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QUANTUM NONSYMMETRIC GRAVITY AND THE SUPERFIBER BUNDLE FORMALISM

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Abstract. The formalism of the principal superfiber-bundle is applied to quantum Nonsymmetric gravitational theory. It is shown that the metric and Fadeev–Popov fields arise as superfield components of the superconnection. Moreover the BRST and anti-BRST transformations are shown to be the gauge transformations of the parameters of the ghost and anti-ghost superfields.

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1. Introduction

The gravitational force is the oldest interaction known to man and the least understood. It is the dimensionful character of the gravitational constant which destroys the productivity of the theory. That is, it is impossible to have a quantum consistent renormalizable theory. Since all the present day experimental data confirm general relativity (GR), any future quantum or other kind of gravitation theory should be an extension rather than a replacement of GR. One alternative is to treat GR as a special case of an extended (or general) theory, namely Nonsymmetric gravitation theory (NGT) [1–8]. The most appealing feature of the theory is that all its theoretical predictions are compatible with the existing classical experimental tests which have confirmed the validity of GR. Moreover, the recently proposed fifth force can be contained naturally in the NGT formalism. Furthermore, the quantum version of this theory has been recently proposed (QNGT) [9] and it turns out that it contains a larger symmetry [10-13] which may resolve the outstanding problem of the renormalizability. Despite of all this progress and as it is the case of gauge theories, one needs a natural geometrical interpretation of QNGT within the framework of the mathematical structure of the principal fiber bundle.

The mathematical structure of gauge theories was displayed by Wu and Yang who established a dictionary of equivalence between the physical terms and mathematical concepts [14]. Thus a pure gauge theory is described by a principal fiber bundle $P(X, G)$, with base space X (the space-time manifold) and structure group G (the gauge group). The gauge fields A_μ and the strength field tensor $F_{\mu\nu}$ are the coefficients of a connection (1-form) in the principal fiber bundle and its curvature (2-form) respectively in a local coordinate system in P. In this scheme, the gauge transformations are fiber bundle equivalence.

When a gauge theory is quantized, one needs to introduce in the Lagrangian a gauge fixing term so that the effect it produces is two-fold:

i) New spinless and anti-commuting fields appear (the so called Fadeev-Popov fields (FP)).

ii) The original gauge invariance is broken and a new invariance called BRST arises.

It is important to mention that FP fields and BRST transformations do not appear in the Wu-Yang's dictionary to provide them with a precise geometrical interpretation.

The aim of this work is to find out a natural geometrical interpretation of the FP fields as well as BRST and anti-BRST transformations of QNGT within the structure framework of the principal fiber bundle. To this end, we use the concepts of supermanifold supergroup and superfiber bundle which have been developed in Ref. [15-19].

In Section 2 we give a brief summary and notions on supermanifold, superforms and superfibers. In Section 3 we use the mathematical structure of the principal superfiber-bundle to give a geometrical description of QNGT and show that the metric and FP fields arise as superfields components of the connection in a local coordinate system. Moreover, the BRST and anti-BRST transformations arise as gauge transformations of the parameters of the ghost and anti-ghost superfields. In Section 4 we draw our conclusions.

2. Supermanifold, Superforms and Superfibers

A Z_2 graded vector space B is a direct sum of two vectors B_0 and B_1

$$B = B_0 \otimes B_1 \tag{2.1}$$

where B_0 and B_1 are called the even and odd parts of B. If B is endowed with a norm (induced norm) it is called a Z_2 graded Banach space. An element

$b \in B$ is said to be homogeneous if $b \in B_0$ or $b \in B_1$ with

$$|b| = \begin{cases} 0 & \text{if } b \in B_0 \\ 1 & \text{if } b \in B_1 \end{cases} \tag{2.2}$$

("| |" means grade). Now, if N is the nilpotent part of B, i. e. $B = \mathcal{R} \otimes \mathcal{N}$, thus one can define the body and soul projections b and S respectively through the application $b\colon B \to \mathcal{R}$ and $S\colon B \to \mathcal{N}$. The Cartesian product S^{m+n} is constructed in a graded B module as follows:

$$S^{m+n} = S^{m,n} \otimes S^{\bar{m},\bar{n}} \tag{2.3}$$

where

$$S^{m,n} = (B_0)^m \otimes (B_1)^n \tag{2.4}$$

and

$$S^{\bar{m},\bar{n}} = (B_0)^m \otimes (B_0)^n \,. \tag{2.5}$$

Then, one can generalize the notion of a manifold which is locally diffeomorph to R^n, to a supermanifold where the vector space $\mathcal{R}^n$ is replaced by the superspace $\mathcal{S}^{m,n}$. Consequently, a supermanifold of dimension (m,n) is defined as a topological space endowed with a superatlas $\{(\mathcal{U}_i, \ominus_i)\}$ where $\mathcal{U}_i$ is an open set of $M = \underset{i\in I}{\cup} \mathcal{U}_i$ and $\ominus_i$ is an homeomorphism of U_i in $\mathcal{S}^{m,n}$. If U is an open set of M, then the set of superdifferentiable functions of U in B is denoted by $B(U)$. A vectorial superfield defined on U is an element χ of a vectorial space endomorphic to $B(U)$ such that:

$$\begin{aligned} \chi(fg) &= (\chi f)g + (-1)^{|\chi||f|} f(\chi g) \qquad && f,g \in B(U) \\ \chi(af) &= (-1)^{|\chi||a|} a(\chi f) && f \in B(U),\ a \in B\,. \end{aligned} \tag{2.6}$$

