

ANALYSIS OF THE STANDING-WAVE SOLUTIONS OF THE ONE-DIMENSIONAL SINE-GORDON EQUATION

N. Martinov, N. Vitanov
Faculty of Physics, University of Sofia,
5, J. Bourchier Blvd, Sofia 1126, Bulgaria

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Abstract. Solutions of the one-dimensional sine-Gordon equation which consists of a function of the product of two real Jacobi elliptic functions are discussed. In this case the studied equation is reduced to a system, which consists of three algebraic equations. Due to this system seven solutions were found. The solutions were studied and classified into four classes — *A*, *B*, *C*, *D*. Class *D* has not been studied yet.

Резюме. Обсуждаются решения одномерного уравнения sine-Gordon, состоящие из сложной функции произведения двух эллиптических функций Якоби. Для этого случая изучаемое уравнение сведено к системе из трёх алгебраических уравнений. Благодаря этой системе найдены семь решений уравнения sine-Gordon. Они подразделены в четырёх классах. Класс *D* состоит из новых решений в виде стоящих волн.

1. One-dimensional sine-Gordon Equation and the Lamb Substitution

The one-dimensional sine-Gordon equation

$$\frac{\partial^2 \phi}{\partial x^2} - \frac{\partial^2 \phi}{\partial t^2} = \sin \phi(x, t) \quad (1)$$

has many applications in physics [3]. Its importance is also due to the fact, that this equation possesses soliton solutions [1]. An approach for finding particular solutions of the sine-Gordon equation is based on the Lamb substitution [5].

$$\phi = 4 \tan^{-1}[\bar{f}(x)\bar{g}(t)]. \quad (2)$$

Due to the substitution (2) two kinds of solutions were found. The first kind includes solutions which consist only of elementary functions [6]. The second kind

consists of solutions which are functions of the product of two real Jacobi elliptic functions. Three solutions belonging to this kind are known [4]:

plasma oscillations:

$$\phi_1 = 4 \tan^{-1} [A \operatorname{cn}(\alpha x; k_1) \operatorname{cn}(\beta t; k_2)] \quad (3)$$

$$k_1^2 = \frac{A^2[\alpha^2(A^2 + 1) + 1]}{\alpha^2(A^2 + 1)^2}; \quad k_2^2 = \frac{A^2[\beta^2(A^2 + 1) - 1]}{\beta^2(A^2 + 1)^2}; \quad \beta^2 - \alpha^2 = \frac{1 - A^2}{1 + A^2};$$

breather oscillations:

$$\phi_2 = 4 \tan^{-1} [A \operatorname{dn}(\alpha x; k_1) \operatorname{sn}(\beta t; k_2)] \quad (4)$$

$$k_1^2 = 1 - \frac{1 - \alpha^2(A^2 + 1)/A^2}{\alpha^2(A^2 + 1)}; \quad k_2^2 = \frac{A^2[1 - \beta^2(A^2 + 1)]}{\beta^2(A^2 + 1)}; \quad \alpha^2 = A^2\beta^2;$$

fluxon oscillations:

$$\phi_3 = 4 \tan^{-1} \left[A \operatorname{dn}(\alpha x; k_1) \frac{\operatorname{sn}(\beta t; k_2')}{\operatorname{cn}(\beta t; k_2)} \right] \quad (5)$$

$$k_1^2 = 1 - \frac{\alpha^2(A^2 - 1)/A^2 - 1}{\alpha^2(A^2 - 1)}; \quad k_2^2 = 1 - \frac{A^2[\beta^2(A^2 - 1) - 1]}{\beta^2(A^2 - 1)}; \quad \alpha^2 = A^2\beta^2.$$

The functions sn, cn, dn are Jacobi elliptic functions [2], k_1 and k_2 are their corresponding elliptic integral modules and A , α and β are parameters.

