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LRS BIANCHI TYPE-I COSMOLOGICAL MODELS WITH PERFECT FLUID

GANESHWAR MOHANTY
School of Mathematical Sciences, Sambalpur University
Jyoti Vihar, Sambalpur-768019, Orissa, India

BIVUDUTTA MISHRA
Icfai Institute of Science and Technology
Bhubaneswar-751023, Orissa, India

Abstract. LRS Bianchi-I space–time filled with perfect fluid is considered. It is shown that the field equations together with the conservation equations are solvable when the metric potential B is assumed as separable function of x and t as $B = f(x)g(t)$. Subsequently the space–time metric with a suitable x-coordinate transformation turns out to be

$$ds^2 = dt^2 - g^2(t)\left[dX^2 + e^{\frac{2X}{k}}(dY^2 + dZ^2)\right]$$

where k is a non-zero arbitrary constant. Three particular cases pertaining to the equations of state $p = 0$ and $\rho = 0$, $p = \frac{\rho}{3} > 0$ and $p + \rho = 0$ are considered and exact cosmological solutions are obtained. Some physical properties of the models are discussed.

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1. Introduction

For simplification and description of the large-scale behavior of the actual universe, locally rotationally symmetric [henceforth refereed as to LRS] Bianchi-I space–time has been widely studied. In order to study problems like the formation of galaxies and the process of homogenization and isotropization of the universe, it is necessary to study problems relating to inhomogeneous and anisotropic space–time. Recently Mazumder [1] has obtained cosmological solutions for LRS Bianchi-I space–time filled with a perfect fluid with arbitrary cosmic scale functions and studied kinematical properties of the particular form of the solution. Hajj-Boutros and Sfeila [2] and Shriram [3] also obtained some solutions for the same field equations by using solution-generation technique. Taub [4] and Tomimura [5] have studied inhomogeneous cosmologies. In all these models the material distribution is that of a perfect fluid. Here we have taken an attempt to solve the field equations when the metric potentials are functions of both space and time variables, corresponding to non-homogeneous anisotropic space-time.

In Section 2, we have derived Einstein's field equations and conservation equations for LRS Bianchi Type-I metric and discussed their consequences. In Section 3, we have considered three different cases pertaining to different viable physical situations governed by three equation of states $p = 0$ and $\rho = 0$, $p = \frac{\rho}{3} > 0$, $p + \rho = 0$ and explicit solutions are obtained in two cases. However it is shown that the case corresponding to dust distribution does not provide solution in explicit form. In Section 4, some physical properties are discussed. It is found that the universe starts evolution with zero volume in the radiating model and from constant volume in false vacuum model. Even though both models behave alike in physical nature, the radiating model approaches homogeneity whereas the false vacuum model does not. The concluding remarks are given in Section 5.

2. Field Equations and their Consequences

The metric for the LRS Bianchi-I space-time is of the form

$$\mathrm{d}s^2 = \mathrm{d}t^2 - A^2\,\mathrm{d}x^2 - B^2(\mathrm{d}y^2 + \mathrm{d}z^2) \tag{1}$$

where A and B are functions of x and t.

The energy-momentum tensor of a perfect fluid is given by

$$T_{ij} = (p + \rho)u_i u_j - p g_{ij}\,, \qquad u_i u^i = 1 \tag{2}$$

where p is the pressure and ρ is mass-energy density.

In view of Eqs (1) and (2), the explicit forms of the field equations

$$R_{ij} - \frac{1}{2} R g_{ij} = -8\pi T_{ij} \tag{3}$$

for the comoving coordinate system $u_i = (0, 0, 0, 1)$ are

$$2\frac{B_{44}}{B} - \frac{B_1^2}{A^2B^2} + \frac{B_4^2}{B^2} = -8\pi p \tag{4}$$

$$\frac{B_{44}}{B} - \frac{B_{11}}{A^2B} + \frac{A_1B_1}{A^3B} + \frac{A_4B_4}{AB} + \frac{A_{44}}{A} = -8\pi p \tag{5}$$

$$2\frac{B_{11}}{A^2B} - 2\frac{A_1B_1}{A^3B} - 2\frac{A_4B_4}{AB} + \frac{B_1^2}{A^2B^2} - \frac{B_4^2}{B^2} = -8\pi\rho \tag{6}$$

$$B_{14} - \frac{B_1A_4}{A} = 0\,. \tag{7}$$

Hereafterwards the subscripts 1 and 4 represent partial derivatives with respect to x and t respectively.

The energy conservation equations $T^{ij}_{;j} = 0$ for the metric (1) take the forms:

$$p_1 = 0 \qquad \text{i. e.} \qquad p = p(t) \tag{8}$$

and

$$\frac{\rho_4}{p+\rho} = -\left(\frac{A_4}{A} + 2\frac{B_4}{B}\right). \tag{9}$$

On integration Eq. (7) yields

$$A = B_1 h(x) \tag{10}$$

where $h(x)$ is an arbitrary function of x.

