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LIGHT SOLITONS IN NONLINEAR MEDIA. SELF-CHANNELLING

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Abstract. The light beam behaviour in nonlinear media is considered in the case when the nonlinear polarization contains susceptibilities of third and fifth order. Taking into account that the last ones can take different signs a number of new solitary solutions for the nonlinear differential equations deduced are found in analytical form. The effects of self-action (self-channelling) are considered. Numerical values and estimations are presented.

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1. Introduction

The intensive development of laser physics and technics has led to much extensive investigations of a wide class of nonlinear light phenomena regarded to the high power light density. In recent 30 years many papers and informative reviews concerning self-action of light beams (self-focusing, self-phase modulation of optical pulses) and phenomena related to the light transmission have been published. The reason is not only in the fact that the plane wave approximation does not give a qualitative information in the cases when nonlinear investigations take place. Both the theoretical and experimental investigations show that the utilization of the nonlinear dependence of the index of refraction on field intensity makes possible new effects possessing remarkable behaviour of the solitons [1–8]. In general, the nonlinear systems are rich soil for investigation of a wide variety of solitons from the point of view of their form and behaviour. Firstly the experimental observation of solitons was reported in [9]. Subsequently a number of papers are devoted to soliton creation and propagation in fibers [8, 17]. By virtue of the soliton conception a generation of optical pulses with an envelope of the sech-profile in subpicosecond range is possible (soliton laser, see [18, 19]). Concerning the problem of light self-action, besides its fundamentality, a number of important applications can take place. In particular, the self-focusing is the limiting factor in the design of high-power laser amplifiers and is closely related with other physical processes in a medium (e. g. it plays an important role in generating plasmas through optical breakdown).

Usually the nonlinear equations governing such problems are studied by means of the nonlinear Schrödinger-like equations (NLSLE) possessing the nonlinearites of third or third and fifth order with respect to the function. Furthermore the last case gives an opportunity to take into account also the effects of a complete or partial saturation due to the higher order nonlinearities [20]. The use of NLSLE is widely discussed in the literature. There exists a lot of papers in which various aspects of the solitary solutions are investigated. Besides the analytical solutions [21–37] criteria for the existence, stability and integrability have been reported and discussed [38–43]. Also pattern formation and spatiotemporal chaos [44, 45] as well as soliton-soliton (soliton-kink) interactions were studied (mainly numerically) [39, 46]. Some investigations demonstrate the existence of stable stationary radially symmetric modes when a two dimensional laser beam propagates in the nonlinear media considered (e. g. [47]).

In our investigations we shall make an attempt to treat the problem beyond the framework of NLSLE. As it will be seen below, in general the equations governing the problems considered contain more higher derivatives (in particular second time derivative). The presence of the higher derivatives can create new solutions and impose restrictions on the soliton propagation velocity.

We start with the Maxwell's equations

$$\begin{aligned} \operatorname{rot}\mathbf{H} &= \frac{1}{c}\frac{\partial \mathbf{D}}{\partial t}, \\ \operatorname{rot}\mathbf{E} &= -\frac{1}{c}\frac{\partial \mathbf{B}}{\partial t} \end{aligned} \tag{1}$$

where $\mathbf{B} = \mathbf{H}$ and the displacement

$$\mathbf{D} = \mathbf{E} + 4\pi\mathbf{P}, \tag{2}$$

the polarization $\mathbf{P}$ being nonlinear function of the electric field $\mathbf{E}$.

When the dielectric medium under consideration possesses a centre of symmetry, $\mathbf{P}$ has to be an odd function of the electric field, i. e. in the expansion

$$\mathbf{P} = \sum_n \chi^{(n)}\mathbf{E}^n \tag{3}$$

only the susceptibilities of odd order must be presented. This is equivalent to the expansion of the refractive index n in a series of $|\mathbf{E}|^2$. As we shall see, the plane wave velocities in such a media depend on the electric field amplitude. In our consideration we shall keep both $\mathbf{E}^2$ and $\mathbf{E}^4$ terms in the refractive index expansion and shall give solitary solutions of the nonlinear differential equations deduced.

Treating the problems we partially shall use the Lagrangian formalism of the field theory. In terms of the adapted method the nonlinear terms enter the Lagrangian as a constituent part of the "potential energy". The light propagation parameters are determined using the extremal behaviour of this "potential energy". The obtained extrema together with the corresponding boundary conditions play the role of dispersion laws for the frequency, which is now a function of both the wave vector and the field amplitude.

The results obtained demonstrate that (because of the different values and signs of the nonlinear susceptibilities) the light beam, varying its intensity profile in the plane normal to the propagation direction, can self-channallize. After that the intensity profile conserves its ("solitary") form along the propagation direction. The light wave itself, depending on the corresponding conditions, can take the shape of a "bell", "inverted bell", "kink" or "anti-kink" and its wave vector can depend on the size (the effective cross-section) of such formations. Also the frequency of the solitary waves can depend on their amplitude (the amplitude drives the phase and vice versa).

Besides the self-modulation effects, in some situations the solitary formations can propagate with frequencies lower than those of the plane waves. The propagation of solitary pulses in thin wave guides (fibres) is considered, too. For some cases the possibility of the solitary formations to propagate with a velocity higher than the light phase velocity (in the corresponding linear medium) and without "smearing" will be shown to take place.

We shall show also that in some cases the nonlinear differential equations governing the problem possess plane wave solutions with the same parameters as the soliton ones. We shall make a comparison of the plane wave solutions and the solitons from the energetically point of view and shall discuss briefly the validity of the approximations used. Finally, we shall give numerical values for some realistic situations.