2. An Approach for Solving the sine-Gordon Equation

The supposition is:

$$\begin{aligned} \bar{f} &= Af[\alpha(x - x_0); k_1] \\ \bar{g} &= g[\beta(t - t_0); k_2]. \end{aligned} \quad (6)$$

f and g are real Jacobi elliptic functions; x_0 and t_0 are arbitrary constants. As a result of (6) the form of (2) is as follows:

$$\phi(x, t) = 4 \tan^{-1} \left\{ Af[\alpha(x - x_0); k_1] g[\beta(t - t_0); k_2] \right\}. \quad (7)$$

If (7) is substituted into (1) the result is:

$$\begin{aligned} (1 + A^2 f^2 g^2) \left(A \frac{d^2 f}{dx^2} g - Af \frac{d^2 g}{dt^2} \right) - 2Afg \left[A^2 \left(\frac{df}{dx} \right)^2 g^2 - A^2 f^2 \left(\frac{dg}{dt} \right)^2 \right] \\ = Afg(1 - A^2 f^2 g^2). \end{aligned} \quad (8)$$

The Jacobi elliptic functions obey to the following equations:

$$(f')^2 = a_1 f^4 + b_1 f^2 + c_1 \quad (9)$$

$$(g')^2 = a_2 g^4 + b_2 g^2 + c_2 \quad (10)$$

where the prime denotes differentiation by the corresponding variable, a_1, b_1, c_1 depend on the module k_1 and a_2, b_2, c_2 depend on the module k_2 .

If (9) and (10) are substituted into (8) then the result is:

$$\begin{aligned} A^3 f^3 g^3 (\alpha^2 b_1 - \beta^2 b_2 - 1) + 2A f^3 g (\alpha^2 a_1 + A^2 \beta^2 c_2) \\ - 2A f g^3 (\beta^2 a_2 + \alpha^2 A^2 c_1) + A f g (\alpha^2 b_1 - \beta^2 b_2 - 1) = 0. \end{aligned} \quad (11)$$

If:

$$\alpha^2 b_1 - \beta^2 b_2 = 1 \quad (12a)$$

$$\alpha^2 a_1 + A^2 \beta^2 c_2 = 0 \quad (12b)$$

$$\beta^2 a_2 + A^2 \alpha^2 c_1 = 0 \quad (12c)$$

then (7) is a solution of the one-dimensional sine-Gordon equation.

System (12) consists of three algebraic equations. Solution (7) of the sine-Gordon equation depends on five parameters A , α , β , k_1 and k_2 connected by the three relations (12). Hence the solution depends on two free parameters.

3. Solutions of the sine-Gordon Equation which Kind is (7)

The three solutions (3), (4) and (5) are well-known. But there are more solutions of the kind of (7).

We study the 12 main Jacobi elliptic functions. The relations between the modules of these functions and the coefficients a , b , c in their generating equations are presented in Table 1.

Table 1. Relations between the modules of the 12 main Jacobi elliptic functions and the coefficients a , b , c in their generating equations

Elliptic functions	a	b	c
1. $\text{cn}(x)$	$-k^2$	$2k^2 - 1$	$1 - k^2$
2. $\text{sn}(x)$	k^2	$-1 - k^2$	1
3. $\text{dn}(x)$	-1	$2 - k^2$	$k^2 - 1$
4. $\text{sn}(x)/\text{dn}(x)$	$1 - k^2$	$2 - k^2$	1
5. $\text{cn}(x)/\text{sn}(x)$	1	$2 - k^2$	$1 - k^2$
6. $\text{sn}(x)/\text{dn}(x)$	$-k^2(1 - k^2)$	$2k^2 - 1$	1
7. $\text{dn}(x)/\text{sn}(x)$	1	$2k^2 - 1$	$-k^2(1 - k^2)$
8. $\text{cn}(x)/\text{dn}(x)$	k^2	$-1 - k^2$	1
9. $\text{dn}(x)/\text{cn}(x)$	1	$-1 - k^2$	k^2
10. $1/\text{sn}(x)$	1	$-1 - k^2$	k^2
11. $1/\text{cn}(x)$	$1 - k^2$	$2k^2 - 1$	$-k^2$
12. $1/\text{dn}(x)$	$k^2 - 1$	$2 - k^2$	-1

