0+h_0) \\ &- \psi(a_0) + \varphi_0'(a_0+h_0)(a_1+h_1) - \varphi_0'(a_0)a_1)\varepsilon\Big](h_0+h_1\varepsilon)^{-1}.\end{aligned}$$

Now we multiply by $\dfrac{h_0 - h_1\varepsilon}{h_0 - h_1\varepsilon}$. We obtain

$$\begin{aligned}&\lim_{h_0\to 0} \frac{\varphi_0(a_0+h_0) - \varphi_0(a_0)}{h_0} + \lim_{h_0\to 0} \frac{\psi(a_0+h_0) - \psi_0(a_0)}{h_0}\varepsilon \\ &\quad + \lim_{h_0\to 0} \frac{\varphi_0'(a_0+h_0) - \varphi_0'(a_0)}{h_0} a_1\varepsilon \\ &\quad + \lim_{[h_0,h_1]\to[0,0]} \frac{\varphi_0'(a_0)h_0h_1 - \big(\varphi_0(a_0+h_0) - \varphi_0(a_0)\big)h_1}{h_0^2} a_1\varepsilon \\ &\quad = \varphi_0'(a_0) + \psi'(a_0)\varepsilon + \varphi_0''(a_0)a_1\varepsilon \\ &\quad + \lim_{[h_0,h_1]\to[0,0]} \frac{\varphi_0'(a_0) - \frac{\varphi_0(a_0+h_0)-\varphi_0(a_0)}{h_0}}{h_0} a_1h_1\varepsilon \\ &\quad = \varphi_0'(a_0) + \big(\psi'(a_0) + \varphi_0''(a_0)a_1\big)\varepsilon.\end{aligned}$$

Thus, existing partial derivatives of φ_0 and φ_1 satisfying conditions (1) enable the dual derivative of a dual function. □

The conditions (1) for the dual differentiability are an analogy with Cauchy-Riemann equations for complex-valued functions of a complex variable. We remark that, in complex calculus, the term "holomorphic function" which is used for a complex function (complex) differentiable in a neighbourhood of each point in its domain, was preceded by the term "synectic function". Yes, let us just mention that Augustin Louis Cauchy was the first mathematician who systematically studied functions from $\mathbb{C}$ to $\mathbb{C}$ and his followers Charles Briot and Jean Claude Bouquet wrote the first textbook focused on complex calculus where they defined the synectic function ([3]); the term synectic function was taken also by Eduard Study ([4]), who was already researching dual functions. Although in modern complex analysis the word synectic has been displaced by the word holomorphic, for dual numbers let us retain, perhaps a little nostalgically, the word synectic.

Definition 1. Dual-valued functions of a dual variable which are (dual) differentiable in a neighbourhood of each point in its domain, are called *synectic functions*.

Example 1. (i) Every constant function $\widehat{\varphi}(x_0 + x_1\varepsilon) = c_0 + c_1\varepsilon$ is synectic with zero derivative.

(ii) The function $\widehat{\varphi}(x_0 + x_1\varepsilon) = x_0^2 + (x_0^3 + 2x_0x_1 + 3)\varepsilon$ is synectic and its dual derivative is $\widehat{\varphi}'(x_0 + x_1\varepsilon) = 2x_0 + (3x_0^2 + 2x_1)\varepsilon$.

(iii) The function $\widehat{\varphi}(x_0 + x_1\varepsilon) = \sin(x_0) + x_1\cos(x_0)\varepsilon$ is also synectic and its dual derivative is $\widehat{\varphi}'(x_0 + x_1\varepsilon) = \cos(x_0) - x_1\sin(x_0)\varepsilon$. This function is called the *dual sine*.

Remark 1. Because dual differentiation is linear and obeys the product, quotient, and chain rules, the sums, products and compositions of synectic functions are synectic, too.

2.2 A consideration of how to visualize dual numbers and dual functions

Now, let us deal with the question of how to appropriately visualize dual numbers and dual functions. Of course, we could visualize dual numbers as points in the Euclidean plane, where the coordinates are the regular and nilpotent components of a dual number. However, we decided to use the arctangent scale for the nilpotent component, just because of its nilpotency property.

