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CUMULANT-LIKE CROSS-CORRELATION FUNCTIONS TO DETERMINE TEMPORAL BEHAVIOR OF TWO SIGNALS

BIMAN DAS and NICOLAS LYGA
Department of Physics, State University of New York
Potsdam, New York 13676, USA

Abstract. In statistics, cumulants of a particular order represent "true" moments of that order since in their definition the effects of all lower orders moments have been washed out. We have obtained expressions for the cumulant-like or "true" cross-correlation functions (equivalent to cumulants rather than moments) by subtracting the lower order correlation functions. It is expected that these expressions will provide clearer cross-correlations of two signals showing some features that may be hidden in the usual cross-correlation functions. The expressions will be useful for experimentalists wishing to get useful temporal information from the digitized time series of two experimental signals.

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1. Introduction

Correlation functions are important in different fields of science and engineering. It is a standard technique to analyze a signal on a statistical basis, that is, to calculate its moments, variance, central moments, cumulants, and determine correlation functions to determine the shape of pulses, spectral bandwidths, short-term memory effects and other temporal behavior in the intensity fluctuations of the signals. Their use in optics has been pioneered by Wolf, Glauber and others [1, 2] and their use is wide [3, 4]. Correlation functions are easy to compute; one can write a simple program to compute correlation functions from the digitized intensity time series of the signals of interest.

Cross-correlation functions determine temporal behavior of two signals showing how the fluctuation of one signal is related to the fluctuation of the other. For example, the second order cross-correlation function in the intensity fluctuations of the two orthogonally polarized components of the output from a thin, long, mirrorless gas laser amplifier (amplified spontaneous emission) was computed [5]. The results showed that the signals were strongly anti-correlated when the medium was heavily saturated. The signals were found to be uncorrelated before the gain saturation set in the medium.

It is known that central moments have been defined in statistics to wash out the effects of the first order moment [6, 7]. The second order central moment μ_2 is given by

$$\mu_2 = \int (I - \langle I \rangle)^2 P(I)\,\mathrm{d}I = \langle I^2 \rangle - \langle I \rangle^2 = M_2 - M_1^2 \tag{1}$$

where $P(I)$ is the probability distribution of the variable I and M_1 and M_2 represent first order and second order moments of the distribution, respectively. Clearly, the central moment of second order represents the "true" moment of second order since in the definition the effects of first order moment are washed out. Similarly, in general, the cumulants have been defined in such a way that in the definition of a cumulant of a particular order, the effects of all lower order moments have been washed out. As a result, a cumulant of a particular order represents the "true" moment of that particular order when representing fluctuations [6, 7]. This is one of the reasons why cumulants are always preferred over moments in computing the statistics of fluctuations of experimental signals. In the case of correlation functions, from their usual definitions one cannot expect the true higher order effect of only that particular order to be present in that order since no attempt has been made in the definition to wash out the effect of lower order correlations functions. However, the technique, as used to define cumulants, was used to define "true" or cumulant-like auto-correlation functions to wash out the effects of all lower-order auto-correlation functions. The relations providing cumulant-like auto-correlation functions of first few orders were reported by Das et al [8, 9]. As is well known, a usual auto-correlation function of order $i + j$ is defined as

$$c^{i,j}(\tau) = \left\langle (I(t) - \langle I \rangle)^i \, (I(t+\tau) - \langle I \rangle)^j \right\rangle . \tag{2}$$

Clearly, in the limit of time delay t tends to zero, the preceding auto-correlation function becomes the central moment of order $i+j$ of the distribution. Because of this, the relationship between cumulants and central moments (or moments) were used to define cumulant-like auto-correlation functions.

2. Cumulant-like Cross-correlation Functions

The objective of this article is to provide expressions for the cumulant-like or "true" cross-correlation functions, which are not known but can be easily determined. We have used the similar technique to determine the cumulant-like cross-correlation function as was used for the cumulant-like auto- correlation functions [8, 9].