2. Formulation of the Problem and Basic Equations

The nonlinear behaviour of the medium can be expressed using the nonlinear polarization $\mathbf{P}$ of the form

$$\mathbf{P} = \kappa\mathbf{E} + \chi^{(3)}\mathbf{E}^3 + \chi^{(5)}\mathbf{E}^5 \tag{4}$$

where $\mathbf{E}$ is the electric field, κ is the linear susceptibility, $\chi^{(3)}$ and $\chi^{(5)}$ are the nonlinear susceptibilities of the third and fifth order, respectively. The light propagation in a lossless medium with a polarization given by (4) and described by Maxwell's equations (1) satisfies the equation

$$\begin{aligned}\left(\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)E &= \frac{4\pi}{c^2}\frac{\partial^2 \mathbf{P}}{\partial t^2} \\ &= \frac{4\pi}{c^2}\left[\kappa\frac{\partial^2 \mathbf{E}}{\partial t^2} + \chi^{(3)}\frac{\partial^2 \mathbf{E}^3}{\partial t^2} + \chi^{(5)}\frac{\partial^2 \mathbf{E}^5}{\partial t^2}\right].\end{aligned} \tag{5}$$

Substituting here the radiation field of the form

$$E = \mathcal{E}(\mathbf{r},\mathbf{t})\,\mathrm{e}^{\mathrm{i}(kz-\omega t)} + \text{c.c.} \tag{6}$$

where $\mathcal{E}(\mathbf{r},\mathbf{t})$ is a complex amplitude, neglecting the second derivatives along the propagation direction in the right-hand side of (5) and keeping the first harmonics only, one obtains the following nonlinear differential equation:

$$\begin{aligned}&\frac{1}{c^{*2}}\frac{\partial^2\mathcal{E}}{\partial t^2} - \frac{\partial^2\mathcal{E}}{\partial x^2} - \frac{\partial^2\mathcal{E}}{\partial y^2} - \frac{\partial^2\mathcal{E}}{\partial z^2} - \mathrm{i}\Big(\frac{2\omega}{c^{*2}}\frac{\partial\mathcal{E}}{\partial t} + 2k\frac{\partial\mathcal{E}}{\partial z}\Big) \\ &- \Big(\frac{\omega^2}{c^{*2}} - k^2\Big)\mathcal{E} - \frac{4\pi\omega^2}{c^2}\Big[3\chi^{(3)}|\mathcal{E}|^2 + 10\chi^{(5)}|\mathcal{E}|^4\Big]\mathcal{E} = 0\end{aligned} \tag{7}$$

where

$$c^* = \frac{c}{\sqrt{1+4\pi\kappa}}. \tag{8}$$

Deriving (7), the time derivatives of $\mathcal{E}^3$ and $\mathcal{E}^5$ are omitted. In general, all the terms creating the first harmonics should appear in (7) but those are assumed to provide small corrections only.

Equation (7) has been analysed in [20] for some cases of self-focusing of a light beam, self-modulation of optical pulses and transmission of the light. In all these cases the dispersion relation

$$ck = \omega\sqrt{1+4\pi\kappa} \tag{9}$$

is supposed to be valid and the cubic nonlinear susceptibility $\chi^{(3)}$ is taken positive. Now we shall consider more general cases of the dispersion relations $\omega(k)$ arising naturally, which include the field amplitude dependence.

It is convenient to use the Lagrangian formalizm of the field theory. Formally, (7) can be deduced from the Lagrangian density of the form

$$\begin{aligned} L = \mathrm{const}\Big\{ \frac{1}{c^{*2}} \left|\frac{\partial \mathcal{E}}{\partial t}\right|^2 - |\nabla\mathcal{E}|^2 \\ - \mathrm{i}\Big[\frac{\omega}{c^{*2}} \left(\frac{\partial \mathcal{E}}{\partial t}\mathcal{E}^* - \mathcal{E}\frac{\partial \mathcal{E}^*}{\partial t}\right) + \mathbf{k}\Big(\mathcal{E}^*\nabla\mathcal{E} - \mathcal{E}\nabla\mathcal{E}^*\Big)\Big] \\ + \left(\frac{\omega^2}{c^{*2}} - k^2\right)|\mathcal{E}|^2 + \frac{2\pi\omega^2}{c^2}\Big[3\chi^{(3)}|\mathcal{E}|^4 + \frac{20}{3}\chi^{(5)}|\mathcal{E}|^6\Big]\Big\}. \end{aligned} \tag{10}$$

In this notation, the energy density of the system considered has the form

$$\begin{aligned} w(x,y,z,t) = \mathrm{const}\Big\{ \frac{1}{c^{*2}} \left|\frac{\partial \mathcal{E}}{\partial t}\right|^2 + |\nabla\mathcal{E}|^2 - \mathrm{i}\mathbf{k}\Big(\mathcal{E}^*\nabla\mathcal{E} - \mathcal{E}\nabla\mathcal{E}^*\Big) \\ + \Big(\frac{\omega^2}{c^{*2}} - k^2\Big)|\mathcal{E}|^2 - \frac{2\pi\omega^2}{c^2}\Big[3\chi^{(3)}|\mathcal{E}|^4 + \frac{20}{3}\chi^{(5)}|\mathcal{E}|^6\Big]\Big\}. \end{aligned} \tag{11}$$

Looking at (10) and (11) one can see that the expression

$$U = -\,\mathrm{const}\Big[\Big(\frac{\omega^2}{c^{*2}} - k^2\Big)|\mathcal{E}|^2 + \frac{2\pi\omega^2}{c^2}\Big(3\chi^{(3)}|\mathcal{E}|^4 + \frac{20}{3}\chi^{(5)}|\mathcal{E}|^6\Big)\Big] \tag{12}$$

can be considered as a “potential energy density” which has to take extremal values as a function of $|\mathcal{E}|$. At the same time, the conditions for the extrema of (12) will play the role of the dispersion relation between ω, k and $\mathcal{E}$.

The simple form of U (12) can be obtained also directly from the term $\mathbf{E}\,\frac{\partial \mathbf{P}}{\partial t}$, which takes part in the expression for the energy of electromagnetic field. For that purpose, when the procedure above is used and the first harmonics in (4) have been kept, the obtained total time derivative

$$\frac{\partial}{\partial t}\left(\kappa|\mathcal{E}|^2 + \frac{3}{2}\chi^{(3)}|\mathcal{E}|^4 + \frac{10}{3}\chi^{(5)}|\mathcal{E}|^6\right)$$

must be excluded.

Also it is easy to see that const in (10–12) has the value

$$\mathrm{const} = \frac{c^2}{4\pi\omega^2}\,. \tag{13}$$

The whole energy of the system has the form

$$W = \int w(\mathbf{r},\mathbf{t})\,\mathrm{d}^3\mathbf{r}. \tag{14}$$

3. The Self-Channelling

The effect of self-channelling can be understood by following elementary considerations. If, for simplicity, the field amplitude E were a constant (E_0), (7) gives the following nonlinear dispersion relation:

$$\nu^2 \equiv \frac{k^2c^2}{\omega^2} = 1 + 4\pi\Big(\kappa + \frac{3}{2}\chi^{(3)}|E_0|^2 + \frac{5}{2}\chi^{(5)}|E_0|^4\Big), \tag{15}$$

i. e. the "refractive index" ν of the medium depends on the field amplitude E_0. Thus, when the light beam intensity varies in the plane normal to the propagation direction the light turns into the high field region focusing and making a channel ("self-channelling").