Due to the system (12) seven solutions whose kind is (7) were found. The solutions ϕ_1 , ϕ_2 and ϕ_3 are well-known plasma, breather and fluxon oscillations.

$$\phi_4 = 4 \tan^{-1} \left\{ A \operatorname{dn}[\alpha(x - x_0); k_1] \frac{\operatorname{cn}[\beta(t - t_0); k_2]}{\operatorname{sn}[\beta(t - t_0); k_2]} \right\} \quad (13)$$

$$k_1^2 = 1 - \frac{1 + \alpha^2(1 - A^2)/A^2}{\alpha^2(1 - A^2)}; \quad k_2^2 = 1 + \frac{1 - \beta^2(1 - A^2)/A^2}{\beta^2(1 - A^2)}; \quad \alpha^2 - \beta^2 = \frac{A^2}{A^2 - 1},$$

$$\phi_5 = 4 \tan^{-1} \left\{ A \operatorname{dn}[\alpha(x - x_0); k_1] \frac{1}{\operatorname{sn}[\beta(t - t_0); k_2]} \right\} \quad (14)$$

$$k_1^2 = \frac{\alpha^2(A^2 + 1)^2 - A^2}{\alpha^2 A^2 (A^2 + 1)}; \quad k_2^2 = \frac{A^2 - \beta^2(A^2 + 1)}{\beta^2 A^2 (A^2 + 1)}; \quad \alpha^2 + \beta^2 = \frac{A^2}{A^2 + 1}.$$

The solutions ϕ_4 and ϕ_5 were studied in connection with the correspondence between the solutions of the sine-Gordon and Poisson-Boltzmann equations [7]

$$\phi_6 = 4 \tan^{-1} \left\{ A \frac{\operatorname{sn}[\alpha(x - x_0); k_1]}{\operatorname{cn}[\alpha(x - x_0); k_1]} \operatorname{dn}[\beta(t - t_0); k_2] \right\} \quad (15)$$

$$k_1^2 = \frac{\alpha^2(1 - A^2)^2 + A^2}{\alpha^2(1 - A^2)}; \quad k_2^2 = \frac{A^2 - \beta^2(1 - A^2)^2}{\beta^2 A^2 (1 - A^2)}; \quad \beta^2 = \alpha^2 A^2,$$

$$\phi_7 = 4 \tan^{-1} \left\{ A \frac{\operatorname{sn}[\alpha(x - x_0); k_1]}{\operatorname{cn}[\alpha(x - x_0); k_1]} \frac{1}{\operatorname{dn}[\beta(t - t_0); k_2]} \right\} \quad (16)$$

$$k_1^2 = \frac{\alpha^2(1 - A^2)^2 + A^2}{\alpha^2(1 - A^2)}; \quad k_2^2 = \frac{\beta^2(1 - A^2)^2 - A^2}{\beta^2(1 - A^2)}; \quad \alpha^2 - \beta^2 = \frac{1}{1 - A^2}.$$

The solutions ϕ_1 - ϕ_5 can be received by generalization of the Lamb substitution due to Riemann theta-functions [9-11]. The solutions ϕ_6 and ϕ_7 can be received by generalization of the Lamb substitution without leaving the area of Jacobi elliptic functions. They do not coincide with the solutions describing plasma, breather and fluxon oscillations.

4. Analysis of the Solutions

The found solutions can be classified into four classes:

Class A: Plasma oscillations. The solution ϕ_1 belongs to this class. If α and β are chosen as independent parameters then

$$A = A(\alpha, \beta), \quad k_1 = k_1(\alpha, \beta), \quad k_2 = k_2(\alpha, \beta).$$

For the solution ϕ_1 :

$$A^2 = \frac{1 + \alpha^2 - \beta^2}{1 - \alpha^2 + \beta^2},$$

$$k_1^2 = \frac{(\alpha^2 + \beta^2 + 1)(\alpha^2 - \beta^2 + 1)}{4\alpha^2}; \quad k_2^2 = \frac{(\alpha^2 + \beta^2 + 1)(\alpha^2 - \beta^2 + 1)}{4\beta^2}. \quad (17)$$

Class B: Breather oscillations. Two solutions belong to this class: ϕ_2 — breather oscillations A and ϕ_5 — breather oscillations B.

