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FORM OF THE SPIN-ORBIT POTENTIAL IN A NONLOCAL SEPARABLE INTERACTION

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Abstract. It is shown that for spin-1/2 particle in a nonlocal field, the large component of the Dirac bispinor can be obtained. A rigorous expression for the spin-orbit potential in case of a nonlocal separable interaction is derived.

1. Introduction

The use of the Schrödinger equation with a nonlocal separable interaction achieved definite success in the theoretical nuclear physics, especially in the physics of the few-nucleon systems. In the separable potential the spin-orbit interaction is usually included such that this interaction is considered phenomenologically without theoretical basis [1–4].

In the present work we derived a rigorous expression for the spin-orbit interaction from the large component of the Dirac bispinor in case of a nonlocal separable interaction. The use of a nonlocal potential in the Dirac equation is helpful in this regard that in such case, as in the case of the Schrödinger equation, the problem is solved exactly.

2. Formulation of the Problem

The Dirac equations for the spinors $\phi(\boldsymbol{r})$ and $\chi(\boldsymbol{r})$, for a spin-1/2 particle moving in an external nonlocal field are written in the form (in units $\hbar = 1$):

$$E\phi - c(\boldsymbol{\sigma}.\boldsymbol{p})\chi = \int V_1(\boldsymbol{r},\boldsymbol{r}')\phi(\boldsymbol{r}')\,d\boldsymbol{r}', \tag{1a}$$

$$(E + 2mc^2)\chi - c(\boldsymbol{\sigma}.\boldsymbol{p})\phi = \int V_2(\boldsymbol{r},\boldsymbol{r}')\chi(\boldsymbol{r}')\,d\boldsymbol{r}' \tag{1b}$$

where $\boldsymbol{p} = -i\boldsymbol{\nabla}$ is the momentum operator and $\boldsymbol{\sigma}(\sigma_1, \sigma_2, \sigma_3)$ is the Pauli vector-matrix where the rest energy mc^2 is excreted from the total energy.

For the central interaction, the parity conservation law requires the following form for the nonlocal separable interaction:

$$V = \begin{pmatrix} V_1(\boldsymbol{r}, \boldsymbol{r}') & 0 \\ 0 & V_2(\boldsymbol{r}, \boldsymbol{r}') \end{pmatrix} \tag{2}$$

where

$$\begin{aligned} V_1(\boldsymbol{r}, \boldsymbol{r}') &= V_{jl}(r)V_{jl}(r')\Omega_{jlM}(\boldsymbol{n})\Omega^*_{jlM}(\boldsymbol{n}') \\ V_2(\boldsymbol{r}, \boldsymbol{r}') &= V_{jl'}(r)V_{jl'}(r')\Omega_{jl'M}(\boldsymbol{n})\Omega^*_{jl'M}(\boldsymbol{n}'). \end{aligned} \tag{3}$$

Here $V_{jl}(r)$ is the "form" of the potential and $\Omega_{jlM}(\boldsymbol{n})$ is the spin spherical function [5]; $j = l \pm 1/2$ is the quantum number of the total angular momentum such that $l + l' = 2j$, $\boldsymbol{n} = \boldsymbol{r}/r$, $\boldsymbol{n}' = \boldsymbol{r}'/r$ are the unit vectors in direction $\boldsymbol{r}$ and $\boldsymbol{r}'$, respectively.

Now let us consider the nonrelativistic limit in set (1). Hence, from (1b) we define the small component as

$$\chi(\boldsymbol{r}') = \frac{c(\boldsymbol{\sigma}.\boldsymbol{p})}{E + 2mc^2}\phi(\boldsymbol{r}) + \frac{1}{E + 2mc^2}\int V_2(\boldsymbol{r}, \boldsymbol{r}')\chi(\boldsymbol{r}')\, d\boldsymbol{r}'. \tag{4}$$