Now we shall consider some particular cases.

When the electric field has the form

$$\mathbf{E}(\mathbf{r},\mathbf{t}) = \mathcal{E}(x,z)\exp \mathrm{i}(kz-\omega t) + \text{c.c.} \tag{16}$$

where the amplitude $\mathcal{E}(x,z)$ is a function of x and z only, we obtain from (7) the following nonlinear differential equation:

$$\frac{\partial^2\mathcal{E}}{\partial x^2} + \frac{\partial^2\mathcal{E}}{\partial y^2} + \mathrm{i}2k\frac{\partial\mathcal{E}}{\partial z} + \Big(\frac{\omega^2}{c^{*2}} - k^2\Big)\mathcal{E} + \frac{4\pi\omega^2}{c^2}\Big(3\chi^{(3)}|\mathcal{E}|^2 + 10\chi^{(5)}|\mathcal{E}|^4\Big)\mathcal{E} = 0 \tag{17}$$

It is convenient to treat the most interesting cases $\chi^{(3)} > 0$, $\chi^{(5)} < 0$ and $\chi^{(3)} < 0$, $\chi^{(5)} > 0$ separately.

3.1. Case $\chi^{(3)} > 0, \chi^{(5)} < 0$

In this case Eq. (17) has the following soliton solutions:

3.1.1. *First type "kink" ("anti-kink") solution*

$$\begin{aligned}\mathcal{E}(x,z) = \mathcal{E}_0\,\mathrm{e}^{\mathrm{i}(qx+pz+\varphi)}\sinh\frac{x-x_0-w(z-z_0)}{\mathcal{L}} \\ \times\Big(1+\operatorname{sech}^2\eta\sinh^2\frac{x-x_0-w(z-z_0)}{\mathcal{L}}\Big)^{-\frac{1}{2}}\end{aligned} \tag{18}$$

where

$$q = \pm\frac{kw}{\sqrt{1+w^2}}\sqrt{1-R} = \pm k\sin\theta\sqrt{1-R}\,, \qquad (p = k\cos\theta)\,, \tag{19}$$

$$p = -k \pm \Delta p\,, \qquad \Delta p = \frac{k}{\sqrt{1+w^2}}\sqrt{1-R} = \pm k\cos\theta\sqrt{1-R}\,, \tag{20}$$

$$R \equiv 1 - \frac{\omega^2}{k^2c^{*2}} - \frac{12\pi\omega^2\chi^{(3)}\mathcal{E}_0^2\cosh^2\eta}{k^2c^2} + \frac{40\pi\omega^2|\chi^{(5)}|\mathcal{E}_0^4\cosh^4\eta}{k^2c^2}\,, \tag{21}$$

$$\mathcal{E}_0 = \pm\frac{3}{2}\tanh\sqrt{\frac{\chi^{(3)}}{5|\chi^{(5)}|(\cosh 2\eta - 2)}}\,, \tag{22}$$

$$\mathcal{L} = \frac{2c(\cosh 2\eta - 2)}{3\omega\chi^{(3)}\sinh 2\eta}\sqrt{\frac{10|\chi^{(5)}|(1+w^2)}{3\pi}}\,, \qquad (\cosh 2\eta > 2)\,. \tag{23}$$

In (18–20) q and Δp are wave vector components, φ is a phase constant, (x_0, z_0) is the centre of the kink formation (18) and w is a constant which determines the angle θ between the wave front and the propagation direction z:

$$w = \tan\theta\,. \tag{24}$$

The parameter $\mathcal{L}$ characterises the effective size of the region where the field amplitude varies significantly due to the nonlinear effects, i. e. $\mathcal{L}$ plays the role of the kink-length.

The parameter η can be determined by following considerations. As it was mentioned in Sec. 2 one can think that the light propagates in a medium with a "potential energy" U given by (12). The extremum condition of U as a function of $\mathcal{E}$ leads in this case to the equation

$$|\mathcal{E}|^4 - \frac{3\chi^{(3)}}{10|\chi^{(5)}|}|\mathcal{E}|^2 + \frac{(k^2c^{*2} - \omega^2)c^2}{40\pi|\chi^{(5)}|\omega^2c^{*2}} = 0\,. \tag{25}$$

The solution of (25) can be written in the form

$$|\mathcal{E}|^2 = \frac{3\chi^{(3)}}{20|\chi^{(5)}|}\left(1 \pm \sqrt{1-Q}\right) \tag{26}$$

where

$$Q \equiv \frac{10|\chi^{(5)}|(k^2c^{*2} - \omega^2)c^2}{9\pi(\chi^{(3)})^2\omega^2c^{*2}}\,. \tag{27}$$

Looking for a stable solution, the sign plus in (26) must be used. In this case the potential energy U has a minimum. If now we take into account that at infinity ($z \to \infty$)

$$|\mathcal{E}|^2 \to \mathcal{E}_0^2\cosh^2\eta \tag{28}$$

and use (26), the following expression for η can be obtained:

$$\eta = \operatorname{arccosh}\left(\frac{3\sqrt{1-Q}}{2\sqrt{1-Q} - 1}\right)^{\frac{1}{2}} \tag{29}$$

where Q is given by (27) and the inequality

$$0 < k^2 c^{*2} - \omega^2 < \frac{9\pi(\chi^{(3)})^2 \omega^2 c^{*2}}{40|\chi^{(5)}|c^2} \tag{30}$$

must be satisfied.

Taking into account (28), Eq. (25) reduces to the condition

$$R = 0 \tag{31}$$

where R is given by (21). Thus the condition (31) represents the dispersion relation between ω, k and the field amplitude $\mathcal{E}_0$ (see (15)).

Using (31) and (24), the wave vector components q and p become equal to:

$$q = \pm \frac{kw}{\sqrt{1+w^2}} = \pm k \sin\theta\,, \tag{32}$$

$$p = -k \pm \frac{k}{\sqrt{1+w^2}} = k(1 \mp \cos\theta) \tag{33}$$

instead of (19) and (20), respectively.