Breather oscillations *A*. In this case:

$$A^2 = \frac{\alpha^2}{\beta^2}; \quad k_1^2 = 1 - \frac{\beta^2(1 - \alpha^2 - \beta^2)}{\alpha^2(\alpha^2 + \beta^2)}; \quad k_2^2 = \frac{\alpha^2(1 - \alpha^2 - \beta^2)}{\beta^2(\alpha^2 + \beta^2)}. \quad (18)$$

Breather oscillations *B*. Here:

$$A^2 = \frac{\alpha^2 + \beta^2}{1 - \alpha^2 - \beta^2}; \quad k_1^2 = 1 - \frac{\beta^2(1 - \alpha^2 - \beta^2)}{\alpha^2(\alpha^2 + \beta^2)}; \quad k_2^2 = \frac{\alpha^2(1 - \alpha^2 - \beta^2)}{\beta^2(\alpha^2 + \beta^2)}. \quad (19)$$

The difference between the breather oscillations *A* and *B* is as follows: If parameters α and β are fixed then the integral elliptic modules k_1 and k_2 for the breather oscillations *A* and *B* are the same. The periods of the Jacobi elliptic functions $T_x = 2K(k_1)/\alpha$ and $T_i = 2K(k_2)/\beta$ are the same too. But the amplitudes are different. Breather oscillations *A* and *B* whose amplitudes are the same, have different periods.

Class C: Fluxon oscillations. Two solutions belong to this class: ϕ_3 — fluxon oscillations *A* and ϕ_4 — fluxon oscillations *B*.

Fluxon oscillations *A*: In this case:

$$A^2 = \frac{\beta^2}{\alpha^2}; \quad k_1^2 = 1 - \frac{\beta^2(\alpha^2 - \beta^2 - 1)}{\alpha^2(\alpha^2 - \beta^2)}; \quad k_2^2 = 1 - \frac{\alpha^2(\alpha^2 - \beta^2 - 1)}{\beta^2(\alpha^2 - \beta^2)}. \quad (20)$$

Fluxon oscillations *B*. Here:

$$A^2 = \frac{\alpha^2 - \beta^2 - 1}{\alpha^2 - \beta^2}; \quad k_1^2 = 1 - \frac{\beta^2(\alpha^2 - \beta^2 - 1)}{\alpha^2(\alpha^2 - \beta^2)}; \quad k_2^2 = 1 - \frac{\alpha^2(\alpha^2 - \beta^2 - 1)}{\beta^2(\alpha^2 - \beta^2)}. \quad (21)$$

The amplitudes of the fluxon oscillations *A* and *B* are different. Against the values of parameters α and β the amplitude of the fluxon oscillations *A* is higher than the amplitude of the fluxon oscillations *B*. The opposite case is possible too.

Class D: This class has not been studied yet. Two solutions belong to it: ϕ_6 and ϕ_7 . These solutions cannot be received from the solutions ϕ_1 – ϕ_5 by transformations between the Jacobi elliptic functions (see [8] Table 21.6.8). For solution ϕ_6 :

$$A^2 = \frac{\alpha^2}{\beta^2}; \quad k_1^2 = \frac{(\alpha^2 - \beta^2)^2 + \beta^2}{\alpha^2(\alpha^2 - \beta^2)}; \quad k_2^2 = \frac{\alpha^2 - (\alpha^2 - \beta^2)^2}{\beta^2(\alpha^2 - \beta^2)}. \quad (22)$$

For solution ϕ_7 :

$$A^2 = \frac{\alpha^2 - \beta^2 - 1}{\alpha^2 - \beta^2}; \quad k_1^2 = \frac{(\alpha^2 - \beta^2)^2 + \beta^2}{\alpha^2(\alpha^2 - \beta^2)}; \quad k_2^2 = \frac{\alpha^2 - (\alpha^2 - \beta^2)^2}{\beta^2(\alpha^2 - \beta^2)}. \quad (23)$$

Two solutions ϕ_6 and ϕ_7 have different amplitudes.