Thus, the set of vectorial superfields defined on M and denoted by $D(M)$ has the following properties:

$D(U)$ is a set left graded B module, i. e.

$$\forall f \in B(U); \qquad \chi \in D(U) \Rightarrow (f\chi) \in D(U) \tag{2.7}$$

with the following bracket operation:

$$[X,Y] = XY - (-1)^{|X||Y|} YX\,. \tag{2.8}$$

Thus, if $(x^\mu, \theta^\alpha)_{\mu=\overline{1,m},\, \alpha=\overline{1,n}}$ is a chart of coordinates over M, then $\frac{\partial}{\partial x^\mu} \in D(U)_0$ and $\frac{\partial}{\partial \theta^\alpha} \in D(U)_1$ allow to write locally the superfield X as

$$X = \sum_{\mu=1}^{m} f^\mu \frac{\partial}{\partial x^\mu} + \sum_{\alpha=1}^{n} f^\alpha \frac{\partial}{\partial \theta^\alpha} \qquad f^\mu,\, f^\alpha \in B(U)\,. \tag{2.9}$$

In the same manner as in the classical differential geometry, one can define a differentiable P-superform ω over a supermanifold M of dimension (m, n) as an application:

$$\omega\colon \underbrace{D(M) \otimes \cdots \otimes D(M)}_{P\text{-times}} \to B(M) \tag{2.10}$$

such that:

i) ω is a $B(M)$ P-linear, i. e.

$$\begin{aligned}\omega(X_1, \ldots, fX_k, \ldots, X_P) \\ = (-1)^{|f|(X_{k-1}|\ldots|X_P|)} f\omega(X_1, \ldots, X_k, \ldots, X_P)\end{aligned} \tag{2.11}$$

ii) ω is a graded antisymmetric, i. e.

$$\begin{aligned}\omega(X_1, \ldots, fX_k, \ldots, X_P) \\ = (-1)^{1+|X_k|.|X_{k+1}|}\omega(X_1, \ldots, X_{k+1}, X_k, \ldots, X_P)\,.\end{aligned} \tag{2.12}$$

We remind that the space of the p-superforms $\Omega^P(M)$ is a graded B module. If ω is even (resp. odd) then: $\omega(X_1, \ldots, X_P) \in B(M)$ with $|\omega| = 0$ (resp. $|\omega| = 1$).

The exterior product "$\wedge$" of two p and p' superforms ω and ω' respectively is defined by

$$\omega \wedge \omega' = (-1)^{pp'+|\omega|.|\omega'|}\omega' \wedge \omega\,. \tag{2.13}$$

The exterior differentiation is defined as an even linear application such that

$$\mathrm{d}\colon \Omega^P(M) \to \Omega^{P+1}(M)$$

with

a) $\mathrm{d}(\omega + \omega') = \mathrm{d}\omega + \mathrm{d}\omega'$

b) $\mathrm{d}(\omega \wedge \omega') = \omega \wedge \mathrm{d}\omega' + (-1)^P\, \mathrm{d}\omega \wedge \omega'$

c) $\mathrm{d}^2 = 0$

d) $\mathrm{d}f = \sum_{A=1}^{m+n} \mathrm{d}x^A \frac{\partial f}{\partial x^A}$

where $f \in \Omega^0(M) = B(M)$ and $(x^A)\,(A = \overline{1, m+n})$ is a chart of the M coordinates. Locally, a p-superform can be expressed as

$$\omega = \sum_{A_1 \ldots A_P} \mathrm{d}x^{A_P} \wedge \cdots \wedge \mathrm{d}x^{A_1} \omega_{A_1 \ldots A_P} \,. \tag{2.14}$$

Now, the notion of a superfiber can be easily obtained by extending the notion of a fiber which necessitates the definition of a Lie supergroup. In fact, a Lie supergroup of dimension (m, n) is a set A such that:

i) The set A is an abstract group.

ii) The set A is a supermanifold of dimension (m, n).

iii) The applications φ_1 and φ_2 defined by

$$\begin{aligned} &\varphi_1 : A \otimes A \to A \\ &\varphi_2 : A \to A \end{aligned} \tag{2.15}$$

where $\varphi_2(a) = a^{-1}$, $a \in A$ and a^{-1} is the inverse element of a, are superanalytic applications.

The infinitesimal right or left translations are superdiffeomorph. Thus, the space of vectorial superfields which are left invariants constitutes a Lie superalgebra g of G, i. e.:

$$\begin{aligned} &\text{i)} \quad [X_i, X_j] = (-1)^{1+|X_i|.|X_j|}[X_i, X_j] && X_i, X_j \in g \\ &\text{ii)} \quad [aX_i, X_j] = a[X_i, .X_j] && X_i, X_j \in g,\ a \in B \\ &\text{iii)} \quad (-1)^{|X_i|.|X_j|}[[X_i, X_k], X_j] + (-1)^{|X_k|.|X_i|}[[X_k, X_j], X_i] && \\ &\qquad + (-1)^{|X_j||.|X_k|}[[X_j, X_i], X_k] = 0 && X_i, X_j, X_k \in g \end{aligned}$$

where $[,]$ is the graded Lie bracket. The homogeneous elements $\{I_A, A = \overline{1, m+n}\}$ constitute a basis of g. If there exists an element f^C_{AB} of B such that

$$[I_A, I_B] = f^C_{AB} I_C \tag{2.16}$$

where $|f^C_{AB}| = |A| + |B| + |C|$. The dual basis θ^A of I_A is given by the Maurer–Cartan equation

$$\mathrm{d}\theta^A = \frac{1}{2}\theta^B \wedge \theta^C f^A_{BC} \,. \tag{2.17}$$

Now, let G and M be a Lie supergroup and supermanifold respectively. $P(M, G)$ is a principal superfiber with a total space P, basis M and a structural group G over a supermanifold with a dimension $\dim(M) + \dim(G)$ if

i) G acts freely at the right over P;

ii) M is a quotient superspace P/G: the superdifferentiable superprojection Π is defined by

$$\Pi \rightarrow M\,;$$

iii) P is locally trivial.