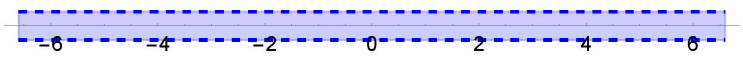

Figure 2. Dual numbers where the arctangent scale is used for the nilpotent component. They are then a kind of "thick" straight line.

Since dual functions are described by two real functions $\varphi_0(x_0, x_1)$ and $\varphi_1(x_0, x_1)$, we can draw their graph as a graph of two functions of two real variables. We keep the arctangent scale for the nilpotent component. So there are two strips, one is placed above (or below) the other. Note that we can visually observe the property of synectivity!

3 Differentiation of Rototranslations

The bijection $V = \begin{pmatrix} 0 & -v_3 & v_2 \\ v_3 & 0 & -v_1 \\ -v_2 & v_1 & 0 \end{pmatrix} \xmapsto{\mathsf{ax}} (v_1, v_2, v_3) = \mathbf{v}$ between a skew-symmetric matrix V and a vector $\mathbf{v}$ will be considered and the vector $\mathbf{v}$ will be called the *axial vector*. Of course, this bijection also works for tensor fields and vector fields. We denote ax^{-1} by xa. Then

(i) $\mathsf{xa}(\mathbf{v})\mathbf{w} = \mathbf{v} \times \mathbf{w}$

(ii) $\mathsf{xa}(\mathbf{v})$ is often written as $\mathbf{v}\times$, see [5]

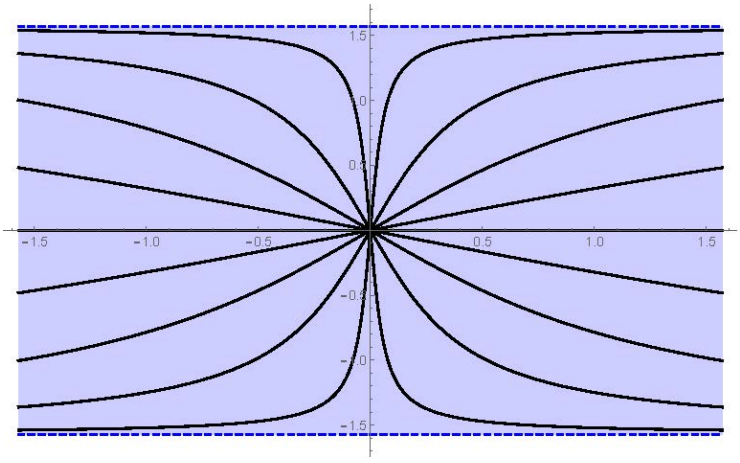

Figure 3. Dual numbers again, somewhat zoomed. Curves represent c-multiples of regular dual numbers $x_0 + x_1\varepsilon$, $c \in \mathbb{R}$.

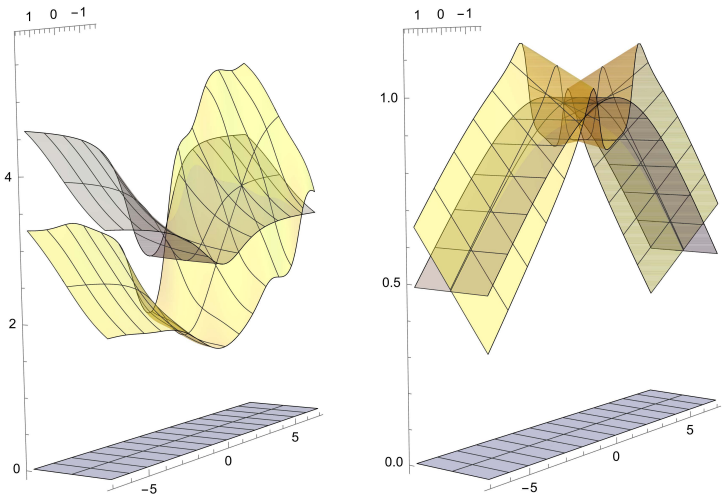

Figure 4. Graphs of two dual functions: a non-synectic one and a synectic one. Here, $\varphi_0(x_0, x_1)$ is colored gray and $\varphi_0(x_0, x_1) + \varphi_1(x_0, x_1)$ is yellow.