If now we calculate the energy of the system (per unit length in x-direction) according to (14) and (13), we obtain that integral (14) contains the divergent term

$$W_1 = \frac{c^2 \mathcal{L}^2 \mathcal{E}_0^2 \cosh^2\eta}{\pi\omega^2} \left(q^2 + k^2 + 2kp\right) \int\limits_{-\infty}^{+\infty} \mathrm{d}\zeta\,. \tag{34}$$

The same problem (a divergence of the total energy) appears also in the cases of usual plane wave solutions with constant amplitudes and therefore a comparison of the soliton energy with the energy corresponding to the plane wave is of interest only. We would like to make use of the fact that (17) has the following plane wave solution:

$$\mathcal{E}_1(x,z) = \mathcal{E}_0 \cosh\eta\, \mathrm{e}^{\mathrm{i}(qx+pz+\varphi)} \tag{35}$$

with the same parameters as for the kink solution (18) and with energy (34). So the energy difference between soliton energies W and W_1 is:

$$\begin{aligned}\triangle W = -\frac{27c(\chi^{(3)})^2\eta}{160\omega|\chi^{(5)}|^{\frac{3}{2}}} \sqrt{\frac{3(1+w^2)}{10\pi}}\, \frac{1-2\cosh 2\eta}{(\cosh 2\eta - 2)^2} \\ \left[1 + \frac{\sinh 2\eta}{2\eta}\, \frac{\cosh 2\eta - 2}{2\cosh 2\eta - 1}\right].\end{aligned} \tag{36}$$

It can be seen that $\triangle W > 0$, i. e. the plane wave solution (35) is energetically more favourable than (18). (In fact $\triangle W$ determines the "self-energy" of the kink.)

Finally, it must be taken into account that the series expansion of the polarization P (4) as well as the corresponding nonlinear equation (17) are valid when the $\mathbf{E}^5$ term is small compared to $\mathbf{E}^3$ one. Looking at (17) we find that the inequality

$$|\mathcal{E}_{\max}| < \sqrt{\frac{3}{10}\left|\frac{\chi^{(3)}}{\chi^{(5)}}\right|} \tag{37}$$

must be satisfied. For the case considered (37) leads to the condition

$$\cosh 2\eta < 5 \tag{38}$$

(see also (22)).

3.1.2. *Second type "kink" ("anti-kink") solution*

$$\mathcal{E} = \mathcal{E}_0\left(1 + \mathrm{e}^{\pm\frac{x - x_0 - w(z - z_0)}{\mathcal{L}}}\right)^{-\frac{1}{2}} \tag{39}$$

where

$$q = \pm\frac{\omega w}{c*}\sqrt{1 + \frac{1 + w^2}{4\mathcal{L}^2}\frac{c^{*2}}{\omega^2}}\,, \qquad p = -k + \frac{q}{w}\,,$$

$$\mathcal{E}_0 = \frac{3}{2}\sqrt{\frac{\chi^{(3)}}{10|\chi^{(5)}|}}\,, \tag{40}$$

$$\mathcal{L} = \frac{c}{3\omega\chi^{(3)}}\sqrt{\frac{10|\chi^{(5)}|(1 + w^2)}{3\pi}}\,. \tag{41}$$

In fact, this solution represents a "kink" (when the sign minus in (39) is used) and "anti-kink" (when sign plus in (39) is taken). The parameters $\mathcal{L}$ and φ have the same meaning as in the Subcase 1 above. Following the same procedure as in the Subcase 1, the extremum condition of U as a function of $\mathcal{E}$ leads to (25), which now is a condition for ω

$$\omega^2 = \frac{k^2c^2}{1 + 4\pi\left[\kappa + \frac{3}{2}\chi^{(3)}|\mathcal{E}_0|^2 - 10|\chi^{(5)}|\mathcal{E}_0|^4\right]}\,. \tag{42}$$

In (48) $\mathcal{E}_0$ is substituted instead of $|\mathcal{E}|$ because the corresponding plane wave solution of (17) can be written in the form

$$\mathcal{E}_1 = \mathcal{E}_0\,\mathrm{e}^{\mathrm{i}(qx + pz + \varphi)}\,. \tag{43}$$

In this case the energy W can be compared with the energy W_1 which corresponds to the situation when the light occupies the region left of the anti-kink centre (or right of the kink centre). Then the solution $\mathcal{E}_1$ is a step function which has a form (43) for the light region and zero for the rest. Taking into account also that the relation

$$p^2 + q^2 + 2kp + k^2 - \frac{\omega^2}{c^{*2}} = \frac{1+\omega^2}{\mathcal{L}^2}$$

takes place, one can find that the energy difference is now

$$\Delta W = W - W_1 = \frac{27c(\chi^{(3)})^2\eta}{160\omega|\chi^{(5)}|^{\frac{3}{2}}}\sqrt{\frac{3(1+w^2)}{10\pi}}\,. \tag{44}$$

Hence, in this case the solution (39) is not favourable energetically in comparison with the step function.

3.1.3. *"Inverse bell" soliton solution*

$$\begin{aligned}\mathcal{E}(x,z) &= \mathcal{E}_0\, \mathrm{e}^{\mathrm{i}(qx+pz+\varphi)} \cosh\frac{x-x_0-w(z-z_0)}{\mathcal{L}} \\ &\times \left(1+\operatorname{cosech}^2\eta\cosh^2\frac{x-x_0-w(z-z_0)}{\mathcal{L}}\right)^{-\frac{1}{2}}\end{aligned} \tag{45}$$

where q and p have the form (32) and (33), respectively,

$$\mathcal{E}_0 = \frac{3}{2}\operatorname{cotanh}\sqrt{\frac{\chi^{(3)}}{5|\chi^{(5)}|}(\cosh 2\eta + 2)}\,, \tag{46}$$

$$\mathcal{L} = \frac{2c(\cosh 2\eta+2)}{3\omega\chi^{(3)}\sinh 2\eta}\sqrt{\frac{10|\chi^{(5)}|(1+w^2)}{3\pi}}\,, \tag{47}$$

$$\eta = \operatorname{arcsinh}\left(\frac{3\sqrt{1-Q}}{1-2\sqrt{1-Q}}\right)^{\frac{1}{2}}\,, \tag{48}$$

Q is given by (27) and for the validity of a stable solution of the type (45) the inequalities

$$\frac{27\pi\omega^2c^{*2}(\chi^{(3)})^2}{40c^2|\chi^{(5)}|} < k^2c^{*2} - \omega^2 < \frac{9\pi\omega^2c^{*2}(\chi(3))^2}{10|\chi^{(5}|c^2}$$

must be satisfied.