A connection Γ in the principal superfiber P is an application which associates to each point $z \in P$ a tangent space $T_z P = H_z$.

3. BRST and Anti-BRST Transformations

At the quantum level of Nonsymmetric Gravitation Theory, the primary fields are represented by the Vierbein, connection and the ghosts related to the general coordinates and the linear group $GA(4, R)$.

If we want to reproduce these fields in the geometrical structure of the superfiber bundle $P(M, G_s)$, one has to identify the basis M with the four dimensional space-time whose metric is nonsymmetric and the structural group G_s with a Lie supergroup allowing the introduction of the primary fields via the superconnection. To be more specific, one has to consider a structural group G_s as

$$G_s = GL(4, R) \otimes GA(4, R) \otimes \mathcal{S}^{0,2} \tag{3.1}$$

where $GL(4, R)$, $GA(4, R)$ and $\mathcal{S}^{0,2}$ represent the general local linear groups and the translation supergroup with dimension $(0, 2)$ respectively. Thus, the Lie superalgebra g_s of the structural group G_s can be written as

$$g_s = gl(4, R) \otimes ga(4, R) \otimes \mathcal{S}^{0,2} \tag{3.2}$$

with generators $T^\mu{}_\nu$, $(T^a{}_b, P_a)$ and F_α ($\mu, \nu = \overline{1,4}$, $a, b = \overline{1,4}$, $\alpha = 0, 2$), corresponding to $gl(4, R)$, $ga(4, R)$ and $\mathcal{S}^{0,2}$, respectively and verifying the following graded Lie superalgebra:

$$\begin{aligned}
[T^\tau{}_\lambda, T^\mu{}_\nu] &= f^{\tau\mu\rho}_{\lambda\nu\sigma} T^\sigma{}_\rho \\
[T^a{}_b, T^c{}_d] &= f^{ace}_{bdf} T^f{}_e \\
[T^a{}_b, P_a] &= f^{ac}_{bd} P_c \\
[P_a, P_b] = \{F_\alpha, F_\beta\} &= [T^a{}_b, T^\mu{}_\nu] = [P_a, F_\alpha] = 0 \\
[T^\mu \nu, P_a] = [T^\mu{}_\nu, F_\alpha] &= [T^a{}_b, F_\alpha] = 0
\end{aligned} \tag{3.3}$$

where

$$\begin{aligned}
f^{\tau\mu\rho}_{\lambda\nu\sigma} &= \delta^\tau{}_\nu \delta^\mu{}_\rho \delta_\lambda{}^\sigma - \delta_\lambda{}^\mu \delta_\nu{}^\sigma \delta_\rho{}^\tau \\
f^{ace}_{bdf} &= \delta^a{}_d \delta^c{}_f \delta^e{}_b - \delta^c{}_b \delta^e{}_d \delta^a{}_f \\
f^{ac}_{bd} &= \delta^a{}_d \delta^c{}_b\,.
\end{aligned}$$

Let Ω be the 2-form supercurvature associated to the superconnection Φ and Φ' an even pseudo-tensorial form of the type (ad, g_s) ("ad" means adjoint representation) defined on the principal superfiber bundle $P(M, G_s)$. Notice that the introduction of the 1-form Φ' is justified by the fact that the imposed constraint on the supercurvature Ω can be obtained by the fact that the covariant differentiation of a pseudo-tensorial form is tensorial. Now, if we introduce a generalized superconnection $\tilde{\Phi}$ such that

$$\tilde{\Phi} = \Phi - \Phi' . \tag{3.4}$$

Then the corresponding generalized supercurvature $\tilde{\Omega}$ gets the form:

$$\tilde{\Omega} = D\tilde{\Phi} = \Omega - \Theta \tag{3.5}$$

with

$$\Omega = D\Phi = d\Phi - \frac{1}{2}[\Phi, \Phi']$$

and

$$\Theta = D\Phi' = d\Phi' + [\Phi, \Phi'] . \tag{3.6}$$

Using the fact that the BRST and anti-BRST are global transformations, one has to consider a principle superfiber bundle $P(M, G_s)$ which is globally trivial with respect to odd supergroup $\mathcal{S}^{0,2}$ translations. If we define $Z = (X^M)_{M=\overline{1,6}} \equiv (x^\mu, \theta^\alpha)_{\mu=\overline{1,4},\alpha=1,2}$ as a local coordinates system on $P(M, G_s)$, then the generalized superconnection and supercurvature take the form

$$\tilde{\Phi} = dZ^M \tilde{\Phi}_M = dZ^M(\Phi_M - \Phi'_M) \tag{3.7}$$

and

$$\tilde{\Omega} = \frac{1}{2} dZ^N \wedge dZ^M \tilde{\Omega}_{MN} = \frac{1}{2} dZ^N \wedge dZ^M (\Omega_{MN} - \Theta_{MN}) \tag{3.8}$$

with

$$\begin{aligned} \Omega_{MN} &= \partial_m \Phi_N + (-1)^{1+|Z^N|.|\Phi_M|} \partial_N \Phi_M - [\Phi_N - \Phi_M] \\ \Theta_{MN} &= \partial_M \Phi'_N + (-1)^{1+|Z^N|.|\Phi'_M|} \partial_N \Phi'_M \\ &\quad - \{[\Phi_N, \Phi'_M] + (-1)^{1+|\Phi_N|.|\Phi'_N|} [\Phi_M, \Phi'_N]\} . \end{aligned} \tag{3.9}$$