The cross-correlation function of order $i + j$ of two signals I_1 and I_2 is defined as [5, 6]

$$c_{12}^{ij}(\tau) = \left\langle (I_1(t) - \langle I_1 \rangle)^i \, (I_2(t+\tau) - \langle I_2 \rangle)^j \right\rangle \tag{3}$$

where t is the time delay. In the above definitions, the mean values of the signals have been subtracted to generalize the use of the expression in cases where the mean values of the signals are not meaningful or cannot be measured. One example of this is an experimental situation where a.c.-coupling of the signals and the presence of amplifier noise of the detection system makes it impossible to determine the absolute average and absolute zero of the signals. The bivariate central moments are defined as [7]

$$\mu_{ij} = \left\langle (I_1(t) - \langle I_1 \rangle)^i \, (I_2(t) - \langle I_2 \rangle)^j \right\rangle \tag{4}$$

where μ_{ij} is the bivariate moment of order $i + j$. From the above definitions for cross-correlation function and bivariate central moment, it is clear that as the time delay τ between the signals I_1 and I_2 tends to zero, the $(i+j)^{\text{th}}$ order cross-correlation function becomes the bivariate central moment of that order of the distribution.

To determine the cumulant-like cross-correlation functions we start from the relationship between the bivariate cumulants and bivariate moments of a distribution as reported by Stuart and Ord [7]

$$k_{11} = \mu_{11} \,. \tag{5}$$

Thus

$$kc_{12}^{11}(\tau) = c_{12}^{11}(\tau) \tag{6}$$

where $kc_{12}^{11}(\tau)$ represents the second order cumulant-like cross-correlation function for signals 1 and 2. In general, kc_{12} represent the cumulant-like or "true" cross-correlation function for signals 1 and 2. Also,

$$k_{21} = \mu_{21} \,. \tag{7}$$

Thus, the third order cumulant-like cross-correlation function is

$$kc_{12}^{21}(\tau) = c_{12}^{21}(\tau). \tag{8}$$

For the fourth order, the relationships between the bivariate cumulants and bivariate central moments are

$$k_{31} = \mu_{31} - 3\mu_{20}\mu_{11} \tag{9}$$

and

$$k_{22} = \mu_{22} - \mu_{20}\mu_{02} - 2\mu_{11}^2 . \tag{10}$$

Therefore, fourth-order cumulant-like cross-correlation functions can be written as

$$kc_{12}^{31}(\tau) = c_{12}^{31}(\tau) - 3c_1^{11}(\tau)c_{12}^{11}(\tau), \tag{11}$$

and also

$$kc_{12}^{22}(\tau) = c_{12}^{22}(\tau) - c_1^{11}(\tau)c_2^{11}(\tau) - 2\left(c_{12}^{11}(\tau)\right)^2 \tag{12}$$

where $c_1^{11}(\tau)$ and $c_2^{11}(\tau)$ are the second order auto-correlation functions of the first and second signals, respectively. Since as $\tau \to 0$, $c_{12}^{31}(\tau)$, $c_1^{11}(\tau)$ and $c_{12}^{11}(\tau)$ become (by their definition) μ_{31}, μ_{20} and μ_{11} respectively, $kc_{12}^{31}(\tau)$ becomes k_{31} as $t \to 0$, as expected. For the fifth order, the relations between bivariate cumulants and bivariate central moments are

$$k_{41} = \mu_{41} - 4\mu_{30}\mu_{11} - 6\mu_{21}\mu_{20} \tag{13}$$

$$k_{32} = \mu_{32} - \mu_{30}\mu_{02} - 6\mu_{21}\mu_{11} - 3\mu_{20}\mu_{12}. \tag{14}$$

Thus the cumulant-like fifth-order cross-correlation functions are

$$kc_{12}^{41}(\tau) = c_{12}^{41}(\tau) - 4c_1^{pq}(\tau)c_{12}^{11}(\tau) - 6c_{12}^{21}(\tau)c_1^{11}(\tau) \tag{15}$$

where $p + q = 3$, and p and q are positive integers, and

$$kc_{12}^{32}(\tau) = c_{12}^{32}(\tau) - 4c_1^{pq}(\tau)c_2^{11}(\tau) - 6c_{12}^{21}(\tau)c_{12}^{11}(\tau) - 3c_1^{11}(\tau)c_{12}^{12}(\tau). \tag{16}$$