(iii) $\mathsf{xa}(\mathbf{v})A$ is the expression of the cross product $\mathbf{v} \times A$ of a vector $\mathbf{v}$ and a matrix A

The construction of ax works for both the real and the dual case.

3.1 Elementary rototranslational fields

Consider for a moment that translations live in a different world: with X, Y and Z coordinates. The translation $\mathbf{t}_X(A)$ by A in the direction of the X-axis is represented by the vector field $\mathbf{t}_X(A) = (A, 0, 0)$. This field can be viewed as the *elementary X-translational field*.

But now, let us connect rotations and translations into the dual space; we take $\widehat{x} = x + X\varepsilon$ and $\widehat{\alpha} = \alpha + A\varepsilon$. Then the *elementary $\widehat{x}$-rototranslational field* $\widehat{RT}_{\widehat{x}}(\widehat{\alpha})$ is defined as

$$\begin{aligned}\widehat{RT}_{\widehat{x}}(\widehat{\alpha}) &= R_x(\alpha) + \varepsilon \mathbf{t}_X(A) \times R_x(\alpha)\\ &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\alpha - \varepsilon A \sin\alpha & -\sin\alpha - \varepsilon A\cos\alpha \\ 0 & \sin\alpha + \varepsilon A\cos\alpha & \cos\alpha - \varepsilon A \sin\alpha \end{pmatrix}.\end{aligned}$$

Analogously, the *elementary $\widehat{y}$-rototranslational field* $\widehat{RT}_{\widehat{y}}(\widehat{\beta})$ and the *elementary $\widehat{z}$-rototranslational field* $\widehat{RT}_{\widehat{z}}(\widehat{\gamma})$ are expressed by

$$\widehat{RT}_{\widehat{y}}(\widehat{\beta}) = \begin{pmatrix} \cos\beta - \varepsilon B \sin\beta & 0 & \sin\beta + \varepsilon B\cos\beta \\ 0 & 1 & 0 \\ -\sin\beta - \varepsilon B\cos\beta & 0 & \cos\beta - \varepsilon B \sin\beta \end{pmatrix}$$

and

$$\widehat{RT}_{\widehat{z}}(\widehat{\gamma}) = \begin{pmatrix} \cos\gamma - \varepsilon C \sin\gamma & -\sin\gamma - \varepsilon C\cos\gamma & 0 \\ \sin\gamma + \varepsilon C\cos\gamma & \cos\gamma - \varepsilon C \sin\gamma & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

respectively.

It is obvious that the entries of these matrices are synectic functions. In fact these entries are nothing but dual cosines, dual sines, 1 and 0. By multiplying matrices $\widehat{RT}_{\widehat{x}}(\widehat{\alpha})$, $\widehat{RT}_{\widehat{y}}(\widehat{\beta})$ and $\widehat{RT}_{\widehat{z}}(\widehat{\gamma})$ together in the appropriate sequence, any rototranslational field can be expressed. Thus (see Remark 1) its entries are also synectic functions.

This can be summarized in the following claim.

Proposition 2. *Every rototranslational field $\widehat{H} = \Phi + \varepsilon \mathbf{t} \times \Phi$ is a rank-2 tensor field whose components are synectic functions.*

3.2 Eigenvectors

We now consider a rototranslational matrix, and in order to investigate the properties of the rototranslation, it is essential to focus on its eigenvectors. We restrict ourselves to the real case, that is, more precisely, to the dual extension of the real numbers. In this context, the situation is analogous to that of ordinary rotation matrices, but it is nonetheless more general.

So, we compute eigenvalues and eigenvectors of the rototranslational matrix $\widehat{RT}_{\widehat{x}}(\widehat{\alpha}) = R_x(\alpha) + \varepsilon \mathbf{t}_X(A) \times R_x(\alpha)$.