Here $|Z^M|$, $|\Phi_M|$, $|\Phi'_M|$ stand for the Grassman grade of Z^M, Φ_M and Φ'_M respectively.

Now as it is mentioned in Section 2 and since $\frac{\partial}{\partial\theta^\alpha}$ is a vertical superfield, one deduces that

$$\tilde{\Omega}_{\alpha\beta} = \tilde{\Omega}_{\alpha\mu} , \tag{3.10}$$

i. e.

$$\Omega_{\alpha\beta} = \Theta_{\alpha\beta} \qquad (\alpha, \beta = 1, 2;\ \mu = \overline{1,4} \tag{3.11}$$

and

$$\Omega_{\alpha\mu} = \Theta_{\alpha\mu} . \tag{3.12}$$

Moreover, and since the superfields components Φ_M, Φ'_M, Ω_{MN} and Θ_{MN} have their values on g_s, one can write:

$$\begin{aligned} \Phi_M &= \Phi^a{}_{bM} T^b{}_a + \Phi^\mu{}_{\nu M} T^\nu{}_\mu + \Phi^a{}_M P_a + \Phi^a{}_M F_\alpha \\ \Phi'_M &= \Phi'^a{}_{bM} T^b{}_a + \Phi'^\mu{}_{\nu M} T^\nu{}_\mu + \Phi'^a{}_M P_a + \Phi'^a{}_M F_\alpha \\ \Omega_{MN} &= \Omega^a{}_{bMN} T^b{}_a + \Omega^\mu{}_{\nu MN} T^\nu{}_\mu + \Omega^a{}_{MN} P_a + \Omega^a{}_{MN} F_\alpha \\ \Theta_{MN} &= \Theta^a{}_{bMN} T^b{}_a + \Theta^\mu{}_{\nu MN} T^\nu{}_\mu + \Theta^a{}_{MN} P_a + \Theta^a{}_{MN} F_\alpha . \end{aligned} \tag{3.13}$$

From the relations (3.10–3.12) and (3.13) one obtains

$$\begin{array}{lll} \Omega^a{}_{b\mu\alpha} = \Theta^a{}_{b\mu\alpha} & \Omega^a{}_{b\mu\beta} = \Theta^a{}_{b\mu\beta} & \Omega^\lambda{}_{\tau\mu\alpha} = \Theta^\lambda{}_{\tau\mu\alpha} \\ \Omega^\lambda{}_{\tau\alpha\beta} = \Theta^\lambda{}_{\tau\alpha\beta} & \Omega^\alpha{}_{\mu\alpha} = \Theta^\alpha{}_{\mu\alpha} & \Omega^\alpha{}_{\alpha\beta} = \Theta^\alpha{}_{\alpha\beta} \\ \Omega^\alpha{}_{\mu\beta} = \Theta^\alpha{}_{\mu\beta} & \Omega^\alpha{}_{\beta\gamma} = \Theta^\alpha{}_{\beta\gamma} . & \end{array} \tag{3.14}$$

Furthermore, the generalized supertorsion $\tilde{\Omega}^\alpha$ of $S^{0,2}$ is given by

$$\tilde{\Omega}^\alpha_{MN} = \Omega^\alpha_{MN} - \Theta^\alpha_{MN} \tag{3.15}$$

where

$$\Omega^\alpha_{MN} = \partial_M \Phi^\alpha{}_M + (-1)^{1+|Z^N|.|\Phi_M|} \partial_N \Phi^\alpha{}_M \tag{3.16}$$

and

$$\Theta^\alpha_{MN} = \partial_M \Phi'^\alpha{}_M + (-1)^{1+|Z^N|.|\Phi_M|} \partial_N \Phi'^\alpha{}_M . \tag{3.17}$$

From Eqs (3.14) and (3.15), the components of the supertortion $\tilde{\Omega}^\alpha$ vanish with respect to the anticommuting directions.. Moreover, the potentials $\Phi^\alpha{}_M$ associated with the odd generators Φ_α are of a pure gauge. Therefore, one can impose the condition

$$\Omega^\alpha_{MN} = 0 . \tag{3.18}$$

Consequently

$$\Theta^{\alpha}{}_{\beta\gamma} = \Theta^{\alpha}{}_{\mu\beta} = 0\,. \tag{3.19}$$

Moreover, requiring that the geometrical structure of the principal superfiber bundle $P(M, G_s)$ has to contain the metric structure of the nonsymmetric space-time M, such that for $\Theta^{\alpha} = 0$, the component of the superfields $\tilde{\Omega}_{\mu\nu}$, allows to obtain the standard results concerning the curvature and torsion of the nonsymmetric space-time metric. Consequently, one has to have

$$\tilde{\Omega}_{\mu\nu} = \omega_{\mu\nu} \tag{3.20}$$

and this implies that

$$\Theta^{\lambda}{}_{\tau\mu\nu} = \Theta^{\alpha}{}_{b\mu\nu} = \Theta^{\alpha}{}_{\mu\nu} = \Theta^{a}{}_{\mu\nu} = 0\,. \tag{3.21}$$