Assuming that the following two conditions are satisfied

$$\frac{V_2(\boldsymbol{r}, \boldsymbol{r}')}{E + 2mc^2} \ll 1, \qquad \frac{E}{2mc^2} \ll 1, \tag{5}$$

substituting (4) into (1a) and carrying out the iteration, for the large component of the Dirac bispinor we obtain:

$$\begin{aligned} E\phi = {} & \frac{c^2p^2}{E + 2mc^2}\phi + \frac{c(\boldsymbol{\sigma}.\boldsymbol{p})}{E + 2mc^2}\int d\boldsymbol{r}' V_2(\boldsymbol{r}, \boldsymbol{r}')\frac{c(\boldsymbol{\sigma}.\boldsymbol{p}')}{E + 2mc^2}\phi(\boldsymbol{r}') \\ & + \frac{c(\boldsymbol{\sigma}.\boldsymbol{p})}{E + 2mc^2}\int d\boldsymbol{r}' V_2(\boldsymbol{r}, \boldsymbol{r}')\frac{1}{E + 2mc^2}\int d\boldsymbol{r}'' V_2(\boldsymbol{r}, \boldsymbol{r}'')\frac{c(\boldsymbol{\sigma}.\boldsymbol{p}'')}{E + 2mc^2}\phi(\boldsymbol{r}'') \\ & + \cdots + \int V_1(\boldsymbol{r}, \boldsymbol{r}')\phi(\boldsymbol{r}')\, d\boldsymbol{r}'. \end{aligned} \tag{6}$$

In the nonrelativistic limit Eq. (6) can be written in the form:

$$\begin{aligned} E\phi = {} & \frac{p^2}{2m}\Big(1 - \frac{E}{2mc^2}\Big)\phi + \int V_1(\boldsymbol{r}, \boldsymbol{r}')\phi(\boldsymbol{r}')\, d\boldsymbol{r}' \\ & + \frac{1}{(2mc)^2}\Big(1 - \frac{E}{2mc^2}\Big)(\boldsymbol{\sigma}.\boldsymbol{p})\int d\boldsymbol{r}' V_2(\boldsymbol{r}, \boldsymbol{r}')\Big(1 - \frac{E}{2mc^2}\Big)(\boldsymbol{\sigma}.\boldsymbol{p}')\phi(\boldsymbol{r}') + \cdots + . \end{aligned} \tag{7}$$

In Eq. (7), the main terms are those which are in the Schrödinger equation for a nonlocal potential, i.e.

$$E\phi = \frac{p^2}{2m}\phi + \int V_1(\boldsymbol{r}, \boldsymbol{r}')\phi(\boldsymbol{r}')\, d\boldsymbol{r}'. \tag{8}$$

The remaining terms are considered small (of order v/c) corrections.

3. Expression for the Spin-orbit Interaction

We shall consider the term:

$$U = \frac{1}{(2mc)^2}(\sigma.p)\int V_2(r,r')(\sigma.p')\phi(r')\,dr' \tag{9}$$

in Eq. (7). It is clear that responsible for the spin-orbit interaction is, namely, this term. Let us transform term (9). Noting that $p' = i\nabla$, we can write the identity:

$$p'[V_2(r,r')\phi(r')] = (p'V_2)\phi(r') + V_2(p'\phi). \tag{10}$$

If we demand that at infinity $V_2\phi$ equals zero, then U may be written as:

$$U = -\int(\sigma.p)\left[(\sigma.p')V_2(r,r')\right]\phi(r')\,dr'. \tag{11}$$

Using the relation

$$(\sigma.r)(\sigma.p) = (r.p) + i(\sigma.[r\times p]) \tag{12}$$

where $i = [r \times p]$ is the angular momentum operator, the expression (12) may be given in the form:

$$(\sigma.n)(\sigma.p) = (n.p) + \frac{i}{r}(\sigma.i), \tag{13}$$

from which

$$(\sigma.p) = (\sigma.n)\left\{-i\frac{\partial}{\partial r} + \frac{i}{r}(\sigma.i)\right\}. \tag{14}$$

Then the product $(\sigma.p)(\sigma.p')$, which enters in (11), is presented in the form:

$$(\sigma.p)(\sigma.p') = -(\sigma.n)(\sigma.n')\left\{-\frac{\partial}{\partial r} + \frac{1}{r}(s.i)\right\}\left\{-\frac{\partial}{\partial r'} + \frac{1}{r'}(s.i')\right\}. \tag{15}$$

Here $i' = [r', p']$ is the angular momentum operator, and $s = \frac{1}{2}\sigma$ is the spin operator.