Solving the equation $\widehat{RT}_{\widehat{x}}(\widehat{\alpha})\widehat{\mathbf{v}} = \widehat{\lambda}\widehat{\mathbf{v}}$ (for nonzero $\widehat{\mathbf{v}}$), we obtain two types of solutions:

(I) $\widehat{\lambda} = 1 + K\varepsilon$ ($K \in \mathbb{R}$), $\widehat{\mathbf{v}} = (h\varepsilon, 0, 0)$ ($h \in \mathbb{R}^*$); these are nilpotent eigenvectors;

(II) $\widehat{\lambda} = 1$, $\widehat{\mathbf{v}} = (h + H\varepsilon, 0, 0)$ ($h \in \mathbb{R}^*$, $H \in \mathbb{R}$); these are regular eigenvectors.

Now, let $\mathbf{t}$ denote a general translation, i.e. by A in the direction of the X-axis, by B in the direction of the Y-axis and by C in the direction of the Z-axis.

Eigenvalues and eigenvectors for the matrix $R_x(\alpha) + \varepsilon \mathbf{t} \times R_x(\alpha)$ come out as follows:

(I) $\widehat{\lambda} = 1 + K\varepsilon$ ($K \in \mathbb{R}$), $\widehat{\mathbf{v}} = (h\varepsilon, 0, 0)$ ($h \in \mathbb{R}^*$); these are nilpotent eigenvectors;

(II) $\widehat{\lambda} = 1$, $\widehat{\mathbf{v}} = \left(h + H\varepsilon, \frac{1}{2}h\left(B\cot\frac{\alpha}{2} + C\right)\varepsilon, \frac{1}{2}h\left(-B + C\cot\frac{\alpha}{2}\right)\varepsilon\right)$ ($h \in \mathbb{R}^*$, $H \in \mathbb{R}$); these are regular eigenvectors.

4 Particle Oriento-Position

We consider a particle oriento-position $\widehat{A} = \alpha + \varepsilon \mathbf{x} \times \alpha$.

The rototranslation between material particles is described by a quantity called curvature tensor and its linearization gives the characteristic curvature and differential vectors underlying the helicoidal modeling in both the sense of the body geometric description and the evolution of a deforming body.

4.1 Curvature

Curvature tensor is defined by its components k_j (i. e. components of the vector along material coordinate ξ^j) by

$$\mathsf{xa}\left(A^{\mathsf{T}} k_j\right) = A^{\mathsf{T}} A_{,j}, \quad j = 1, 2, 3,$$

where $A_{,j}$ is the derivative of oriento-position A with respect to ξ^j.

Compared to [5], the formulas we derive for curvature vectors are indeed for general non-singular A.

Starting with

$$k_j = \left(A^{\mathsf{T}}\right)^{-1} \mathsf{ax}\left(A^{\mathsf{T}} A_{,j}\right) \tag{2}$$

and separating regular and nilpotent parts, $k_j = k_j^{\mathrm{ang}} + \varepsilon k_j^{\mathrm{lin}}$, we obtain

$$k_j^{\mathrm{ang}} = \left(\alpha^{\mathsf{T}}\right)^{-1} \mathsf{ax}\left(\alpha^{\mathsf{T}} \alpha_{,j}\right) \tag{3}$$

$$k_j^{\mathrm{lin}} = \left(\alpha^{\mathsf{T}}\right)^{-1} \mathsf{ax}\left(\alpha^{\mathsf{T}}\mathsf{xa}\left(x_{,j}\right)\alpha\right) + \mathsf{xa}(x)\left(\alpha^{\mathsf{T}}\right)^{-1} \mathsf{ax}\left(\alpha^{\mathsf{T}}\alpha_{,j}\right) \tag{4}$$

In what follows, we make use of the following linear algebraic claim.