In this case, the energy W of the soliton formation is divergent too. Taking into account that (17) has another plane-wave solution of the form

$$\mathcal{E}_1(x,z) = \mathcal{E}_0\sinh\eta\,\mathrm{e}^{\mathrm{i}(qx+pz+\varphi)} \tag{49}$$

with the same parameters, as for the solution (45), for the energy difference one can find $\Delta W \equiv W - W_1$

$$\Delta W = -\frac{27c(\chi^{(3)})^2\eta}{160\omega|\chi^{(5)}|^{\frac{3}{2}}}\sqrt{\frac{3(1+w^2)}{10\pi}}\,\frac{1+2\cosh 2\eta}{(2+\cosh 2\eta)^2} \times\left(1-\frac{\sinh\eta}{\eta}\,\frac{2+\cosh 2\eta}{1+2\cosh 2\eta}\right). \tag{50}$$

The analysis of (50) shows that $\Delta W < 0$, i. e. as in the Subcases 1 and 2 above, the solitary formation (45) is energetically more favourable in comparison with the plane-wave one (49).

It must be noted that because of the series expansion of the polarization P (4) (see also the corresponding equation (17)), the inequality (37) leads to a condition which is always satisfied.

The intensity profile of the electric field has the form

$$|E|^2 \sim \frac{\cosh^2\frac{x-x_0-w(z-z_0)}{\mathcal{L}}}{1+\operatorname{cosech}^2\eta\cosh^2\frac{x-x_0-w(z-z_0)}{\mathcal{L}}},$$

i. e. represents an "inverse bell". Thus, "dark" channel is formed (the light is expelled from the channel).

3.1.4. *"Bell"-soliton solution*

$$\mathcal{E}_1(x,z) = \mathcal{E}_0\,\mathrm{e}^{\mathrm{i}(qx+pz+\varphi)}\left(1+\operatorname{sech}\eta\cosh\frac{x-x_0-w(z-z_0)}{\mathcal{L}}\right)^{-\frac{1}{2}} \tag{51}$$

where

$$q = \pm\frac{\omega}{c^*}\frac{w}{\sqrt{1+w^2}}\sqrt{1+\frac{1+w^2}{\mathcal{L}^2}\frac{c^{*2}}{\omega^2}} = \pm\sin\theta\sqrt{\frac{\omega^2}{c^{*2}}+\frac{1}{4\mathcal{L}^2\cos^2\eta}},$$

$$p = -k \pm \Delta p,$$

$$\Delta p = \frac{\omega}{c^*}\frac{1}{\sqrt{1+w^2}}\sqrt{1+\frac{1+w^2}{4\mathcal{L}^2}\frac{c^{*2}}{\omega^2}} = \sqrt{\frac{\omega^2\cos^2\theta}{c^{*2}}+\frac{1}{4\mathcal{L}^2}},$$

$$\mathcal{E}_0 = \pm\frac{3}{2}\tanh\eta\sqrt{\frac{\chi^{(3)}}{10|\chi^{(5)}|}}, \tag{52}$$

$$\mathcal{L} = \frac{c\coth\eta}{3\omega\chi^{(3)}}\sqrt{\frac{|\chi^{(5)}|(1+w^2)}{3\pi}} = \frac{c\coth\eta}{3\omega\chi^{(3)}\cos\theta}\sqrt{\frac{10|\chi^{(5)}|}{3\pi}}, \tag{53}$$

$$\theta \equiv \arctan w. \tag{54}$$

The solution (51) represents a solitary formation of the "bell"-form with an effective width $\mathcal{L}$. Now the parameter η can be determined after giving the electric field amplitude $\mathcal{E}_0$ (or the maximal electric field amplitude $\mathcal{E}_{\max}$). The later can be expressed e. g. in terms of the beam power density (per unit area) P_0. Another possibility is to introduce the "normalization constant" I as follows:

$$I = \int_{-\infty}^{\infty} |\mathcal{E}|^2 \, \mathrm{d}x \tag{55}$$

which also can be expressed by the beam power density per unit length in y-direction. In the following considerations the parameter η will be determined when $\mathcal{E}_0$ (or $\mathcal{E}_{\max}$) is given and the inequality (37) is satisfied.

Using (14), (11) and (12), the following expression for the energy of the system W can be obtained:

$$\begin{aligned} W = {} & \frac{27c(\chi^{(3)})^2\eta}{160\omega|\chi^{(5)}|^{\frac{3}{2}}} \sqrt{\frac{3(1+w^2)}{10\pi}} \tanh\eta \Bigg[\frac{\sin\eta}{\eta} + \frac{4(\sinh\eta - \eta)}{\tanh^3\eta} \\ & - \frac{(\cosh^3\eta - 2\cosh^2\eta + 1)\cosh\eta}{1+\cosh\eta} \Bigg] . \end{aligned} \tag{56}$$

In the case considered the inequality (37) leads to the condition

$$\tanh\eta / \sqrt{1 + \operatorname{sech}\eta} < \frac{2}{\sqrt{3}} \, . \tag{57}$$

The analysis shows that when $\eta > \eta_0$ ($\eta \approx 1.173$), the energy value W is negative and the soliton formation is energetically favourable. One can see from (6), (51) and (52) that the field intensity profile has the form

$$|E|^2 = \frac{9\chi^{(3)}\tanh^2\eta}{10|\chi^{(5)}|} \left(1 + \operatorname{sech}\eta \cosh \frac{x - x_0 - w(z - z_0)}{\mathcal{L}} \right)^{-1} .$$

Thus, the soliton formation (51) is acting as a wave guide channel along the direction determined by θ (54), i. e. in this case the light concentrates in the region with a "cross-section" $2\mathcal{L}$ forming a "filament".

