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STRING COSMOLOGY IN THE LYRA GEOMETRY

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Abstract. A class of cosmological solutions of massive strings are obtained in Kantowski–Sach space–time based on Lyra's geometry. The different equations of state for string model namely: (i) $\rho = \rho(\lambda)$ (barotropic equation of state); (ii) $\rho = \lambda$ (geometric strings); (iii) $\rho = (1 + w)\lambda$ (Takabayashi string) are considered.

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1. Introduction

The study of the universe as a whole has been an interesting subject for theoretical physicists since the appearance of the theory of general relativity. The string theory has been an useful concept before the creation of the particle in the universe [1–5]. Cosmic strings have been considered in the study of early universe cosmology. They may be one of the sources of density perturbations that are required for formation of large scale structure in the universe [3].

The existence of a large scale network of strings in the early universe does not contradict the present day observations of the universe. Moreover, the galaxy formation can be explained by the density fluctuations of the vacuum strings [4].

Stachel [1] considered massless strings. The energy–momentum tensor for the massive strings has been first formulated by Letelier [2], who considered the massive strings being formed by geometric strings with particles attached along its extension. So the total energy-momentum tensor for a cloud of massive strings can be written as [6]

$$T_b^a = \rho V_a V^b - \lambda x_a x^b \tag{1}$$

where ρ is the rest energy density for a cloud of strings with particles attached to them. Thus we have

$$\rho = \rho_p + \lambda \tag{2}$$

ρ_p is the particle density, λ is the strings tension density, V_a is the four-velocity for the cloud of particles and x^a is the four-vector representing the strings direction which essentially is the direction of anisotropy.

Thus

$$V_a V^a = -1 = -x_a x^a \quad \text{and} \quad V_a x^b = 0$$
$$\text{in } (-,+++) \text{ signature}.$$

Since the discovery of general relativity by Einstein, there have been numerous modifications of it. Lyra [7] has suggested a modification of Riemannian geometry by introducing a gauge function into the structureless manifold that bears a close resemblance to Weyl's geometry. Subsequent investigations were done by several authors in scalar tensor theory and cosmology within the framework of Lyra geometry [8].

But ,as far our knowledge there has not been any work in literature where Lyra's geometry to be considered in string cosmology.

So it is interesting to study cosmological model in string theory within the framework of Lyra's geometry.

In this paper, we shall study string cosmology in Kantowski-Sach space-time based on Lyra's geometry in normal gauge, i. e. displacement vector

$$\Phi_i = (\beta(t), 0, 0, 0)\,. \tag{3}$$

2. Basic Equation

The metric ansatz for the spatially homogenous Kantowski-Sach space-time is taken in the following form:

$$ds^2 = -\,dt^2 + A^2\,dx^2 + B^2(d\theta^2 + \sin^2\theta\,d\phi^2) \tag{4}$$

where $A = A(t)$, $B = B(t)$.

The field equations in normal gauge for Lyra's manifold, obtained by Sen [8], are

$$R_{ab} - \frac{1}{2} g_{ab} R + \frac{3}{2}(\phi_a \phi_b - \frac{1}{2} g_{ab} \phi_\alpha^\alpha) = -\chi T_{ab} \tag{5}$$

where ϕ_a is the displacement field of vector defined in [3] and other symbols have their usual meaning as in the Riemannian geometry.

The field equation (5) for the metric (4) reduces

$$\frac{2\dot{A}\dot{B}}{AB} + \frac{\dot{B}^2}{B^2} + \frac{1}{B^2} - \frac{3}{4}\beta^2 = \rho \tag{6}$$

$$\frac{2\ddot{B}}{B} + \frac{\dot{B}^2}{B^2} + \frac{1}{B^2} + \frac{3}{4}\beta^2 = \lambda \tag{7}$$

$$\frac{\ddot{A}}{A} + \frac{\ddot{B}}{B} + \frac{\dot{A}\dot{B}}{AB} + \frac{3}{4}\beta^2 = 0\,. \tag{8}$$

The physical quantities, namely the proper volume R^3, the expansion scalar θ and the shear σ^2 have the following expressions for the above ansatz:

$$R^3 = AB^2 \tag{9}$$

$$\theta = \frac{3\dot{R}}{R} = \frac{\dot{A}}{A} + \frac{2\dot{B}}{B} \tag{10}$$

$$\sigma^2 = \frac{2}{3}\left(\frac{\dot{B}}{B} - \frac{\dot{A}}{A}\right)^2 . \tag{11}$$

The energy conditions [6] restrict ρ and ρ_p be positive semidefinite, but λ may take values positive, negative or zero as well.