Proposition 3. *For a third order matrix M and a three-dimensional vector v $\mathsf{ax}\left(M^{\mathsf{T}}\mathsf{xa}\left(v\right)M\right) = \operatorname{adj} M \cdot v$ holds.*

Proof. As $\mathsf{xa}\left(v\right)$ is skew-symmetric, $\left(M^{\mathsf{T}}\mathsf{xa}\left(v\right)M\right)^{\mathsf{T}} = M^{\mathsf{T}}\left(\mathsf{xa}^{(}v)\right)^{\mathsf{T}} M = M^{\mathsf{T}}\left(-\mathsf{xa}(v)\right)M = -M^{\mathsf{T}}\mathsf{xa}(v)M$ and $M^{\mathsf{T}}\mathsf{xa}\left(v\right)M$ is skew-symmetric, too. The use of the ax map is therefore correct. It is enough to show that the left hand side is in the well known form of the product of a adjugate matrix with a vector, which we do by direct calculation. So both sides equal

$$\begin{aligned}\Big(&(m_{22}m_{33}-m_{23}m_{32})v_1+(m_{13}m_{32}-m_{12}m_{33})v_2+(m_{12}m_{23}-m_{13}m_{22})v_3,\\ &(m_{23}m_{31}-m_{21}m_{33})v_1+(m_{11}m_{33}-m_{13}m_{31})v_2+(m_{13}m_{21}-m_{11}m_{23})v_3,\\ &(m_{21}m_{32}-m_{22}m_{31})v_1+(m_{12}m_{31}-m_{11}m_{32})v_2+(m_{11}m_{22}-m_{12}m_{21})v_3\Big).\end{aligned}$$

□

Thanks to the proposition, we can simplify the formulas above. In particular, if α is a constant matrix, then $\alpha_{,j} = 0$ and $k_j^{\mathrm{ang}} = o$, $k_j^{\mathrm{lin}} = \det\alpha\left(\alpha\alpha^{\mathsf{T}}\right)^{-1} x_{,j}$.

Let us further note this fact.

Proposition 4.

$$k_j^{\mathrm{lin}} - \mathsf{xa}(x)k_j^{\mathrm{ang}} = x_{,j} \textit{ for all } j = 1,2,3 \iff \alpha \textit{ is orthonormal.}$$

Proof. Using the previous proposition, we compute

$$k_j^{\mathrm{lin}} - \mathsf{xa}(x)k_j^{\mathrm{ang}} = \left(\alpha^{\mathsf{T}}\right)^{-1}\operatorname{adj}\alpha \cdot x_{,j}.$$

It is clear that $\left(\alpha^{\mathsf{T}}\right)^{-1}\operatorname{adj}\alpha \cdot x_{,j} = x_{,j}$ for an orthonormal α. In the opposite direction, let us assume that $\left(\alpha^{\mathsf{T}}\right)^{-1}\operatorname{adj}\alpha$ is the identity matrix. First, $\det\operatorname{adj}\alpha = (\det\alpha)^{n-1}$ in general, so for $n = 3$ it gives $\det\operatorname{adj}\alpha = (\det\alpha)^2$. For determinants, the equation $\left(\alpha^{\mathsf{T}}\right)^{-1}\operatorname{adj}\alpha = I$ gives $\frac{1}{\det\alpha}\cdot(\det\alpha)^2 = 1$, hence $\det\alpha = 1$. Now, $\left(\alpha^{\mathsf{T}}\right)^{-1}\operatorname{adj}\alpha = \det\alpha\left(\alpha^{\mathsf{T}}\right)^{-1}\alpha^{-1} = \left(\alpha^{\mathsf{T}}\right)^{-1}\alpha^{-1} = I$ and this is orthogonality. □

It is demonstrated in [5] that the self-based curvature obtained using the arm operator the nilpotent part of curvature is just the gradient of x. This is true, but only for orthonormal α, as Proposition 4 shows.