3.1.5. *Second "inverse bell" soliton solution*

$$\mathcal{E}(x,z) = \mathcal{E}_0 \, \mathrm{e}^{\mathrm{i}(qx+pz+\varphi)} \left(1 - \operatorname{sech} \frac{x - x_0 - w(z - z_0)}{\mathcal{L}} \right)^{\frac{1}{2}} \tag{58}$$

where q and p are given by (32)and (33), respectively ,

$$\mathcal{E}_0 = \pm\frac{3}{4}\sqrt{\frac{\chi^{(3)}}{5|\chi^{(5)}|}}\,,$$

$$\mathcal{L} = \frac{c^*}{2}\sqrt{\frac{5(1+w^2)}{\omega^2 - k^2 c^{*2}}} = \frac{2c}{3\omega\chi^{(3)}}\sqrt{\frac{10|\chi^{(5)}|(1+w^2)}{3\pi}}$$

and

$$\omega^2 - k^2 c^{*2} = \frac{27\pi(\chi^{(3)})^2\omega^2 c^{*2}}{32|\chi^{(5)}|c^2}\,.$$

Taking into account that (43) is also a solution of (17) with the same parameters as for (58), for the energy difference $\Delta W \equiv W_1 - W_2$ (the "self-energy" of the solitary formation) one can find

$$\Delta W = \frac{81(4+\pi)}{2560}\sqrt{\frac{1+w^2}{30\pi}}\,\frac{c}{\omega}\,\frac{\left(\chi^{(3)}\right)^2}{|\chi^{(5)}|^{\frac{3}{2}}}\,.$$

Such a manner a "dark" channel formed is not energetically favourable in comparison with the plane wave (43).

When the field amplitude varies slowly in z-direction so as

$$\left|\frac{\partial^2\mathcal{E}}{\partial z^2}\right| \ll k_z\left|\frac{\partial\mathcal{E}}{\partial z}\right|\,, \tag{59}$$

after neglecting the second derivative in (17) the following nonlinear Schrödinger-like equation with nonlinearities of third and fifth power with regard to the field amplitude is obtained:

$$\mathrm{i}2k\frac{\partial\mathcal{E}}{\partial z} = \left(k^2 - \frac{\omega^2}{c^{*2}}\right)\mathcal{E} - \frac{\partial^2\mathcal{E}}{\partial x^2} - \frac{4\pi\omega^2}{c^2}\left(3\chi^{(3)}|\mathcal{E}|^2 + 10\chi^{(5)}|\mathcal{E}|^4\right)\mathcal{E}\,. \tag{60}$$

This equation has the following solitary solutions.

3.1.6. *"Kinks" and "anti-kinks" of the type (18)*

The corresponding parameters are now

$$q = kw\,, \tag{61}$$

$$p = -\frac{1}{2}kw^2\,, \tag{62}$$

$\mathcal{E}_0$, $\mathcal{L}$, η and Q are given by (22), (23) (where $w = 0$), (29) and (27), respectively. Also the conditions $\omega < kc^*$ and (38) must be satisfied.

Eq. (60) has also a plane-wave solution of the type (35) with the parameters of the kink. The calculation of the energies corresponding to both formations gives for the energy difference $\Delta W = W - W_1$ the value (36), where $w = 0$.

3.1.7. *"Kink" and "anti-kink" solutions of the form (39)*
The corresponding parameters are

$$q = kw\,,$$
$$p = \frac{1}{2k}\left[\left(1 + 4\pi\kappa + \frac{27\pi}{40}\frac{(\chi^{(3)})^2}{\chi^{(5)}}\right)\frac{\omega^2}{c^2} - (1+w^2)k^2\right],$$

$\mathcal{E}_0$ is given by (40) and $\mathcal{L}$ determines from (41) after setting $w = 0$.

As in the Subsubsection 2 above this anti-kink (or kink) can be compared with the "step" solution when e. g. $\mathcal{E}_1$ has the form (43) if $-\infty < x - x_0 - vt < 0$ and $\mathcal{E}_1 = 0$ if $0 < x - x_0 - vt < \infty$. The calculation gives for the energy difference $\Delta W = W - W_1$ the expression (44) (in which $w = 0$). Obviously, also when (59) takes place the step function is energetically more favourable than the anti-kink (or kink).

3.1.8. *"The "inverse bell" soliton solution of the type (45),*
The parameters q, p and $\mathcal{E}_0$ have the form (61), (62) and (46), respectively and $\mathcal{L}$ is given by (47), where $w = 0$. At that if

$$0 < k^2c^{*2} - \omega^2 < \frac{27\pi(\chi^{(3)})^2\omega^2c^{*2}}{40|\chi^{(5)}|c^2}\,,$$

then

$$\eta = \operatorname{arcsinh}\left(\frac{3\sqrt{1-Q}}{1-2\sqrt{1-Q}}\right)^{\frac{1}{2}}$$

and if

$$\frac{27\pi(\chi^{(3)})^2\omega^2c^{*2}}{40|\chi^{(5)}|c^2} < k^2c^{*2} - \omega^2 < \frac{9\pi(\chi^{(3)})^2\omega^2c^{*2}}{10|\chi^{(5)}|c^2}\,,$$

then

$$\eta = \operatorname{arcsinh}\left(\frac{3\sqrt{1-Q}}{2\sqrt{1-Q}-1}\right)^{\frac{1}{2}}.$$

For cases Q is given by (27).

Taking into account that (43) is also a solution of (60) (with the same parameters as for the solitary formation), for the energy difference $\Delta W = W - W_1$ can be obtained the expression (44), where $w = 0$.

3.1.9. *The "bell"-soliton of the type (51)*

$$p = \frac{k}{2}\left(\frac{\omega^2}{k^2 c^{*2}} - \frac{1}{4k^2\mathcal{L}^2} - 1 - w^2\right),$$

q, $\mathcal{E}_0$ and $\mathcal{L}$ are given by (61) , (52) and (53) (where $w = 0$), respectively. The energy of this solitary formation W has the form (56), in which $w = 0$. In the case considered the condition (37) leads to (57).

3.2. The Case $\chi^{(3)} < 0$, $\chi^{(5)} > 0$

If $\chi^{(3)} < 0$ and $\chi^{(5)} > 0$ Eq. (17) has the following soliton solutions.