The different equations of state for string model [6] are

I.	$\rho = \rho(\lambda)$	Barotropic Equation of State;
II.	$\rho = \lambda$	Geometric String;
III.	$\rho = (1 + w)\lambda$	Takabayashi String, i. e. P-string.

In the following section we shall determine the exact solutions of the field equations using the above equations of state for string model in Lyra's geometry.

3. Solutions and Discussions

Case I: Barotropic Equation of State $\rho = \rho(\lambda)$

In this case we take displacement vector as constant, i. e. $\beta = \text{const}$. To solve the field equations one note that there are three field equations connecting four unknowns. So one more relation connecting these variables is needed. Hence we assume $A = \mu B^n$ (μ and n are constants) between the scale factors for unique solution of the field equations.

Using this relation we get from (8) the following:

$$\frac{\ddot{B}}{B} + \frac{n^2}{n+1}\frac{\dot{B}^2}{B^2} = -\frac{3\beta^2}{4(n+1)}\,. \tag{12}$$

This differential equation can be rewritten in the integral form

$$\int \frac{\mathrm{d}B}{\sqrt{DB^{\frac{-2n^2}{n+1}} - \frac{3\beta^2 B^2}{4(n^2+n+1)}}} = \pm(t - t_0) \tag{13}$$

where D is the integration constant. The above integral can be solved only when $\frac{2n^2}{n+1}$ is an even integer, i. e. when $\frac{2n^2}{n+1} = 2m$ $(m > 1)$.

Hence we obtain

$$B = (b \sin E(t - t_0))^{\frac{1}{m+1}} \tag{14}$$

where $E = \frac{3\beta^2}{4(n^2+n+1)}$ and $b^2 = \frac{4(n^2+n+1)D}{3\beta^2}$. The other parameters have the following expressions:

$$A = \mu \left[b \sin E(t - t_0)\right]^{\frac{n}{m+1}} \tag{15}$$

$$\rho = \frac{(2n+1)E^2}{(m+1)^2} \tan^2 E(t - t_0) + \left[b \sin E(t - t_0)\right]^{-\frac{2}{m+1}} - \frac{3}{4}\beta^2 \tag{16}$$

$$\lambda = \frac{3E^2}{(m+1)^2} \tan^2 E(t - t_0) + \frac{E^2}{m+1} \sec^2 E(t - t_0) + \left[b \sin E(t - t_0)\right]^{-\frac{2}{m+1}} + \frac{3}{4}\beta^2 \tag{17}$$

$$\theta = \frac{(n+2)E}{m+1} \tan E(t - t_0) \tag{18}$$

$$\sigma^2 = \frac{2}{3} E^2 (n-2)^2 \tan^2 E(t - t_0) \tag{19}$$

$$R^3 = \mu \left[b \sin E(t - t_0)\right]^{\frac{n+2}{m+1}} \tag{20}$$

$$p_p = \frac{2(n-1)E^2}{(m+1)^2} \tan^2 E(t - t_0) - \frac{E^2}{m+1} \sec^2 E(t - t_0) - \frac{3}{2}\beta^2 \tag{21}$$

We see that the Universe starts at an initial epoch $t = t_0$ which is a point singularity. At this instant, the physical quantities $\theta, \sigma \to 0$.

We see that as time proceeds the volume increases but at $t = t_0 + \frac{\pi}{2E}$ the volume will attain its maximum value and after that as time increases, the volume decreases and at $t \to t_0 + \frac{\pi}{E}$ the volume will be zero. At epoch of maximum volume $\sigma^2, \theta \to \infty$.

Case II: Geometric String $\rho = \lambda$

Taking the following combination of equations (6) - (7) + 2(8), we get:

$$\frac{2\ddot{A}}{A} + \frac{4\dot{A}\dot{B}}{AB} = 0\,. \tag{22}$$

After integrating we obtain

$$B^2 = \frac{B_0}{\dot{A}}\,, \qquad B_0 \text{ is integration constant.} \tag{23}$$

Therefore for any $A(t)$ from (24) one can obtain $B(t)$.

Sub Case I: Exponential Form

Let

$$A = A_0\, \mathrm{e}^{nt}\,, \qquad A_0 \text{ and } n \text{ are arbitrary constants.} \tag{24}$$

Hence we get

$$B^2 = \frac{B_0}{A_0 n}\, \mathrm{e}^{-nt} \tag{25}$$

$$\theta = 0 \tag{26}$$

$$\sigma^2 = \frac{3}{2} n^2 \tag{27}$$

$$\rho = \frac{5n^2}{8} + \frac{A_0 n}{B_0}\, \mathrm{e}^{nt} \tag{28}$$

$$\beta^2 = -2n^2 \tag{29}$$

$$R^3 = \frac{\mu B_0}{n}\,. \tag{30}$$

This model is not physically interesting since the proper volume of the Universe is constant. Also one can note that β is imaginary.