Finally, for the sixth order, the relationships between bivariate cumulants and bivariate central moments are

$$k_{51} = \mu_{51} - 5\mu_{40}\mu_{11} - 10\mu_{31}\mu_{20} - 10\mu_{30}\mu_{21} + 30(\mu_{20})^2\mu_{11} \tag{17}$$

$$\begin{aligned} k_{42} = \mu_{42} &- \mu_{40}\mu_{02} - 8\mu_{31}\mu_{11} - 4\mu_{30}\mu_{12} - 6\mu_{22}\mu_{20} \\ &- 6(\mu_{21})^2 - 6(\mu_{20})^2 + 24\mu_{20}(\mu_{11})^2 \end{aligned} \tag{18}$$

$$\begin{aligned} k_{33} = \mu_{33} &- 3\mu_{31}\mu_{02} - \mu_{30}\mu_{03} - 9\mu_{22}\mu_{11} - 9\mu_{21}\mu_{12} \\ &- 3\mu_{20}\mu_{13} + 18\mu_{20}\mu_{11}\mu_{02} + 12(\mu_{11})^3. \end{aligned} \tag{19}$$

Thus, the sixth order cumulant-like cross-correlation functions are

$$\begin{aligned} kc_{12}^{51}(\tau) &= c_{12}^{51}(\tau) - 5c_1^{mn}(\tau)c_{12}^{11}(\tau) - 10c_{12}^{31}(\tau)c_1^{11}(\tau) \\ &\quad - 10c_1^{pq}(\tau)c_{12}^{21}(\tau) + 30(c_1^{11}(\tau))^2 c_{12}^{11}(\tau) \end{aligned} \tag{20}$$

where $m + n = 4$, $p + q = 3$, and m, n, p and q are positive integers.

$$\begin{aligned} kc_{12}^{42}(\tau) &= c_{12}^{42}(\tau) - c_1^{mn}(\tau)c_1^{11}(\tau) - 8c_{12}^{31}(\tau)c_{12}^{11}(\tau) - 4c_1^{pq}(\tau)c_{12}^{12}(\tau) \\ &\quad - 6c_{12}^{22}(\tau)c_1^{11}(\tau) - 6(c_{12}^{21}(\tau))^2 + 6(c_1^{11}(\tau))^2 \\ &\quad + 24c_1^{11}(\tau)(c_{12}^{11}(\tau))^2 \end{aligned} \tag{21}$$

where $m + n = 4$, $p + q = 3$, and m, n, p and q are positive integers.

$$\begin{aligned} kc_{12}^{33}(\tau) &= c_{12}^{33}(\tau) - 3c_{12}^{31}(\tau)c_2^{11}(\tau) - c_1^{pq}(\tau)c_1^{rs}(\tau) - 9c_{12}^{22}(\tau)c_{12}^{11}(\tau) \\ &\quad - 9c_{12}^{21}(\tau)c_1^{12}(\tau) - 3c_1^{11}(\tau)c_{12}^{13}(\tau) \\ &\quad + 18c_1^{11}(\tau)c_{12}^{11}(\tau)c_2^{11}(\tau) + 12(c_{12}^{11}(\tau))^3 \end{aligned} \tag{22}$$

where $p + q = 3$, $r + s = 3$, and p, q, r and s are positive integers.

As the time delay $\tau \to \infty$ the signals 1 and 2 become independent, and the cross-correlation function of the signal (as defined by Eq. (3)) becomes the product of the corresponding central moments of the signals. We define $KC_{12}^{ij}(\tau)$ as

$$KC_{12}^{ij}(\tau) = kc_{12}^{ij}(\tau) - \left\langle (I_1(\tau) - \langle I_1 \rangle)^i \right\rangle \left\langle (I_1(t) - \langle I_1 \rangle)^j \right\rangle \tag{23}$$

where $KC_{12}^{ij}(\tau)$ is the cumulant-like cross-correlation function of order $i + j$ whose asymptotic value is zero. $KC_{12}^{ij}(\tau)$ will provide important information about the temporal behavior of two signals.