3.2.1. *"Kinks" and "anti-kinks"*

$$\begin{aligned} \mathcal{E}(x,z) = \mathcal{E}_0\, \mathrm{e}^{\mathrm{i}(qx+pz+\varphi)} \sinh \frac{x - x_0 - w(z - z_0)}{\mathcal{L}} \\ \times \left(1 + \sec^2\eta \sinh^2 \frac{x - x_0 - w(z - z_0)}{\mathcal{L}}\right)^{-\frac{1}{2}} \end{aligned} \quad (63)$$

where q and p have the form (32) and (33), respectively,

$$\mathcal{E}_0 = \pm\frac{3}{2}\tan\sqrt{\frac{|\chi^{(3)}|}{5\chi^{(5)}(2-\cos 2\eta)}}\,, \quad (64)$$

$$\mathcal{L} = \frac{2c(2-\cos 2\eta)}{3\omega|\chi^{(3)}|\sin 2\eta}\sqrt{\frac{10\chi^{(5)}(1+w^2)}{3\pi}} = \frac{2c(2-\cos 2\eta)}{3\omega|\chi^{(3)}|\sin 2\eta\cos\theta}\sqrt{\frac{10\chi^{(5)}}{3\pi}}\,, \quad (65)$$

$$\eta = \arccos\left(\frac{3\sqrt{1-Q}}{1+2\sqrt{1-Q}}\right)^{\frac{1}{2}}, \quad (66)$$

$$Q \equiv \frac{10\chi^{(5)}(k^2c^{*2} - \omega^2)c^2}{9\pi(\chi^{(3)})^2\omega^2c^2}\,. \quad (67)$$

For the case considered $\omega > kc^*$.

The parameters above are obtained on the analogy of the Subcase 1 in the previous section ($\chi^{(3)} > 0$, $\chi^{(5)} < 0$). The main difference is the change of the hyperbolic functions with trigonometrical ones. Therefore a periodicity of the soliton parameters with respect to η takes place.

Eq. (17) has also a plane-wave solution of the form

$$\mathcal{E}_1(x,z) = \mathcal{E}_0 \cos\eta\, \mathrm{e}^{\mathrm{i}(qx+pz+\varphi)} \quad (68)$$

with the same parameters as for the "kink" (63). The energy difference $\Delta W \equiv W - W_1$ (W and W_1 correspond to (63) and (68), respectively) is equal to

$$\begin{aligned} \Delta W = \frac{27c(\chi^{(3)})^2\eta}{160\omega(\chi^{(5)})^{\frac{3}{2}}} \sqrt{\frac{3(1+w^2)}{10\pi}} \frac{1-2\cos 2\eta}{(2-\cos 2\eta)^2} \\ \times \left(1 + \frac{\sin 2\eta}{2\eta} \frac{2-\cos 2\eta}{1-2\cos 2\eta}\right). \end{aligned} \tag{69}$$

The inequality (37) leads to the condition $\cos 2\eta > 5$ which always is satisfied.

The presence of periodical functions of η shows that there exist many values of η for which ΔW can be negative as well as positive. E. g. if $\eta < \eta_0 = \arccos \frac{1}{2} \approx 0.523$, then $\Delta W < 0$, i. e. the soliton formation is energetically more favourable than the plane-wave one.

The intensity profile of the electric field has the form

$$\begin{aligned} |E|^2 = \frac{9|\chi^{(3)}|}{10\chi^{(5)}} \frac{\tan^2\eta}{2-\cos 2\eta} \sinh \frac{x - x_0 - w(z-z_0)}{\mathcal{L}} \\ \times \left(1 + \sec^2\eta \sinh^2 \frac{x - x_0 - w(z-z_0)}{\mathcal{L}}\right)^{-1}. \end{aligned} \tag{70}$$

One can see that (70) has a minimum (zero) at the centre of the formation. So the light is expelled from the channel formed in the direction $\theta = \arctan w$, i. e. the dark channel (dark filament) is formed with a crosssection $\sim 2\mathcal{L}$.

3.2.2. *"Bell"-soliton solution*

$$\begin{aligned} \mathcal{E}(x,z) = \mathcal{E}_0\, \mathrm{e}^{\mathrm{i}(qx+pz+\varphi)} \cosh \frac{x - x_0 - w(z-z_0)}{\mathcal{L}} \\ \times \left(cosec^2\eta \cosh^2 \frac{x - x_0 - w(z-z_0)}{\mathcal{L}} - 1\right)^{-\frac{1}{2}} \end{aligned} \tag{71}$$

where

$$q = \pm \frac{kw}{\sqrt{1+w^2}} \sqrt{1 + \frac{4(1+w^2)}{k^2\mathcal{L}^2}} = \pm k \sin\theta \sqrt{1 + \frac{4}{k^2\mathcal{L}^2\cos^2\theta}}\,, \tag{72}$$

$$\begin{aligned} p = -k \pm \Delta p\,, \\ \Delta p = \frac{k}{\sqrt{1+w^2}} \sqrt{1 + \frac{4(1+w^2)}{k^2\mathcal{L}^2}} = k\cos\theta \sqrt{1 + \frac{4}{k^2\mathcal{L}^2\cos^2\theta}}\,, \end{aligned} \tag{73}$$

$$\mathcal{E}_0 = \frac{1}{2} \cot\eta \sqrt{\frac{3|\chi^{(3)}|}{10\chi^{(5)}}}\,, \tag{74}$$

$$\mathcal{L} = \frac{2c(2+\cos 2\eta)}{\omega|\chi^{(3)}|\sin 2\eta}\sqrt{\frac{10\chi^{(5)}(1+w^2)}{3\pi}}\,, \tag{75}$$

$$\eta = \arcsin(1-Q)^{\frac{1}{4}}\,. \tag{76}$$

The condition $Q < 1$ leads to the inequality

$$0 < \omega^2 - k^2c^{*2} < \frac{9\pi(\chi^{(3)})^2\omega^2c^{*2}}{10\chi^{(5)}c^2}\,. \tag{77}$$

Besides (71), Eq. (17) has a monochromatic plane-wave solution of the form

$$\mathcal{E}_1(x,z) = \mathcal{E}_0\,\mathrm{e}^{\mathrm{i}(qx+pz+\varphi)} \tag{78}$$

where q, p, η and $\mathcal{E}_0$ are given by (72), (73), (76) and (74), respectively.

The energy W, corresponding to the formation (71) is divergent. Making the same steps as for the "kink"-formation (63) and taking into account (78) which energy W_1 contains the same divergent terms as W, one obtains

$$\begin{aligned}\Delta W \equiv W - W_1 = -\frac{9}{160}\sqrt{\frac{1+w^2}{30\pi}}\,\frac{c}{\omega}\,\frac{(\chi^{(3)})^2}{(\chi^{(5)})^{\frac{3}{2}}} \\ \times\left[6(1+4\cos^2\eta) + (1-\frac{\sin 2\eta}{2\eta})(3\cot^2\eta - 11)\right].\end{aligned} \tag{79}$$

One can see that $\Delta W < 0$, i. e. the solitary formation is energetically more favourable than (78). It must be noted that the condition (37) is always satisfied (it leads to the trivial inequality $1 < \sqrt{2}$).