In order to avoid the mathematical complexities due to nonlinear nature of the field equations, we assume B as a separable function of t and x in the form:

$$B = g(t)f(x)\,. \tag{11}$$

With the help of Eqs (10) and (11), Eqs (4) and (5) yield

$$\frac{1}{h^2f^2} = 8\pi p(t)g^2 + 2gg_{tt} + g_t^2 \tag{12}$$

$$\frac{h_x}{h^3ff_x} = -8\pi p(t)g^2 - 2gg_{tt} - g_t^2\,. \tag{13}$$

Hereafterwards the suffix x and t represent ordinary derivative with respect to x and t respectively.

From above two Eqs (12) and (13), we obtain

$$hf = k \tag{14}$$

where k is a non-zero real constant.

In view of Eq. (14), Eqs (12) and (13) reduce to a single equation, i. e.

$$\frac{1}{k^2} = 8\pi p(t)g^2 + 2gg_{tt} + g_t^2 \tag{15}$$

so, also

$$A = kg\frac{f_x}{f} \qquad \text{and} \qquad B = gf\,. \tag{16}$$

With the help of the aforesaid metric potentials obtained in (16) and a suitable transformation the metric (1) becomes

$$\mathrm{d}s^2 = \mathrm{d}t^2 - g^2(t)\left[\mathrm{d}X^2 + \mathrm{e}^{\frac{2X}{k}}(\mathrm{d}Y^2 + \mathrm{d}Z^2)\right]\,. \tag{17}$$

On substitution of Eq. (16) in Eq. (9), we obtain

$$\frac{\rho_4}{\rho + p} = -3\frac{g_t}{g} \tag{18}$$

which is a linear differential equation of first order in ρ.

On integration, it yields

$$\rho = -\frac{3}{g^3}\int p(t)g^2 g_t\,\mathrm{d}t + M(x) \tag{19}$$

where $M(x)$ is an arbitrary function of x.

On substitution of Eq. (16), Eq. (6) yields

$$-8\pi\rho = \frac{3}{k^2g^2} - \frac{3g_t^2}{g^2}\,. \tag{20}$$

Equations (19) and (20) imply

$$M(x) = M \quad \text{(absolute constant)}\,.$$

Now, on further substitution of Eq. (19) in Eq. (20), we obtain

$$2gg_{tt} = \frac{1}{k^2} + 8\pi Mg^2 - 8\pi p(t)g^2 - g_t^2\,. \tag{21}$$

Substitution of Eq. (15) in Eq. (21) gives

$$M = 0\,, \qquad \text{since} \qquad g(t) \neq 0\,. \tag{22}$$

Thus the matter density depends on t only.

The magnitude for scalar expansion θ and shear scalar σ for the model (17) are

$$\theta = 3\frac{g_t}{g} \qquad \text{and} \qquad \sigma^2 = 0\,. \tag{23}$$

It is clear that $\frac{\sigma^2}{\theta^2} = 0$, which confirms the isotropic nature of the space- time, which we have already obtained in Eq. (17). The spatial volume is

$$V = g^3 \, e^{\frac{2X}{k}} .$$

Also the rotation w and acceleration turn out to be zero.

Equations (15) and (20) lead to the well known Raychaudhuri equation [6, 7] in the form:

$$\theta_t + \frac{\theta^2}{3} + 4\pi(\rho + 3p) = 0 . \tag{24}$$

3. Cosmological Models

In this section, we intend to construct cosmological models relating to various equations of state.

Case I: Dust Model ($p = 0, \rho > 0$)

In this case Eq. (18) yields

$$\rho = \frac{c}{g^3} \tag{25}$$

where c is the constant of integration.

Substituting Eq. (25) in Eq. (20), we obtain

$$g_t = \sqrt{\frac{1}{k^2} + \frac{8\pi c}{3g}}$$

which on integration yields

$$k^3 \left\{ \sqrt{\frac{g}{k^2} + \frac{8\pi c}{3g}} \sqrt{\frac{g}{k^2}} - \frac{8\pi c}{3} \log \left[\sqrt{\frac{g}{k^2} + \frac{8\pi c}{3g}} + \sqrt{\frac{g}{k^2}} \right] \right\} = t + k_1 \tag{26}$$

where k_1 is the constant of integration.

Case II: Radiating Model $\left(p = \frac{\rho}{3} > 0\right)$

In this case Eq. (18) yields

$$\rho = \frac{c}{g^4} \tag{27}$$

where c is a positive constant of integration for the radiating model.

Substituting Eq. (27) in Eq. (20), we obtain

$$g_t = \sqrt{\frac{1}{k^2} + \frac{8\pi c}{3g^2}}$$

which on integration yields

$$g = \frac{1}{k}\sqrt{(t + k_1)^2 - \frac{8\pi c k^4}{3}} \tag{28}$$

where k_1 is the constant of integration.