Sub Case II: Power Law

Let

$$A = A_0 t^n \qquad (A_0 \text{ and } n \text{ are constants}). \tag{31}$$

Hence B takes the form as

$$B^2 = \frac{B_0}{A_0 n} t^{1-n}\,. \tag{32}$$

The physical parameters are

$$R^3 = \frac{B_0}{n} t \tag{33}$$

$$\theta = \frac{1}{t} \tag{34}$$

$$\sigma^2 = \frac{1}{6} \frac{(3n-1)^2}{t^2} \tag{35}$$

$$\rho = \frac{n-1}{t^2} + \frac{2A_0 n}{B_0} t^{n-1} \tag{36}$$

$$\beta^2 = \frac{3n-1}{t^2} \, . \tag{37}$$

From the above solutions we note that at the initial epoch ($t = 0$) $R^3 \rightarrow 0$ while ρ, θ, σ^2 diverse.

If $n > 1$ then $A \rightarrow 0$ but $B \rightarrow \infty$ as $t \rightarrow 0$. So it is a line singularity. If $n < 1$ then $A, B \rightarrow 0$ as $t \rightarrow 0$. So it is a point singularity. At a later stage, when $t \rightarrow \infty$, R^3 also tends to infinity and ρ becomes insignificant, but other parameters tend to zero. The gauge function is large in the beginning but decreases with the evolution of the model.

Case III: Takabayashi String, i. e. P-string

Here the equation of state $\rho = (1 + w)\lambda$ where $w > 0$ is a constant small enough for string dominant era and large for particle dominant era.

Further using the polynominal relation $A = \mu B^n$ between the metric coefficients, from the field equations we get

$$\frac{\ddot{B}}{B} + a \frac{\dot{B}^2}{B^2} = \frac{b}{B^2} \tag{38}$$

where $a = \dfrac{2n^2 + 2n + wn^2 - w}{2n + wn - w}$, $b = \dfrac{w}{2n + wn - w}$. The above second-order differential equation has a first integral of the form

$$\dot{B}^2 = \frac{b}{(2a-1)B} + DB^{-2a} , \qquad D \text{ is integration constant.} \tag{39}$$

This differential equation can be rewritten in the integral form

$$\int \frac{dB}{\sqrt{\dfrac{b}{(2a-1)B} + DB^{-2a}}} = \pm(t - t_0) \tag{40}$$

t_0 is another integration constant.

From the above integral equation B can be obtained in a closed form only for $a = 0$, $a = 1$ and $D = 0$.

For $a = 0$ and $a = 1$ we get the forms of B respectively

$$\frac{2}{\sqrt{D}}\left[\frac{\sqrt{B}\sqrt{B-C^2}}{2} + \frac{C^2}{2}\ln(\sqrt{B}+\sqrt{B-C^2})\right] = \pm(t-t_0) \quad (41)$$

where $C^2 = \dfrac{b}{D}$ and

$$\frac{2(bB-2D)}{3b^2}\sqrt{bB+D} = \pm(t-t_0)\,. \quad (42)$$

But here we cannot get explicit forms of B in terms of t.

For $D = 0$ we get

$$B = \sqrt{\frac{3b}{4a-2}}(t-t_0)^{\frac{2}{3}}\,. \quad (43)$$

The other parameters have the following expressions:

$$A = \mu\left(\frac{3b}{4a-2}\right)^{\frac{n}{2}}(t-t_0)^{\frac{2n}{3}} \quad (44)$$

$$R^3 = \mu\left(\frac{3b}{4a-2}\right)^{\frac{n+2}{2}}(t-t_0)^{\frac{2(n+2)}{3}} \quad (45)$$

$$\theta = \frac{2(n+2)}{3(t-t_0)} \quad (46)$$

$$\sigma^2 = \frac{8(n-1)^2}{27(t-t_0)^2} \quad (47)$$

$$\frac{3}{4}\beta^2 = \frac{1}{2+w}\left[\frac{4}{9}(2a+2aw+2n-w)\frac{1}{(t-t_0)^2} - \frac{(4a-2)(w+2b+2bw)}{3b(t-t_0)^{\frac{4}{3}}}\right] \quad (48)$$

$$\lambda = \frac{8(n+1-a)}{3(2+w)(t-t_0)^2} + \frac{4(b+1)(2a-1)}{3b(2+w)(t-t_0)^{\frac{4}{3}}}\,. \quad (49)$$

The instant $t = t_0$ is the point singularity while the physical variables diverge. This is the starting point of the string model. Again at a later stage, when $t \to \infty$, R^3 also tends to infinity, but all physical parameters, including gauge function, become insignificant. So string concept will not linger for infinite time.

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