5. Special Cases

In these cases $k_{1,2} = 0$ or 1. If $k = 0$, then:

$$\begin{aligned} \operatorname{sn}(x; 0) &= \sin(x) \\ \operatorname{cn}(x; 0) &= \cos(x) \\ \operatorname{dn}(x; 0) &= 1. \end{aligned} \quad (24)$$

If $k = 1$, then

$$\begin{aligned} \operatorname{sn}(x; 1) &= \tanh(x) \\ \operatorname{cn}(x; 1) = \operatorname{dn}(x; 1) &= 1/\cosh(x). \end{aligned} \tag{25}$$

The assumption below is that $x_0 = 0$ and $t_0 = 0$.

Class A: Plasma oscillations. If $k_1 = 1$ and $k_2 = 0$, then $\beta^2 = 1 - \alpha^2$ and $A = \alpha/(1 - \alpha^2)$. In this case solution ϕ_1 is reduced to:

$$\phi_1 = \phi_{1,1} = 4 \tan^{-1} \left\{ \frac{\alpha \cos[(1 - \alpha^2)^{1/2}t]}{(1 - \alpha^2)^{1/2} \cosh(\alpha x)} \right\}. \tag{26}$$

This is well-known breather solution of the one-dimensional sine-Gordon equation.

Class B: Breather oscillations. Breather oscillations A: Two cases are possible here:

(i) $k_1 = 1, k_2 = 0$. In this case $\beta^2 = 1 - \alpha^2$ and ϕ_2 is reduced to the breather solution of the sine-Gordon equation:

$$\phi_2 = \phi_{2,1} = 4 \tan^{-1} \left\{ \frac{\alpha \sin[(1 - \alpha^2)^{1/2}t]}{(1 - \alpha^2)^{1/2} \cosh(\alpha x)} \right\}. \tag{27}$$

(ii) $k_1^2 = 1 - (1 - \alpha^2)/\alpha^2; k_2^2 = 1$. Then $\beta^2 = \alpha - \alpha^2$ and the solution ϕ_2 is reduced to the solution:

$$\phi_2 = \phi_{2,2} = 4 \tan^{-1} \left\{ \sqrt{\frac{\alpha}{1 - \alpha}} \operatorname{dn} \left[\alpha x; \sqrt{1 - \frac{(1 - \alpha)^2}{\alpha^2}} \right] \tanh \left[\sqrt{\alpha - \alpha^2} t \right] \right\}. \tag{28}$$

This is a very interesting combination between a special function and an elementary one. The solution (28) is a time-aperiodical solution of the sine-Gordon equation.

Breather oscillations B. If $\beta^2 = 1 - \alpha^2$, then $A = \infty$, i.e. the breather oscillations B cannot be reduced to the breather solution of the studied equation. This is another difference between the breather oscillations A and B.

The following special case is possible: $k_1^2 = 1 - (1 - \alpha)^2/\alpha^2; k_2^2 = 1$. Then solution ϕ_3 is reduced to:

$$\phi_3 = \phi_{3,1} = 4 \tan^{-1} \left\{ \sqrt{\frac{\alpha}{1 - \alpha}} \operatorname{dn} \left[\alpha x; \sqrt{1 - \frac{(1 - \alpha)^2}{\alpha^2}} \right] \operatorname{cotan} \left[\sqrt{\alpha - \alpha^2} t \right] \right\}. \tag{29}$$

There is a combination between a special and an elementary function again. But now the elementary function is not tanh.

Class C: Fluxon oscillations.

Fluxon oscillations A: Two special cases are possible here.