Considering the intensity profile of the electric field

$$\begin{aligned}|E|^2 = \frac{9|\chi^{(3)}|}{20\chi^{(5)}}\cot^2\eta\cosh^2\frac{x-x_0-w(z-z_0)}{\mathcal{L}} \\ \times\left(\operatorname{cosec}^2\eta\cosh^2\frac{x-x_0-w(z-z_0)}{\mathcal{L}} - 1\right)^{-1}\end{aligned}$$

(a "bell"-form) one can see that alight guide-channel is formed.

3.2.3. *Second "bell"-soliton solution"*

$$\mathcal{E}(x,z) = \mathcal{E}_0\,\mathrm{e}^{\mathrm{i}(qx+pz+\varphi)}\left(1+\operatorname{sech}\frac{x-x_0-w(z-z_0)}{\mathcal{L}}\right)^{\frac{1}{2}} \tag{80}$$

where q and p have the form (32) and (33), respectively,

$$\mathcal{E}_0 = \pm\frac{3}{4}\sqrt{\frac{|\chi^{(3)}|}{5\chi^{(5)}}}\,, \tag{81}$$

$$\mathcal{L} = \frac{2c}{3\omega|\chi^{(3)}|}\sqrt{\frac{10\chi^{(5)}(1+w^2)}{3\pi}} \,. \tag{82}$$

As well as above, there exist also a solution of (17) of the form

$$\mathcal{E}_1(x,z) = \mathcal{E}_0\, \mathrm{e}^{\mathrm{i}(qx+pz+\varphi)} \tag{83}$$

with the parameter p, q and $\mathcal{E}_0$ characterizing (80). The energy difference $\Delta W = W - W_1$ is now given by

$$\Delta W = \frac{27(4-\pi)}{2560}\sqrt{\frac{3(1+w^2)}{10\pi}}\,\frac{c}{\omega}\,\frac{(\chi^{(3)})^2}{(\chi^{(5)})^{\frac{3}{2}}} \,. \tag{84}$$

Obviously $\Delta W > 0$, i. e. the solitary formation (80) is not energetically favourable in comparison with (65).

3.2.4. *"Inverse bell"-soliton solution*

$$\mathcal{E}(x,z) = \mathcal{E}_0\, \mathrm{e}^{\mathrm{i}(qx+pz+\varphi)}\left(1 - \mathrm{sech}\,\frac{x - x_0 - w(z-z_0)}{\mathcal{L}}\right)^{\frac{1}{2}} \tag{85}$$

where q, p, $\mathcal{E}_0$ and $\mathcal{L}$ have the same values as in the Subcase 3 above (formulae (32), (33), (81) and (82), respectively). Calculating the energies W and W_1 corresponding to (85) and (83), respectively, we find that in this case ΔW (the self-soliton energy) is given by

$$\Delta W = \frac{27(4+\pi)}{2560}\sqrt{\frac{3(1+w^2)}{10\pi}}\,\frac{c}{\omega}\,\frac{(\chi^{(3)})^2}{(\chi^{(5)})^{\frac{3}{2}}} \,. \tag{86}$$

As in the previous Subcase 3, $\Delta W > 0$ and we conclude that the plane-wave solution (83) is more favourable than (85).

It is interesting to note that the solutions (80) and (85) can be represented as a superposition of a "kink" and an "anti-kink" (39) which length $\mathcal{L}$ is taken to be $\mathcal{L}/2$.

When the electric field amplitude $\mathcal{E}$ varies so slowly that (59) is satisfied, Eq. (17) turns into (60). Then for $\chi^{(3)} < 0$, $\chi^{(5)} > 0$ the following soliton solutions take place.

3.2.5. *"The solitary formation of the "kink"-type (63)*
Now q, p, $\mathcal{E}_0$ and $\mathcal{L}$ have the form (61), (62), (64) and (65) (where $w = 0$), respectively,

$$\eta = \arccos\left(\frac{3\sqrt{1+Q}}{1+2\sqrt{1+Q}}\right)^{\frac{1}{2}}, \tag{87}$$

(Q being given by (67)) and

$$0 < \omega^2 - k^2 c^{*2} < \frac{9\pi\chi(3)^2\omega^2 c^{*2}}{10\chi^{(5)}c^2}\,. \tag{88}$$

In this case the energy of the solitary formation obtained can be compared with the plane-wave solution (68), which also takes place (with the same parameters as the "kink"). The energy difference $\Delta W = W - W_1$ has the form (69), where $w = 0$. Thus $\Delta W < 0$ when $\eta < \arccos(1/2)$ and the solitary formation is energetically more favourable than plane-wave one (see the text below (69)).

3.2.6. *Second "bell"-soliton solution (80)*

$$q = \pm\frac{\omega}{c} w\sqrt{1 + 4\pi\kappa - \frac{27\pi(\chi^{(3)})^2}{32\chi^{(5)}}}\,, \tag{89}$$

$$p = -k + w/q\,, \tag{90}$$

$\mathcal{E}_0$ and $\mathcal{L}$ have the values (81) and (82) (where $w = 0$), respectively. Compare the energy W of this "bell-solitary formation with the energy W_1 of the plane wave (83) which also takes place here, one obtains for the energy difference $\Delta W = W - W_1$ the result given by (84) (with $w = 0$). Since $\Delta W > 0$, the plane wave is energetically more favourable than the solitary formation.

3.2.7. *"Inverse bell" soliton solution (85)*
q, p, $\mathcal{E}_0$ and $\mathcal{L}$ have the same values as in the previous Subcase 6 but now the energy difference $\Delta W \equiv W - W_1$ with respect to the plane wave solution (83) is given by (86), in which w is taken to be equal to zero.

4. Conclusions

The results obtained in this paper demonstrate that a wide variety of solitary solutions can exist in nonlinear media when the polarisation has the form (4). For the most interesting cases when the susceptibilities $\chi^{(3)}$ and $\chi^{(5)}$ have opposite signs there exist a number of "kinks" ("anti-kinks"), "bell" and "inverse bell" solutions which lead to a number of different kind of optical channels. (As one can see from the following papers solitary pulses can also exist and propagate by the same circumstances.) The results obtained in virtue of Lagrangian formalism afford an opportunity to treat the different realisations of the self-action effects from the energetically point of view. The analytical solutions obtained for the nonlinear equations deduced as well as the Lagrangian treatment of the problem can be used also in others areas of the physical sciences, in particular in the classical and quantum field theory.

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