Using the condition (3.18) one can consider that the superconnection Φ has its values on $gl(4, R) \otimes ga(4, R)$ algebras and its local representation takes the form

$$\Phi = \mathrm{d}Z^{M}\Phi_{M} = \mathrm{d}x^{\mu}\Phi_{\mu} + \mathrm{d}\theta^{\alpha}\eta_{\alpha} \tag{3.22}$$

where

$$\begin{aligned} \Phi_{M} &= \Phi^{a}{}_{b\mu}T^{b}{}_{a} + \Phi^{\lambda}{}_{\tau\mu}T^{\tau}{}_{\lambda} + \Phi^{a}{}_{\mu}P_{a} \\ \eta_{\alpha} &= \eta^{a}{}_{b\alpha}T^{b}{}_{a} + \eta^{\lambda}{}_{\tau\alpha}T^{\tau}{}_{\lambda} + \eta^{a}{}_{\alpha}P_{a}\,. \end{aligned} \tag{3.23}$$

Notice that the superfields η_{α} have an odd Grassman grade, therefore they can represent the ghosts superfields. In particular, the components of the ghosts superfields evaluated at $\theta^{\alpha} = 0$ will be the ghosts associated with the general coordinates and local linear groups. Thus we obtain

$$\eta^{a}{}_{\alpha} \qquad \eta^{\lambda}{}_{\tau\alpha} = \partial_{\tau}\eta^{\lambda}{}_{\alpha} + \Phi^{\lambda}{}_{\tau\rho}\eta^{\rho}\alpha\,. \tag{3.24}$$

Regarding the superconnection Φ, one can get the ordinary primary fields of QNGT as follows:

$$\begin{aligned} W_{\mu\nu}{}^{\rho} &= \Phi_{\mu\nu}{}^{\rho}|_{\theta^{\alpha}=0} & \omega^{a}{}_{b\mu} &= \Phi^{a}{}_{b\mu}|_{\theta^{\alpha}=0} \\ e^{a}{}_{\mu} &= \Phi^{a}{}_{\mu}|_{\theta^{\alpha}=0} & C^{a}{}_{b\alpha} &= \eta^{a}{}_{b\alpha}|_{\theta^{\alpha}=0} \\ & C^{\mu}{}_{\alpha} = -\eta^{\mu}{}_{\alpha}|_{\theta^{\alpha}=0} \end{aligned} \tag{3.25}$$

The anholonomic and holonomic components of the superconnection $\Phi^{\rho}{}_{\mu\nu}$ and $\Phi^{a}{}_{b\nu}$ are related by the supervierbein $\Phi^{a}{}_{\mu}$ as follows:

$$\Phi^{\rho}{}_{\mu\nu} = \Phi^{\rho}{}_{a}(\partial_{\nu}\Phi^{a}{}_{\mu} + \Phi^{b}{}_{\mu}\Phi^{a}{}_{b\nu}) \tag{3.26}$$

consequently the relations (3.14) are not indepedents and the components $\Theta^{\lambda}{}_{\tau\alpha\beta}$ can be obtained from the components $\Theta^{a}{}_{b\alpha\beta}$ and $\Theta^{a}{}_{\mu\alpha}$. Note that through the relation (3.9), the knowledge of the Θ components allows to determine the Φ' components. For example, from the Eqs (3.19) and (3.21), one has as trivial solutions

$$\Phi'^{\alpha}{}_{\beta} = \Phi'^{\alpha}{}_{\mu} = \Phi'^{a}{}_{b\mu} = \Phi'^{a}{}_{\mu} = \Phi'^{\lambda}{}_{\tau\mu} = 0 . \tag{3.27}$$

The determination of the other components of Φ' such as $\Phi'^{\tau}{}_{\lambda\alpha}$, $\Phi'^{a}{}_{b\alpha}$ and $\Phi'^{a}{}_{\alpha}$ can be easily obtained from the remaining Θ components. Straightforward simplifications lead to

$$\begin{aligned} \Theta^{\lambda}_{\tau\alpha\beta} &= -\tfrac{1}{2}\left[\eta^{\sigma}{}_{\alpha}, \eta^{\rho}{}_{\beta}\right]\Omega^{\lambda}{}_{\tau\sigma\rho} \\ \Theta^{a}{}_{b\mu\alpha} &= -\eta^{\tau}{}_{\alpha}\Omega^{a}{}_{b\tau\mu} - D_{\mu}(\Phi^{a}_{b\tau}\eta^{\tau}{}_{\alpha}) \\ \Theta^{a}_{b\alpha\beta} &= \begin{cases} -2\eta^{\tau}_{\alpha}\partial_{\tau}\eta^{a}_{b\beta} & \text{if } \alpha = \beta \\ 0 & \text{if } \alpha \neq \beta \end{cases} \\ \Theta^{a}_{\mu\alpha} &= -\eta^{\tau}{}_{\alpha}\Omega^{a}{}_{\tau\mu} - D_{\mu}(\eta^{\tau}{}_{\alpha}\Phi^{a}{}_{\tau}) \\ \Theta^{a}{}_{\alpha\beta} &= 0 \end{aligned} \tag{3.28}$$

where $D_{\mu.} = \partial_{\mu.} + [\Phi_{\mu,.}]$ represents the covariant superderivative with values on $ga(4, R)$. Notice that the last relation (3.28) is a direct consequence of the condition $\Omega^{a}\alpha\beta = 0$ (Eq. (3.24)).