(i) $k_1 = k_2 = 1$. The $\beta^2 = \alpha^2 - 1$ and solution ϕ_3 is reduced to:

$$\phi_3 = \phi_{3,1} = 4 \tan^{-1} \left\{ \sqrt{\frac{\alpha^2}{\alpha^2 - 1}} \frac{\sinh[t\sqrt{\alpha^2 - 1}]}{\cosh(\alpha x)} \right\}. \tag{30}$$

This solution of the sine-Gordon equation describes a soliton-antisoliton collision.

(ii) $k_1^2 = 1 - (\alpha - 1)^2/\alpha^2$; $k_2 = 0$. Then $A^2 = \alpha/(\alpha - 1)$ and the solution ϕ_3 is reduced to:

$$\phi_3 = \phi_{3,2} = 4 \tan^{-1} \left\{ \sqrt{\frac{\alpha}{1-\alpha}} \operatorname{dn} \left[\alpha x; \sqrt{1 - \frac{(\alpha-1)^2}{\alpha^2}} \right] \tan \left[\sqrt{\alpha^2 - \alpha t} \right] \right\}. \quad (31)$$

Here the elementary function is not a hyperbolical one. It is a trigonometrical function.

Fluxon oscillations *B*: If $\beta^2 = \alpha^2 - 1$, then $A = \infty$, i.e. fluxon oscillations *B* cannot be reduced to the soliton-antisoliton collision. That is another difference between the fluxon oscillations *A* and *B*.

The following special case is possible: $k_1^2 = 1 - (\alpha - 1)^2/\alpha^2$; $k_2 = 0$. Here:

$$\phi_4 = \phi_{4,1} = 4 \tan^{-1} \left\{ \sqrt{\frac{\alpha}{\alpha-1}} \operatorname{dn} \left[\alpha x; \sqrt{1 - \frac{(\alpha-1)^2}{\alpha^2}} \right] \cotan \left[\sqrt{\alpha^2 - \alpha t} \right] \right\}. \quad (32)$$

Class D. The solution ϕ_6 . If $k_1 = k_2 = 1$, then $\beta^2 = \alpha^2 - 1$, $A^2 = (\alpha^2 - 1)/\alpha^2$ and

$$\phi_6 = \phi_{6,1} = 4 \tan^{-1} \left[\sqrt{\frac{\alpha^2 - 1}{\alpha^2}} \frac{\sinh(\alpha x)}{\cosh[(\alpha^2 - 1)^{1/2} t]} \right]. \quad (33)$$

This is the 4π -impulse solution of the sine-Gordon equation. It describes soliton-soliton collision.

Solution ϕ_7 . If $\beta^2 = \alpha^2 - 1$ then $A = 0$ and solution ϕ_7 cannot be reduced to the 4π -impulse solution. Another special cases are inadmissible too. This is the difference between solutions ϕ_6 and ϕ_7 .

Several differences exist between the classes of solutions *C* and *D*. The amplitudes of the solutions are different. The solutions ϕ_3 and ϕ_4 (class *C*) possess special cases whose kind correspond to (31) and (32) whereas the solutions ϕ_6 and ϕ_7 cannot be reduced to solutions which consist of a product of a Jacobi elliptic function and an elementary function.

The solution ϕ_7 cannot be reduced to any special case.

6. Physical Application of the Solutions of Class *D*

The solutions ϕ_6 and ϕ_7 describe a new kind of oscillations in Josephson junction. The solutions belonging to classes *A*, *B* and *C* perform the following boundary condition: $\partial\phi/\partial x = 0$ if $x = 0$ and $x = 1$. This condition corresponds to open Josephson line [12]. Such boundary condition is not possible from the solutions ϕ_6 and $\phi_{7\infty}$, i.e. the corresponding oscillations cannot exist in open Josephson line. The solutions ϕ_6 and ϕ_7 perform another boundary condition: $\phi = 0$ if $x = 0$ and $x = 1$. Then $\alpha = 4nK(k_1)/l$; $n = \pm 1, \pm 2, \pm 3, \dots$ and the solutions are as follows:

$$\phi_6 = 4 \tan^{-1} \left\{ A \frac{\operatorname{sn} \left[\frac{4nK(k_1)}{l} x; k_1 \right]}{\operatorname{cn} \left[\frac{4nK(k_1)}{l} x; k_1 \right]} \operatorname{dn}(\beta t; k_2) \right\} \quad (34)$$

$$\phi_7 = 4 \tan^{-1} \left\{ A \frac{\operatorname{sn} \left[\frac{4nK(k_1)}{l} x; k_1 \right]}{\operatorname{cn} \left[\frac{4nK(k_1)}{l} x; k_1 \right]} \frac{1}{\operatorname{dn}(\beta t; k_2)} \right\}. \quad (35)$$