3. Remarks and Conclusion

The cumulant-like auto-correlation functions as defined in Ref. [9] were computed for the intensity fluctuations of a heavily saturated source of 3.51 μm radiation coming out from a thin, 3 m long, mirrorless, gas laser amplifier (amplifying spontaneous emission). This output radiation had shown a high degree of directionality and coherence. The intensity of the spontaneously emitted field in any mode displayed rapid and significant fluctuations over the average intensity due to interference effects between the incoherently excited sources even under heavy gain saturation [5-7]. A fast transient digitizer (Lecroy model TR8828C) recorded the digitized intensity time-series of the heavily saturated, linearly polarized, output of the radiation. The intensity time-series is shown in Fig. 1. The digitizer provided 8-bit resolution and digitized at the maximum rate of 200 MHz. Figures 2 and 3 show the auto-correlation function and cumulant-like auto-correlation function of 4[th] order, respectively, for the heavily saturated source of amplified spontaneous emission computed from the intensity-time series. In Fig. 2, $C^{22}(\tau) = c^{22}(\tau) - c^{22}(\infty)$, where

$$c^{22}(\tau) = \left\langle (I(t) - \langle I \rangle)^2 \, (I(t+\tau) - \langle I \rangle)^2 \right\rangle . \tag{24}$$

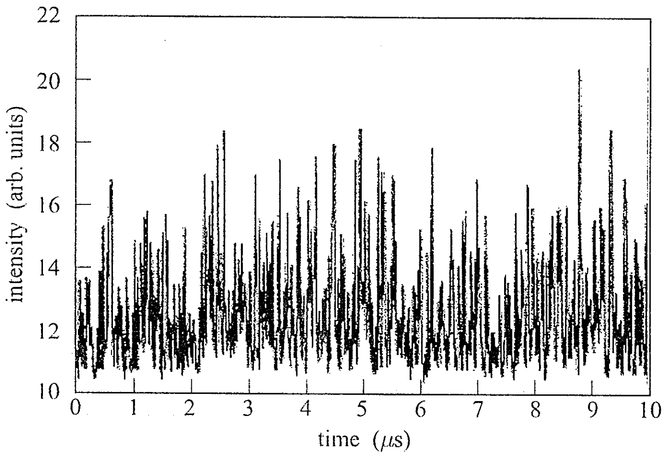

Fig. 1. Intensity-time series of a 3.51 μm linearly polarized heavily saturated amplified spontaneous emission coming from Xe-He

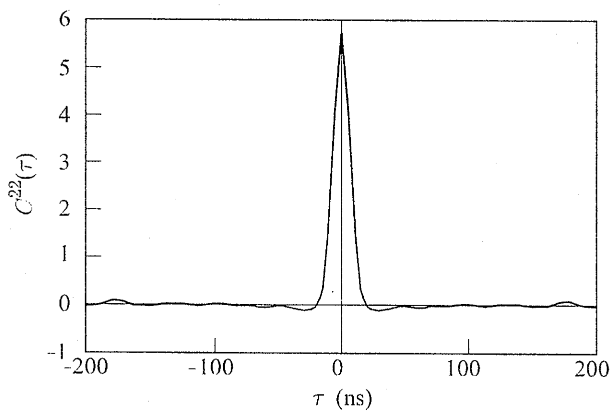

Fig. 2. Fourth order auto-correlation function $C^{22}(\tau) = c^{22}(\tau) - c^{22}(\infty)$ for a heavily saturated source of 3.51 μm amplified spontaneous emission from Xe-He