To introduce explicitly the physical fields $\Phi_{M}{}^{A}|_{\theta^{\alpha}=0}$ given by Eq. (3.25), where A represents the group index, we consider the development of the superfield components Φ^{A} is powers θ^{α} and end up to the following forms:

$$\begin{aligned} \Phi^{\lambda}{}_{\tau\mu} &= W^{\lambda}{}_{\tau\mu} + \theta^{\alpha}k^{\lambda}{}_{\tau\mu\alpha} + \frac{1}{2}\theta^{\alpha}\theta^{\beta}S^{\lambda}{}_{\tau\mu\beta\alpha} \\ \Phi^{a}_{b\mu} &= \omega^{a}{}_{b\mu} + \theta^{\alpha}R^{a}{}_{b\mu\alpha} + \frac{1}{2}\theta^{\alpha}\theta^{\beta}J^{a}{}_{b\mu\beta\alpha} \\ \Phi^{a}_{\mu} &= e^{a}{}_{\mu} + \theta^{\alpha}L^{a}{}_{\mu\alpha} + \frac{1}{2}\theta^{\alpha}\theta^{\beta}N^{a}{}_{\mu\beta\alpha} \\ \eta^{a}_{b\delta} &= C^{a}{}_{b\delta} + \theta^{\alpha}q^{a}{}_{b\delta\alpha} + \frac{1}{2}\theta^{\alpha}\theta^{\beta}X^{a}{}_{\delta\beta\alpha} \\ \eta^{a}_{b} &= C^{a}{}_{\delta} + \theta^{\alpha}V^{a}{}_{\delta\alpha} + \frac{1}{2}\theta^{\alpha}\theta^{\beta}t^{a}{}_{\delta\beta\alpha} \end{aligned} \tag{3.29}$$

where $S^{\lambda}{}_{\tau\mu\beta\alpha}$ $J^{a}{}_{b\mu\beta\alpha}$, $N^{a}{}_{\mu\alpha}$, $X^{a}{}_{\delta\beta\alpha}$ and $t^{a}{}_{\delta\beta\alpha}$ are antisymmetric with respect to the indices α and β. Using (3.9), (3.14), (3.24), (3.25), and (3.28), one can determine the components equations given by (3.29). In particular

$$
\begin{aligned}
R^a{}_{b\mu\alpha} &= \partial_\mu C^a{}_{b\alpha}\omega^a{}_{d\mu} - C^\tau{}_\alpha \partial_\tau \omega^a{}_{b\mu} - C^a{}_{d\alpha}\omega^d{}_{b\mu} - \partial_\mu C^\tau{}_\alpha \omega^a{}_{b\tau} \\
k^\lambda{}_{\tau\mu\alpha} &= -\partial_\mu \partial_\tau C^\lambda{}_\alpha - W^\lambda{}_{\tau\rho}\partial_\tau C^\rho{}_\alpha - W^\lambda{}_{\rho\mu}\partial T C^\rho{}_\alpha \\
&\quad - C^\rho{}_\alpha \partial_\rho W^\lambda{}_{\tau\mu} + W^\rho{}_{\tau\mu}\partial_\rho C^\lambda{}_\alpha \\
L^a{}_{\mu\alpha} &= -C^a{}_{b\alpha} e^b{}_\mu - C^\tau{}_\alpha \partial_\tau e^a{}_\mu - \partial_\mu C^\tau{}_\alpha e^a{}_\tau \\
q^a{}_{b\alpha\alpha} &= C^\tau{}_\alpha \partial_\tau C^a{}_{b\alpha} + C^a{}_{d\alpha} C^d{}_{b\alpha} \\
V^\tau{}_{\alpha\alpha} &= C^\rho{}_\alpha \partial_\rho C^\tau{}_\alpha \\
q^a{}_{b12} + q^a{}_{b21} &= C^a{}_{d2} + C^a{}_{b1} - C^d{}_{b2} C^a{}_{d1} \\
V^\tau{}_{12} + V^\tau{}_{21} &= C_1^\rho \partial_\rho C_2^\tau + C_2^\rho \partial_\rho C_1^\tau \,.
\end{aligned}
\tag{3.30}
$$

One notices that the only independent fields are represented by $\omega^a{}_{b\mu}$, $e^a{}_\mu$, $C^a{}_{b1} = C^a{}_d$, $C^\mu{}_1 = C^\mu{}_1$, $C^\mu{}_2 = q^a{}_{b21} = B^a{}_b$ and $V^\tau{}_{21} = B^\tau$ corresponding to the connection, Vierbein, (anti)ghosts associated with general and local coordinates transformations and auxiliary fields.