Here the supposition is $x_0 = 0$ and $t_0 = 0$.

The corresponding electric potential and magnetic field are as follows:

1. Solution ϕ_6 :

$$V = -2 \frac{\Phi_0 \Omega_0}{\pi c} k_2^2 \frac{A \frac{\operatorname{sn}(\alpha x; k_1)}{\operatorname{cn}(\alpha x; k_1)} \operatorname{sn}(\beta t; k_2) \operatorname{cn}(\beta t; k_2)}{1 + A^2 \frac{\operatorname{sn}^2(\alpha x; k_1)}{\operatorname{cn}^2(\alpha x; k_1)} \operatorname{dn}^2(\beta t; k_2)} \quad (36)$$

$$B = 2 \frac{\Phi_0}{\pi \Lambda_0} \frac{A \frac{\operatorname{dn}(\alpha x; k_1)}{\operatorname{cn}^2(\alpha x; k_1)} \operatorname{dn}(\beta t; k_2)}{1 + A^2 \frac{\operatorname{sn}^2(\alpha x; k_1)}{\operatorname{cn}^2(\alpha x; k_1)} \operatorname{dn}^2(\beta t; k_2)} \quad (37)$$

2. Solution ϕ_7 :

$$V = 2 \frac{\Phi_0 \Omega_0}{\pi c} k_2^2 \frac{A \frac{\operatorname{sn}(\alpha x; k_1)}{\operatorname{cn}(\alpha x; k_1)} \frac{\operatorname{sn}(\beta t; k_2) \operatorname{cn}(\beta t; k_2)}{\operatorname{dn}^2(\beta t; k_2)}}{1 + A^2 \frac{\operatorname{sn}^2(\alpha x; k_1)}{\operatorname{cn}^2(\alpha x; k_1)} \operatorname{dn}^2(\beta t; k_2)} \quad (38)$$

$$B = 2 \frac{\Phi_0}{\pi \Lambda_0} \frac{A \frac{\operatorname{dn}(\alpha x; k_1)}{\operatorname{cn}(\alpha x; k_1)} \frac{1}{\operatorname{dn}(\beta t; k_2)}}{1 + A^2 \frac{\operatorname{sn}^2(\alpha x; k_1)}{\operatorname{cn}^2(\alpha x; k_1)} \operatorname{dn}^2(\beta t; k_2)} \quad (39)$$

Here $\Phi_0 = \pi \hbar c / 2$ is a magnetic flux quantum, $\Lambda_0 = (\Phi_0 / 2\pi L I_0)^{1/2}$ is a Josephson depth, $\Omega_0 = (2\pi I_0 / C \Phi_0)^{1/2}$ is a Josephson frequency. I_0 is the maximum value of the density of the current. L and C are inductivity and capacitance of the Josephson line and $\alpha = 4nK(k_1)/l$.

7. Conclusions

The solutions of the one-dimensional sine-Gordon equation whose kind is (7) are arranged into four classes. Class *A* consists of one solution and class *B*, *C* and *D* consist of two solutions. The main difference between the solutions which belong to the same class is that they have different amplitudes.

Two kinds of special cases of the solutions of the sine-Gordon equation are possible. In the first kind are the cases which consist only of elementary functions:

the breather, the soliton-antisoliton collision and the 4π -impulse. In the second kind are those cases which consist of product of a Jacobi elliptic function and an elementary function. In these cases the Jacobi elliptic function is a function of variable x and the elementary function is a function of variable t .

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