Thus, it is clear that the spin-orbit interaction in case of a nonlocal interaction takes the form:

$$U_{ls}(r,r') = \frac{(\sigma.n)(\sigma.n')}{2mc^2}\left\{\frac{1}{r'}(i.s)\frac{\partial}{\partial r}V_2(r,r') + \frac{1}{r}(i.s)\frac{\partial}{\partial r'}V_2(r,r')\right\}. \tag{16}$$

When the interaction is local then, as it is known, $V_2(r,r')$ is presented in the form:

$$V_2(r,r') = V(r)\delta(r-r') \tag{17}$$

and Eq. (16) reduces to the spin-orbit interaction for the local potential

$$U_{ls}(r) = \frac{1}{2m^2c^2}\frac{1}{r}\frac{\partial V_2(r)}{\partial r}(s.i). \tag{18}$$

Let us now consider a nonlocal central separable interaction. Noting that the potential must take the form of (3), the interaction (16) will be:

$$U_{l's}(\boldsymbol{r},\boldsymbol{r}') = \frac{1}{2m^2c^2}\left\{\frac{\partial V_{l'}}{\partial r}\left[(\boldsymbol{\sigma}.\boldsymbol{n})\Omega_{jl'M}(\boldsymbol{n})\right]\frac{V_{l'}(r')}{r'}(\boldsymbol{i}'.\boldsymbol{s})\left[(\boldsymbol{\sigma}.\boldsymbol{n}')\Omega_{jl'M}(\boldsymbol{n}')\right]\right.$$
$$\left. + \frac{V_l(r)}{r}(\boldsymbol{i}.\boldsymbol{s})\left[(\boldsymbol{\sigma}.\boldsymbol{n})\Omega_{jl'M}(\boldsymbol{n})\right]\frac{\partial V_{l'}(r')}{\partial r'}\left[(\boldsymbol{\sigma}.\boldsymbol{n}')\Omega_{jl'M}(\boldsymbol{n}')\right]\right\}. \tag{19}$$

Accounting the formula [5]

$$(\boldsymbol{\sigma}.\boldsymbol{n})\Omega_{jl'M}(\boldsymbol{n}) = -\Omega_{jlM}(\boldsymbol{n}), \quad (l+l'=2j). \tag{20}$$

We finally obtain

$$U_{ls}(\boldsymbol{r},\boldsymbol{r}') = \frac{1}{2m^2c^2}\left\{\frac{\partial V_{l'}}{\partial r}\Omega_{jlM}(\boldsymbol{n})\frac{V_{l'}(r')}{r'}(\boldsymbol{i}'.\boldsymbol{s})\Omega_{jlM}(\boldsymbol{n}')\right.$$
$$\left. + \frac{V_{l'}(r)}{r}(\boldsymbol{i}.\boldsymbol{s})\Omega_{jlM}(\boldsymbol{n})\frac{\partial V_{l'}(r')}{\partial r'}\Omega_{jlM}(\boldsymbol{n}')\right\} \tag{21}$$

which is the desired spin-orbit potential energy in case of a nonlocal separable interaction.

It should be noted that in the separable interaction it is not necessary to carry out the iteration in Eq. (4). In this case $\chi(r)$ can be expressed directly through $\phi(r)$ in the following form:

$$\chi(\boldsymbol{r}) = \frac{c(\boldsymbol{\sigma}.\boldsymbol{p})}{E+2mc^2}\phi(\boldsymbol{r}) + \frac{1}{E+2mc^2}\frac{V_2(\boldsymbol{r})\int V_2(\boldsymbol{r}')\frac{c(\boldsymbol{\sigma}.\boldsymbol{p}')}{E+2mc^2}\phi(\boldsymbol{r}')\,d\boldsymbol{r}'}{1-\int V_2(\boldsymbol{r}')\frac{1}{E+2mc^2}V_2(\boldsymbol{r}')\,d\boldsymbol{r}'}. \tag{22}$$

Substituting (22) into (1a), accounting that $E \ll 2mc^2$, we can obtain (21).

The result obtained in this work can be used to calculate the spin-orbit interaction for a concrete system.

References

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