Now, the action at the right on the principal superfiber bundle $P(M, G_s)$ of odd translations of the group $S^{0,2}$ can be written as

$$R(\theta^\alpha)\colon (x^\mu, \zeta^\alpha) \to (x^\mu, \zeta^\alpha + \theta^\alpha)\,. \tag{3.31}$$

Let $(Q)_{\alpha=1,2}$ be the operational representation of the generators F_α of $S^{0,2}$. Thus the translation at the right $R(\theta^\alpha)$ is generated by $\theta^\alpha Q_\alpha$ and for an infinitesimal displacement one can write

$$R(\theta^\alpha) = l + \theta^\alpha Q_\alpha\,. \tag{3.32}$$

Using the properties of the superconnection (3.22) and the relations (3.31) and (3.32), we obtain

$$\Phi_M{}^A(x^\mu, \zeta^\alpha + \theta^\alpha) = R(\theta^\alpha)\Phi_M{}^A(x^\mu, \zeta^\alpha)R^{-1}(\theta^\alpha)\,. \tag{3.33}$$

After straightforward simplifications, the expression (3.33) gets the form:

$$
\begin{aligned}
\Phi_M{}^A(x^\mu, \zeta^\alpha + \theta^\alpha) &= \Phi_M{}^A(x^\mu, \zeta^\alpha) + \theta^\alpha [Q_\alpha, \Phi_M{}^A(x^\mu, \zeta^\alpha)] \\
&\quad + \frac{1}{2}\theta^\alpha\theta^\beta [Q_\beta, [Q_\alpha, \Phi_M{}^A(x^\mu, \zeta^\alpha)]]
\end{aligned}
\tag{3.34}
$$

consequently, for $\zeta^\alpha = 0$, one has

$$
\begin{aligned}
\Phi^{\lambda}{}_{\tau\mu} &= W^{\lambda}{}_{\tau\mu} + \theta^\alpha [Q^\alpha, W^{\lambda}{}_{\tau\mu}] + \frac{1}{2}\theta^\alpha\theta^\beta [Q_\beta, [Q_\alpha, W^{\lambda}{}_{\tau\mu}]] \\
\Phi^{a}_{b\mu} &= \omega^{a}{}_{b\mu} + \theta^\alpha [Q^\alpha, \omega^{a}{}_{b\mu}] + \frac{1}{2}\theta^\alpha\theta^\beta [Q_\beta, [Q_\alpha, \omega^{a}{}_{b\mu}]] \\
\Phi^{a}_{\mu} &= e^{a}{}_{\mu} + \theta^\alpha [Q^\alpha, e^{a}{}_{\mu}] + \frac{1}{2}\theta^\alpha\theta^\beta [Q_\beta, [Q_\alpha, e^{a}{}_{\mu}]] \\
\eta^{a}_{b\sigma} &= C^{a}{}_{b\sigma} + \theta^\alpha [Q^\alpha, C^{a}{}_{b\sigma}] + \frac{1}{2}\theta^\alpha\theta^\beta [Q_\beta, [Q_\alpha, C^{a}{}_{b\sigma}]] \\
\eta^{\mu}_{\delta} &= C^{\mu}{}_{\delta} + \theta^\alpha [Q^\alpha, C^{\mu}{}_{\delta}] + \frac{1}{2}\theta^\alpha\theta^\beta [Q_\beta, [Q_\alpha, C^{\mu}{}_{\delta}]] \, .
\end{aligned}
\tag{3.35}
$$

Using (3.29) and (3.35) as well as $Q_1 = Q$ and $Q_2 = \bar{Q}$ (BRST and anti-BRST charges respectively). We obtain the following QNGT BRST transformations:

$$
\begin{aligned}
\delta W^{\lambda}{}_{\tau\mu} &\equiv [Q, W^{\lambda}{}_{\tau\mu}] = k^{\lambda}{}_{\tau\mu} = -\partial_\mu\partial_\tau C^\lambda - W^{\lambda}{}_{\rho\mu}\partial_\tau C^\rho - W^{\lambda}{}_{\tau\rho}\partial_\mu C^\rho \\
&\quad + W^{\rho}{}_{\tau\mu}\partial_\tau C^\lambda - C^\rho\partial_\rho W^{\lambda}{}_{\tau\mu} \\
\delta\omega^{a}{}_{b\mu} &\equiv [Q, \omega^{a}{}_{b\mu}] = R^{a}{}_{b\mu 1} = \partial_\mu C^{a}{}_{b} + C^{d}{}_{b}\omega^{a}{}_{d\mu} - C^{a}{}_{d}\omega^{d}{}_{b\mu} \\
&\quad - C^\tau\partial_\tau\omega^{a}{}_{b\mu} - \partial_\mu C^\tau\omega^{a}{}_{b\mu} \\
\delta e^{a}{}_{\mu} &\equiv [Q, e^{a}{}_{\mu}] = L^{a}{}_{\mu 1} = -C^{a}{}_{b}e^{b}{}_{\mu} - C^\tau\partial_\tau e^{a}{}_{\mu} - \partial_\mu C^\tau e^{a}{}_{\tau} \\
\delta C^{a}{}_{b} &\equiv [Q, C^{a}{}_{b}] = C^{a}{}_{f}C^{f}{}_{b} + C^\tau\partial_\tau C^{a}{}_{b} \\
\delta C^\tau &\equiv [Q, C^\tau] = C^\rho\partial_\rho C^\tau \\
\delta\bar{C}^\tau &\equiv [Q, \bar{C}^\tau] = B^\tau \\
\delta\bar{C}^{a}{}_{b} &\equiv [Q, \bar{C}^{a}{}_{b}] = B^{a}{}_{b} \\
\delta B^{a}{}_{b} &= 0 \\
\delta B^\tau &\equiv [Q, B^\tau] = 0 \, .
\end{aligned}
\tag{3.36}
$$