Figure 3 shows the result of the computation of the cumulant-like auto-correlation function $D^{22}(\tau)$, which was defined as [8, 9] $D^{22}(\tau) = d^{22}(\tau) - d^{22}(\infty)$, where $d^{22}(\tau) = c^{22}(\tau) - 3\left(c^{11}(\tau)\right)^2$. Clearly in the definition of

$d^{22}(\tau)$, the effect of lower order auto-correlation function is subtracted. The auto-corelation time (full width at half maximum of the auto-correlation function) obtained from both figures 2 and 3 is same and about 20 ns. Both these correlation functions show single peak at zero delays, but only $D^{22}(\tau)$ sharply falls to the negative values and shows shoulders. It is interesting to see the similarities and differences between the plots of $D^{22}(\tau)$ and $C^{22}(\tau)$. In general, computation of cumulant-like auto-correlation functions had shown some features that were not clear in the usual auto-correlation functions.

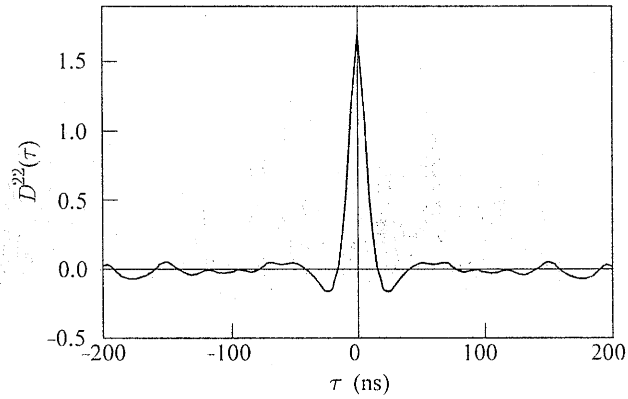

Fig. 3. Fourth order cumulant-like auto-correlation function $D^{22}(\tau) = d^{22}(\tau) - d^{22}(\infty)$ where $d^{22}(\tau) = c^{22}(\tau) - 3(c^{11}(\tau))^2$ for a heavily saturated source of 3.51 μm amplified spontaneous emission from Xe-He

Cross-correlation functions determine the temporal behavior in the fluctuations of two signals and show how the fluctuation of one signal is related to the fluctuation of the other. For example, the second order cross-correlation function in the intensity fluctuations of the two selected orthogonally polarized components of the output from a cw d.c.-excited, thin, 200 m long, mirrorless Xe-He gas laser amplifier (amplified spontaneous emission) was determined by Adams and Abraham [5] for various values of the small signal gain of the medium and for different degrees of homogeneous broadening. Their experimental results showed that the signals were strongly anticorrelated when the medium was heavily saturated. The signals were found to be uncorrelated before the gain saturation set in the medium. This anti-correlation was attributed to the strong coupling between the propagating beams as the medium became saturated. A plot of the second order cross correlation $\langle I_1 I_2\rangle - \langle I_1\rangle\langle I_2\rangle$ (normalized to the product of the variances of the two signals) of orthogonally polarized copropagating beams as a function of amplifier length in a similar system by our research group showed that its value became more negative as the medium became heavily saturated [3]. One would expect the result to be zero if the signals were independent. Higher order cross-correlation functions are more sensitive to large pulses than lower order cross-correlation functions. They can be determined from the experimental intensity-time series of the two signals of the above-mentioned optical system (amplified spontaneous emission) or other experimental optical systems. In a similar way the auto-correlation functions are determined from the intensity time-series of a signal.

In this article we have provided expressions for the computations of cumulant-like or "true" cross-correlation functions to two signals. We obtained the expressions by subtracting the effects of lower order auto and cross-correlation functions starting from the relationships between bivariate moments and bivariate cumulants given by Stuart and Ord [7]. It is expected that the relations provided in this article for the cumulant-like cross-correlation functions will provide clearer cross correlation between two signals. These relations will provide important temporal information about two signals showing some features that may be hidden in the usual cross-correlation functions. The relations can be useful for experimentalists wishing to determine temporal behavior of two experimental signals from their intensity-time series.

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