Proposition 5. *For orthonormal α, $A_{,j} = \mathsf{xa}(k_j)A$.*

Proof. For orthonormal α, $A^{\mathsf{T}} = A^{-1} = \operatorname{adj} A$. Using this fact, the definition of curvature and Proposition 3, we derive

$$A_{,j} = A\,\mathsf{xa}(A^{\mathsf{T}}k_j) = A\,\mathsf{xa}\left(\mathsf{ax}\left(\left(A^{\mathsf{T}}\mathsf{xa}\,(k_j)\,A\right)\right)\right) = AA^{\mathsf{T}}\mathsf{xa}\,(k_j)\,A = \mathsf{xa}\,(k_j)\,A.$$

□

For given k_j, $j = 1, 2, 3$, we observe a matrix system of partial differential equations $A_{,j} = T_j A, \qquad j = 1, 2, 3$, for dual functions depending on three real variables ξ_1, ξ_2 and ξ_3 with a skew-symmetric coefficient matrices $T_j = \mathsf{xa}\,(k_j)$.

4.2 The PDE system

We solve the system $A_{,j} = T_j A, \qquad j = 1, \ldots, n$, for skew-symmetric matrices of coefficients T_j. Our goal is to understand the $n = 3$ case, but let us first look at the simpler $n = 2$ case.

For every column of the matrix A we have to solve

$$\begin{pmatrix} v'_x(x,y) \\ w'_x(x,y) \end{pmatrix} = \begin{pmatrix} 0 & \alpha(x,y) \\ -\alpha(x,y) & 0 \end{pmatrix} \begin{pmatrix} v(x,y) \\ w(x,y) \end{pmatrix},$$

$$\begin{pmatrix} v'_y(x,y) \\ w'_y(x,y) \end{pmatrix} = \begin{pmatrix} 0 & \beta(x,y) \\ -\beta(x,y) & 0 \end{pmatrix} \begin{pmatrix} v(x,y) \\ w(x,y) \end{pmatrix}.$$

Equations

$$\begin{aligned} v'_x &= \alpha w \\ w'_x &= -\alpha v \end{aligned} \tag{5}$$

give $v'_x v + w'_x w = 0$ and, by integration, $v^2(x,y) + w^2(x,y) = c^2 \quad$ for all x, y. Hence there is a function $\varphi(x,y)$ such that

$$\begin{aligned} v &= c\cos\varphi \\ w &= c\sin\varphi \end{aligned}$$

and equations (5) give $\varphi(x,y) = \int_{c_1}^{x} \alpha(h,y)\mathrm{d}h \quad$ for arbitrary c_1. Analogously, $\varphi(x,y) = \int_{c_2}^{x} \beta(x,h)\mathrm{d}h \quad$ for arbitrary c_2. But this means that a solution does not exist, in general, for arbitrary functions $\alpha(x,y)$, $\beta(x,y)$. However, the condition $\alpha'_y = \beta'_x$ allows a solution. In particular, for $\alpha(x,y) = \alpha(x)$ and $\beta(x,y) = \beta(y)$ we can express $v(x,y) = c\cos\left(\int_0^x \alpha(h)dh + \int_0^y \beta(h)dh + d\right)$, $w(x,y) = c\sin\left(\int_0^x \alpha(h)dh + \int_0^y \beta(h)dh + d\right)$.

Note that shift by $\frac{\pi}{2}$ (e.g. choosing $d = \frac{\pi}{2}$) alows to use effectively $-\sin, \cos$ instead $\cos, \sin$, respectively. It has a special meaning for a case of columns

of A such that A is orthogonal: then the solution is the exponential of another matrix.

For instance, with $\eta(x,y)$ standing for $\int_0^x \alpha(h)dh + \int_0^y \beta(h)dh$, the matrix

$$A = \begin{pmatrix} \cos\eta(x,y) & -\sin\eta(x,y) \\ \sin\eta(x,y) & \cos\eta(x,y) \end{pmatrix}$$

equals

$$\exp M, \quad \text{where } M = \begin{pmatrix} 0 & -\eta(x,y) \\ \eta(x,y) & 0 \end{pmatrix}.$$

We thus become acquainted with the difficulty of the problem, and we believe that we also indicate a certain direction for its solution. This modeling is useful not only in kinematics, but also in the study of motion in physical systems as such.

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