Similarly, one can get the anti-BRST transformations by using the following substitutions:

$$
\begin{aligned}
X &\to X, \text{ for } X = W, \omega, e \\
X &\to \bar{X}, \text{ for } X = Q, C^{a}{}_{b}, C^\tau, B^{a}{}_{b}, B^\tau \\
\bar{X} &\to X, \text{ for } X = C^{a}{}_{b}, C^\tau
\end{aligned}
\tag{3.37}
$$

and the relations:

$$
\begin{aligned}
\bar{B}^{a} &= q^{a}{}_{b12} = -B^{a}{}_{b} + \bar{C}^{a}{}_{d}C^{d}{}_{a} - \bar{C}^{d}{}_{b}C^{a}{}_{d} \\
\bar{B}^\tau &= V^{\tau}{}_{12} = -B^\tau + C^\rho\partial_\rho\bar{C}^\tau + \bar{C}^\rho\partial_\rho C^\tau \, .
\end{aligned}
\tag{3.38}
$$

Regarding the gauge fixing Lagrangian L^{gf} (including ghosts) one can reproduce it from the geometrical structure of the superfiber bundle by as summing that it is represented by the component $\theta^\alpha = 0$ of a superlagrangian L^{gf}_s determined by the superconnection Φ as follows:

$$L^{gf}_s = \partial_1 \Phi_2 \partial^\mu \Phi_\mu + \partial^\mu \Phi_2 \partial_1 \Phi_\mu + \partial_1 \Phi_2 \partial_1 \Phi_2 \,. \tag{3.39}$$

Since the superconnection Φ has its values in the Lie algebra $gl(4, R) \otimes ga(4, R)$, one can write the superlagrangian L^{gf}_s in the form

$$L^{gf}_s = L^{gf}_{s_t} + L^{gf}_{s_d} \tag{3.40}$$

where $L^{gf}_{s_t}$ (resp. $L^{gf}_{s_d}$) is associated with the tangents (resp. space-time) indices. Using the fact that

$$\partial_1 \Phi^a{}_{b2}|_{\theta^\alpha=0} = B^a{}_b = [Q, \bar{C}^a{}_b] \tag{3.41}$$

and

$$\partial_1 \Phi^a{}_{b\mu}|_{\theta^\alpha=0} = [Q, \omega^a{}_{b\mu}] \,, \tag{3.42}$$

we obtain

$$L^{gf}_{s_t}|_{\theta^\alpha=0} = B^a{}_b \partial^\mu \omega^a{}_{b\mu} + \partial^\mu \bar{C}^a{}_b [Q, \omega^b{}_{a\mu}] + B^a{}_b B^b{}_a \,. \tag{3.43}$$

Regarding the superlagrangian $L^{gf}_{s_d}$, one notices that the auxiliary field B^λ and anti-ghost $\bar{C}^\lambda$ of the general coordinates transformations are introduced through $\Phi^\lambda{}_{\tau 2}$ using the relations (3.24) and the expression (3.25). Consequently, the expression of $L^{gf}_{s_d}$ can be obtained from the prescription (3.39) by substituting the component superfield $\Phi^\lambda{}_{\tau 2}$ by the anti-ghost superfield η^λ_2 and using the necessary contraction of components the $\Phi^\lambda{}_{\tau\mu}$. This gives

$$L^{gf}_{s_d} = -\partial_1 \eta^\lambda_2 \partial^\mu \Phi^\lambda{}_{\lambda\mu} - \partial^\mu \eta^\lambda_2 \partial_1 \Phi^\lambda{}_{\lambda\mu} + (\partial_1 \eta^\lambda_2)^2 \,. \tag{3.44}$$

Using the fact that

$$\partial_1 \eta^\lambda_2)|_{\theta^\alpha=0} = [Q, \bar{C}^\lambda] = B^\lambda \tag{3.45}$$

and

$$\partial_1 \Phi^\lambda{}_{\lambda\mu}|_{\theta^\alpha=0} = [Q, \hat{g}_{\mu\lambda}] \,, \tag{3.46}$$

we get

$$L^{gf}_{s_d}|_{\theta^\alpha=0} = B^\lambda \partial^\mu \hat{g}_{\mu\lambda} + \partial^\mu \bar{C}^\lambda [Q, \hat{g}_{\mu\nu}] + B^\lambda B_\lambda \tag{3.47}$$

where $\hat{g}_{\mu\nu} = \sqrt{-g}\, g_{\mu\nu}$ and

$$[Q, \hat{g}_{\mu\nu}] = -\partial_\tau (C^\tau \hat{g}_{\mu\nu}) - \hat{g}_{\mu\tau} \partial_\lambda C^\tau - \hat{g}_{\tau\lambda} \partial_\mu C^\tau \,. \tag{3.48}$$

4. Conclusions

The principal superfiber bundle formalism is used to give a suitable geometrical interpretation of the quantized Nonsymmetric gravity. We remark that every superfiber bundle is also a fiber bundle in such a way that the structure of the principal fiber bundle is valuable not only for unquantized, but also for the quantized gauge theory. The metric tensor and Fadeev–Popov fields appear as superfields which are coefficients of a particular kind of connections in the superfiber bundle-soul flat connections coming from the usual connections in the principal fiber bundle describing the unquantized theory. The BRST and anti-BRST transformations are acting over a given connection, through gauge transformations, in the superfiber bundle of parameters the ghost and anti-ghost superfields respectively, which appear in the particular connection on which they act. For the case of soul-flat connections the BRST and anti-BRST transformations are also translations along the anti-commuting variables which act in the base supermanifold. Moreover they generate an $S^{0,2}$ supergroup.

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