With suitable transformations in this case, one can find the model as

$$\mathrm{d}s^2 = \mathrm{d}T^2 - \left(T^2 - \frac{8\pi c k^4}{3}\right)\left[\mathrm{d}X^2 + \mathrm{e}^{2X}\left(\mathrm{d}Y^2 + \mathrm{d}Z^2\right)\right] . \tag{29}$$

Case III: False Vacuum Model ($p + \rho = 0$)

Davies [8], Blome and Priester [9], Hogan [10], Kaiser and Stebbins [11] have studied some of the physical aspects of this equation of state. The analogous model in the static case is quite evident and is the well known de Sitter model (Tolman [12]).

In this case Eq. (18) yields

$$\rho = C(x) \tag{30}$$

where $C(x)$ is an arbitrary function of x.

Subsequently substitution of Eq. (30) in Eq. (20), we get

$$-8\pi C(x) = \frac{3}{k^2 g^2} - \frac{3g_t^2}{g^2}$$

which gives

$$C(x) = C \quad \text{(absolute constant)} .$$

On physical ground C must be a positive constant.

Hence, we obtain

$$g_t = \sqrt{\frac{1}{k^2} + \frac{8\pi C}{3g^2}}$$

which on integration yields a solution as

$$g = \frac{1}{2\sqrt{\frac{8\pi C}{3}}}\left[\mathrm{e}^{\left(\sqrt{\frac{8\pi C}{3}}t + C_1\right)} - \frac{1}{k^2}\,\mathrm{e}^{-\left(\sqrt{\frac{8\pi C}{3}}t + C_1\right)}\right] \tag{31}$$

where C_1 is the constant of integration.

With suitable transformations in this case, one can find the model in the form:

$$ds^2 = dT^2 - (k^2 e^{UT} - e^{-UT}) \left[dX^2 + e^{2X} (dY^2 + dZ^2)\right] \tag{32}$$

where $U = \sqrt{\frac{8\pi C}{3}}$.

4. Some Physical Features

For the case $p = \frac{\rho}{3}$, the scalar expansion $\theta = u^i_{;i}$ calculated as $\theta = \frac{9T}{3T^2 - 8\pi ck^4}$, from which it is evident that $\theta \to 0$ as $T \to \infty$ and $\theta \to \infty$ as $T \to \sqrt{\frac{8\pi c}{3}} k^2 = (\text{say } T_0)$.

The universe is expanding as $\theta > 0$ for $T > T_0$ and contracting as $\theta < 0$ for $T < T_0$. But the rate of expansion is decelerated with increase of time since $\theta_t < 0$ for all T. In this case it is evident from Eq. (24) that the strong energy condition (Hawking–Penrose, [13]) is satisfied. It is obvious from Eqs (27) and (29) that the evolution of the universe starts with a singularity at T_0. This may correspond to a Bigbang singularity. It has also been observed that $\frac{\rho}{\theta^2} = \frac{ck^4}{9T^2} \to 0$ as $T \to \infty$. Thus the cosmological model approaches homogeneity with increase of time.

The spatial volume is found to be $V = e^{2X}\left(T^2 - \frac{8\pi ck^4}{3}\right)^{\frac{3}{2}}$. So, $V = 0$ at $T_0 = \sqrt{\frac{8\pi ck^4}{3}}$, which shows that the universe starts evolution with zero volume at $T = T_0$ and expands with T.

The scalar expansion θ for the case $p + \rho = 0$ calculated as $\theta = 3U\left(\frac{k^2 e^{UT} + e^{-UT}}{k^2 e^{UT} - e^{-UT}}\right)$, from which it is evident that $\theta \to 3U$ (constant) as $T \to \infty$ and $\theta \to 3U\frac{k^2+1}{k^2-1}$ (constant) as $T \to 0$.

The universe is expanding with increase of time, but the rate of expansion is decelerated with increase of time, since $\theta_t < 0$, for all T. It has also been observed that $\frac{\rho}{\theta^2} \to \frac{C}{9U^2}$ (constant) as $T \to \infty$ and $\frac{\rho}{\theta^2} \to \frac{C}{9U^2}\left(\frac{k^2-1}{k^2+1}\right)^2$

(constant) as $T \to 0$. Since $\frac{\rho}{\theta^2}$ increases with time, the model does not approach homogeneity. However, since $\sigma = 0$ it approaches isotropy.

The spatial volume is $V = \frac{1}{8U^3k^6}\left(k^2\, e^{UT} - e^{-UT}\right)^3 e^{\frac{2X}{k}}$. Here $V \to \infty$ as $T \to \infty$ and $V \to \frac{(k^2-1)^3}{8U^3k^3}$ as $T \to 0$. Thus the universe starts evolution from constant volume and expands continuously with time.

5. Conclusions

We have presented exact Bianchi Type-I cosmological models for perfect fluid equations of state $p = \frac{\rho}{3}$ and $p + \rho = 0$. It is observed that the models satisfy strong energy condition and confirm the isotropic nature of the space-time, since $\sigma^2 = 0$. However in general the Raychaudhuri equation holds good for all the models as a consequence of strong energy condition. The major difference between the models is that the radiation model leads to homogeneity with a Bigbang like singularity at $T = T_0$ whereas the false vacuum model does not lead to homogeneity without singularity at initial epoch.

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